Improved Boundary Elements for Interface Cracks

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Outline

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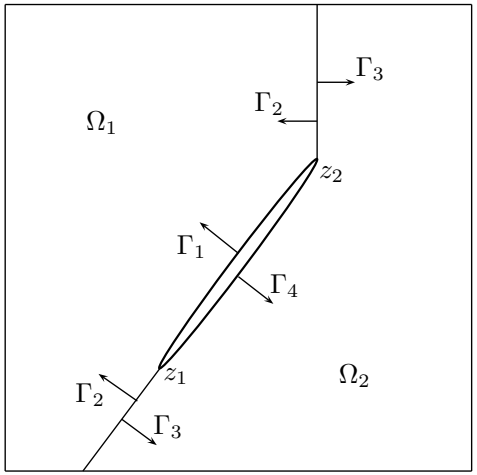
- interface crack
- bounday integral equation
- regularity of solution
- improved Galerkin BEM

interface crack problems

Costabel, M., Stephan, E.P., J.Math.Anal.Appl. (1985)
v.Petersdorff, T., Math.Meth.Appl.Sci. (1989)



Ω_1 bounded Lipschitz domain $\subset \mathbb{R}^2$,
$\Omega_2 = \mathbb{R}^2 \backslash \overline{\Omega_1}$

Find $u_j \in H^1_{loc}(\Omega_j),\ j = 1, 2$

$$\left(\Delta + k_j^2\right) u_j = 0 \text{ in } \Omega_j,\ j = 1, 2 \tag{1}$$

$$u_1|_{\Gamma_2} = u_2|_{\Gamma_3} + v_2, \qquad \frac{1}{\rho_1} \frac{\partial u_1}{\partial n}\bigg|_{\Gamma_2} = -\frac{1}{\rho_2} \frac{\partial u_2}{\partial n}\bigg|_{\Gamma_3} + \psi_2 \tag{2}$$

$$\frac{1}{\rho_1} \frac{\partial u_1}{\partial n}\bigg|_{\Gamma_1} = \psi_1 \qquad \frac{1}{\rho_2} \frac{\partial u_2}{\partial n}\bigg|_{\Gamma_4} = \psi_4 \tag{3}$$

$\operatorname{Im} k_j^2 \geqslant 0,\ k_j \neq 0;\ \rho_j > 0.$

compatibility conditions:

$$\left(u_1|_{\Gamma_1}\,,\,u_1|_{\Gamma_2}\right)\in H^{1/2}(\partial\Omega_1)\iff u_1|_{\Gamma_i}\in H^{1/2}(\Gamma_i),\ u_1|_{\Gamma_1}-u_1|_{\Gamma_2}\in\tilde{H}^{1/2}(\mathbb{R}^+)$$

$$\left(\frac{\partial u_1}{\partial n}\Big|_{\Gamma_1}\,,\,\frac{\partial u_1}{\partial n}\Big|_{\Gamma_2}\right)\in H^{-1/2}(\partial\Omega_1)\iff \frac{\partial u_1}{\partial n}\Big|_{\Gamma_i}\in H^{-1/2}(\Gamma_i),\ \frac{\partial u_1}{\partial n}\Big|_{\Gamma_1}+\frac{\partial u_1}{\partial n}\Big|_{\Gamma_2}\in\tilde{H}^{-1/2}(\mathbb{R}^+)$$

Theorem 1

For given boundary data $\psi_1, v_2, \psi_2, \psi_4$ *in*

$$Y=\big\{(\psi_1,v_2,\psi_2,\psi_4)\in H^{-1/2}(\Gamma_1)\times H^{1/2}(\Gamma_2)\times H^{-1/2}(\Gamma_2)\times H^{-1/2}(\Gamma_4)\,|\,(\psi_1+\psi_4,\psi_2)\in H^{-1/2}(\partial\Omega_1)\big\} \tag{4}$$

there exists a unique solution $u_1\in H^1_{loc}(\Omega_1)$, $u_2\in H^1_{loc}(\Omega_2)$ *of* (1)-(3).

$\left(u_1|_{\Gamma_1}\,,\rho_1^{-1}\ \frac{\partial u_1}{\partial n}\Big|_{\Gamma_2}\,,\,u_1|_{\Gamma_2}\,,\,u_2|_{\Gamma_4}\right)$ *belongs to affine space*

$$X=\big\{(w_1,\phi_2,w_2,w_4)\in H^{1/2}(\Gamma_1)\times H^{-1/2}(\Gamma_2)\times H^{1/2}(\Gamma_2)\times H^{1/2}(\Gamma_4)\,|\,(w_1,w_2)\in H^{1/2}(\partial\Omega_1),(w_4,w_2-v_2)\in H^{1/2}(\partial\Omega_2),(\psi_1,\phi_2)\in H^{-1/2}(\partial\Omega_1)\big\}. \tag{5}$$

boundary integral equation

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$$H\zeta = g \tag{6}$$

for unknown boundary data $\zeta = (w_1, \phi_2, w_2, w_4) \in X$ where g is determined by given data $(\psi_1, v_2, \psi_2, \psi_4) \in Y$

$$H = \begin{bmatrix} \rho_1^{-1} D_{11} & K'_{12} & \rho_1^{-1} D_{12} & 0 \\ -K_{21} & \rho_1 V_{22} + \rho_2 V_{33} & -K_{22} + K_{33} & K_{34} \\ \rho_1^{-1} D_{21} & K'_{22} - K'_{33} & \rho_1^{-1} D_{22} + \rho_2^{-1} D_{33} & \rho_2^{-1} D_{34} \\ 0 & -K'_{43} & \rho_2^{-1} D_{43} & \rho_2^{-1} D_{44} \end{bmatrix} \tag{7}$$

$$V_{ij}\psi_j(x) = 2\int_{\Gamma_j} G_r(x,y)\psi_j(y)\mathrm{d}y, \quad K'_{ij}\psi_j(x) = 2\frac{\partial}{\partial n_x}\int_{\Gamma_j} G_r(x,y)\psi_j(y)\mathrm{d}y,$$

$$K_{ij}v_j(x) = 2\int_{\Gamma_j} \frac{\partial}{\partial n_y} G_r(x,y)v_j(y)\mathrm{d}y, \qquad D_{ij}v_j = -2\frac{\partial}{\partial n_x}\int_{G_j} \frac{\partial}{\partial n_y} G_r(x,y)v_j(y)\mathrm{d}y.$$

$(r = 1 \text{ for } i,j \leqslant 2,\ r = 2 \text{ for } i,j \geqslant 3)\ G_j(x,y) = \frac{\mathrm{i}}{4} H_0^{(1)}(k_j|x-y|) \qquad (x \in \Gamma_i)$

equivalence

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Theorem 2

Interface crack problem (1)-(3) *is equivalent to integral eqn.* (6).

Proof:
„⇒“ let $u_j \in H^1_{loc}(\Omega_j)$ solve (1)-(3) then

$$\zeta = \left(u_1|_{\Gamma_1} , \rho_1^{-1} \left. \frac{\partial u_1}{\partial n} \right|_{\Gamma_2} , u_2|_{\Gamma_2} , u_2|_{\Gamma_4} \right) \in X$$

solves (6).

„⇐“ let $\zeta = (w_1, \phi_2, w_2, w_4)$ solve (6) then w_1, w_2, ψ_1, ϕ_2 define $u_1 \in H^1_{loc}(\Omega_1)$ and $w_3 := w_2 - v_2$, w_4, $\phi_3 := -\phi_2 + \psi_2$, ψ_4 define $u_2 \in H^1_{loc}(\Omega_2)$ (by Green's representation formula) with u_1, u_2 satisfying (1)-(3).

strong ellipticity

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Theorem 3

(I) Given $(\psi_1, v_2, \psi_2, \psi_4) \in Y$ *there exists exactly one solution* $\zeta \in X$ *of integral equation* (6)*.*
(II) There exist a compact operator $T_0 : \tilde{X} \to Y$ *and* $\gamma > 0$ *such that for all* $\zeta \in \tilde{X}$

$$\operatorname{Re} \left\langle (H + T_0)\, \zeta, \overline{\zeta} \right\rangle_{Y \times \tilde{X}} \geqslant \gamma \, \|\zeta\|^2_{\tilde{X}} \tag{8}$$

where $\tilde{X} = Y'$ *(homogeneous space corresp. to* X*) with duality* $\langle \cdot, \cdot \rangle_{y \times \tilde{X}}$*.*

Galerkin procedure

affine subspaces $S_h^0 \subset X$ for $h \in (0, h_0)$ s. t. $\overline{\bigcup_{h>0} S_h^0} = X$.

Find $\zeta_h \in S_h^0$ s. t. $\forall\ \xi_h \in \tilde{S}_h^0 = \{\text{homogeneous space corresp. to } S_h^0\} \subset \tilde{X}$

$$\langle H\zeta_h - g, \xi_h \rangle = 0 \tag{9}$$

Theorem 4

$\exists h_0 > 0\ \forall\ h \in (0, h_0)$ *Galerkin equations* (9) *have a unique solution* ζ_h *and* $\exists C > 0$

$$\|\zeta - \zeta_h\|_{\tilde{X}} \leqslant C \inf_{\xi_h \in S_h^0} \|\zeta - \xi_h\|_{\tilde{X}}\,. \tag{10}$$

Choose $S^{t,k}$-systems of finite elements for $S_h^0 : t \in \mathbb{N},\ k \in \mathbb{N},\ t > k$

$$S_h := S_h^{t,k}(\Gamma_1) \times S_h^{t-1,k-1}(\Gamma_2) \times S_h^{t,k}(\Gamma_2) \times S_h^{t,k}(\Gamma_4) \subset H^{1/2}(\Gamma_1) \times H^{-1/2}(\Gamma_2) \times H^{1/2}(\Gamma_2) \times H^{1/2}(\Gamma_4) \tag{11}$$

$S_h^0 =$ affine subspace of S_h satisfying inhomogeneous compatibility conditions at crack tips:

$$w_1^h(z_i) = w_2^h(z_i),\ w_2^h(z_i) = w_4^h(z_i) + v_2(z_i),\ i = 1, 2.$$

regularity of exact solution of integral eq. (6)

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$$Y^s = \left\{ (\psi_1, v_2, \psi_2, \psi_4) \in Y,\ v_2 \in H^{1/2+s}(\Gamma_2), \psi_j \in \mathcal{H}^{-1/2+s}(\Gamma_j),\ j = 1, 2, 4 \right\}$$

$$X^s = \left\{ (w_1, \phi_2, w_2, w_4) \in X,\ w_j \in H^{1/2+s}(\Gamma_j),\ j = 1, 2, 4,\ \phi_2 \in \mathcal{H}^{-1/2+s}(\Gamma_2) \right\}$$

where $\mathcal{H}^s(\Gamma_j) := \prod_{l=1}^{n_j} H^s\left(\Gamma_j^l\right)$, Γ_j^l, piece of polygonal Γ_j

Theorem 5

For given $(\psi_1, v_2, \psi_2, \psi_4) \in Y^s$ *the solution* ζ *of* (6) *is* $(s \notin A = \{\text{singular exponents } \alpha\})$

$$\zeta = \sum_{j=1}^{J} \sum_{l=1}^{L_j^*} C_{jl} \zeta_{jl} + \zeta_s, \quad \zeta_s \in X^s,\ \zeta_{jl} = \left(w_1^{jl}, \phi_2^{jl}, w_2^{jl}, w_4^{jl} \right) \tag{12}$$

$$w^{jl}(x) = |x - z_j|^{\alpha} \log^r |x - z_j|\, \chi_j(x), \quad \phi^{jl}(x) = |x - z_j|^{\alpha - 1} \log^{r'} |x - z_j|\, \chi_j(x)$$

$z_j,\ j = 1 \ldots J$ *are crack tips, corners of polygon,* $\chi_j(x)$ *cut-off functions.* $\alpha \in \mathbb{N}$ *or a zero of* $\det \hat{H}_{(j)}(\mathrm{i}\alpha)$. $H_{(j)}$ *boundary integral op. describing problem near* z_j *with Mellin symbol* $\hat{H}_{(j)}(\lambda)$.

singularities

corner z_j in crack with angle ω_j: singularities of Neumann problems, $\alpha = k\frac{\pi}{\omega_j},\ k \in \mathbb{N}$ (see [1]).
corners in interface: transcendental eq. $(\rho_1+\rho_2)^2 \sin^2 \pi\alpha = (\rho_1-\rho_2)^2 \sin^2(\pi-\omega_j)\alpha$ [2].
at crack tips: compute $\det \hat{H}(\lambda) \Rightarrow$ transcendental equation

$$(\rho_1+\rho_2)^2 \sin^2 \pi 2\alpha = (\rho_1-\rho_2)^2 \sin^2(\pi-\omega)2\alpha \tag{13}$$

$\Rightarrow$ singular exponents are half the exponents of a corner in interface.
$\omega = \pi$ or $\rho_1 = \rho_2 \Rightarrow \alpha = \frac{1}{2}$ (crack singularity).
$\omega \neq \pi$, $\omega' := \min\{\omega, 2\pi-\omega\},\ \alpha_* = \frac{\pi}{2(2\pi-\omega')} > \frac{1}{4},\ \alpha^* = \min\{\frac{\pi}{2\omega}, \frac{\pi}{(2\pi-\omega')}\} \leqslant \frac{3}{4}$.
(13) has 2 real solutions $\alpha_* < \alpha_1 < \frac{1}{2} < \alpha_2 < \alpha^*$ and $\alpha_1 \to \alpha_*,\ \alpha_2 \to \alpha^*$ as $\frac{\rho_1}{\rho_2} \to 0$ or ∞.

[1] Costabel,M; Stephan,E.P.: Boundary integral equations for mixed boundary value problems in polygonal domains and Galerkin approximation. Banach Center Publications 15 (1985) 175-251.

[2] Costabel,M.; Stephan,E.P.: A direct boundary integral equation method for transmission problems, J. Math. Anal. Appl. 106 (1985) 367-413.

convergence rates of BEM

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$$\|\zeta - \zeta_h\|_{\tilde{X}} = \mathcal{O}(h^{\alpha_0 - \varepsilon}),\ \alpha_0 := \min A,\ \frac{1}{4} < \alpha_0 \leq \frac{1}{2}$$

improvements

mesh refinement at crack tips with grading parameter $\beta \geq 1$

$\beta > \frac{3}{2\alpha_0}$, $t = 2$, $k = 1$: convergence rate $\mathcal{O}(h^{3/2-\varepsilon})$

augmenting BEM by singular fcts. with exponents $< \alpha_1 \in A$

$$\mathcal{O}(h^{\alpha_1 - \varepsilon}) \text{ if } t > \alpha_1$$

p-version of BEM

$\mathcal{O}(p^{-2\alpha_0+2\varepsilon})$ twice the rate of h-version with uniform mesh

hp-version of bem: (geometric mesh refinement near crack tips/corners + increase of polynomial degreee)

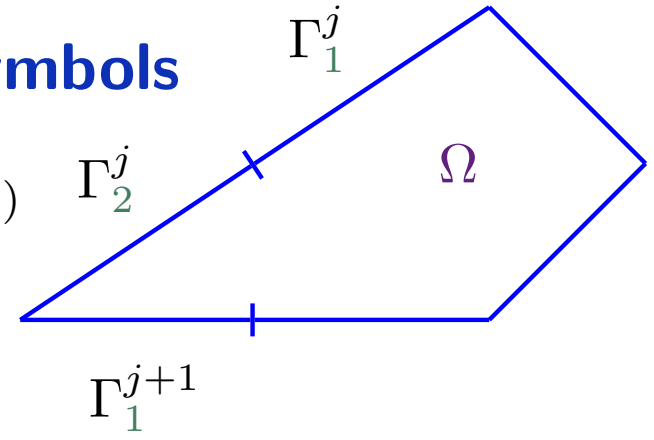
$\Rightarrow$ exponential convergence: ζ in countably normed space

$$\|\zeta - \zeta_h\|_{\tilde{X}} \lesssim e^{-bN^{1/2}}, \quad N = \text{dof } \tilde{S}_h^0$$

Heuer, N., Stephan, E.P., The hp-version of BEM on polygon. JIEA (1996)

Happy 65th birthday, Martin!

Mellin symbols



$$\begin{array}{lcl} \Gamma_2^j & \mapsto & (0,1) \\ t_j & \mapsto & 0 \end{array}$$

Dirichlet: $\triangle u = 0$ in Ω
$u = f$ on $\partial\Omega$ (1)

Given $f \in B_\beta{}^2(\Gamma)$ find $\phi \in$? :

$$V\phi(x) := -\frac{1}{\pi}\int_\Gamma \phi(y)\log|x-y|\,ds_y = f(x) \quad , x \in \Gamma \tag{2}$$

Boundary integral equation (single layer) $\phi \in B_\beta{}^1(\Gamma)$

solution in countably normed space not only standard Sobolev space

$$V : H^{-1/2}(\Gamma) \xrightarrow{invertible} H^{1/2}(\Gamma) \quad \text{if} \quad \text{cap}(\Gamma) < 1$$

$u(x) = V\phi(x),\ x \in \Omega,$ solves (1)

Boundary element method

Galerkin:(Standard h-version)

h mesh size. Find $\phi_h \in S_h^0(\Gamma) = \{$ pw. constants on $\Gamma\}$:

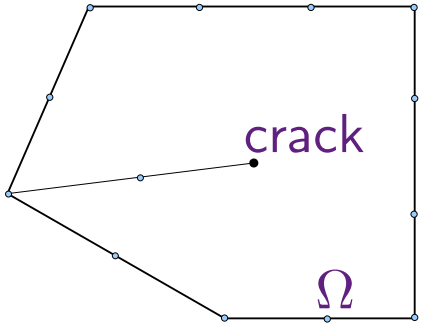
$$\langle V\phi_h, \chi_h\rangle := \int_\Gamma \Big(-\frac{1}{\pi}\int_\Gamma \phi_h(y)\ln|x-y|\,ds_y\Big)\chi_h(x)\,ds_x = \langle f(x), \chi_h\rangle \qquad (3)$$

$$\forall \chi_h \in S_h^0(\Gamma)$$

ϕ smooth solution of (2), ϕ_h Galerkin solution of (3)

$$\|\phi-\phi_h\|_{H^{-\frac{1}{2}}(\Gamma)} \le ch^{\frac{3}{2}}\|\phi\|_{H^1(\Gamma)}$$

$$\|\phi-\phi_h\|_{H^{-\frac{1}{2}}(\Gamma)} = \mathcal{O}(h^{\frac{1}{2}}) \text{ for crack}$$



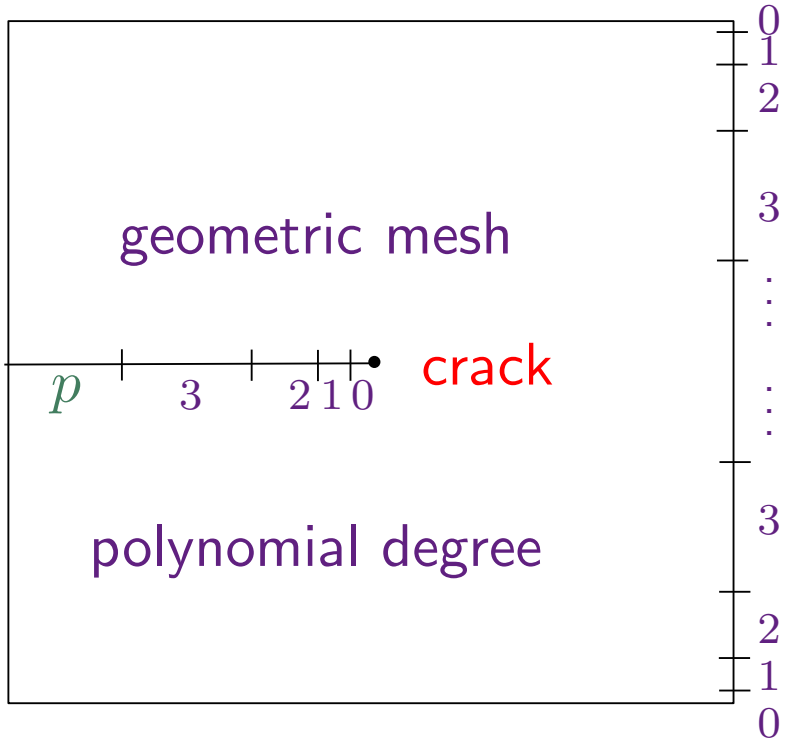
quasi-uniform mesh

Costabel, Stephan, *IMACS, 1981*

Banach Centre Publ. 15 (1985), 175-251

exponential convergence of bem

$$S_{h,p}(\Gamma_\sigma) = \{ \text{ pw. polynomials of degree } p \text{ on subintervals of geom. mesh } \Gamma_\sigma \}$$



$$\|\phi - \phi_{h,p}\|_{H^{-\frac{1}{2}}(\Gamma)} \leq Ce^{-b\sqrt{N}}$$

exponentially fast convergence (even for corner singularities cracks....)

$N = \dim S_h$
$C, b > 0$ constants (indep. of N)

key (for error analysis): $\phi \in B^1_\beta(\Gamma)$

N. Heuer, E. P. Stephan, *The hp-version of the BEM on polygon, J. Integral Equations Appl. 8 (1996) 173-212*

$I = (0,1)$ intervall, Γ polygon with J edges

Weighted Sobolev space

$$H^{m,l}_{\beta}(I) = \{\phi \in H^{l-1}(I), j = l, \ldots, m : |\phi|_{H^{j,l}_{\beta}(I)} := \|x^{\beta+j-l}\phi^{(j)}\|_{L^2(I)} < \infty\}$$

$$B^{l}_{\beta}(\Gamma) = \Pi^{J}_{j=1}\Pi^{2}_{k=1} B^{l}_{\beta}(\Gamma^{j}_{k}) \cap H^{l-1}(\Gamma)$$

$$B^{l}_{\beta}(I) = \{\phi \in H^{m,l}_{\beta}(I), m = l, l+1, \ldots;\ \exists C \geq 0, d \geq 1\ \forall j = l, l+1, \ldots$$
$$|\phi|_{H^{j,l}_{\beta}(I)} \leq C d^{(j-l)}(j-l)!\} \quad (l \geq 1,\ \text{integer}).$$

Countably normed space

$$x^{\mu} \in B^{l}_{\beta}(0,R),\ \mu > l - \frac{1}{2} - \beta$$

Mellin transform: $x = e^{-t} \longmapsto$ Fourier tranform $t \to \lambda$

$$\hat{\phi}(\lambda) := \int_0^\infty x^{i\lambda-1}\phi(x)\,dx = \int_{-\infty}^{\infty} \phi(e^{-t})e^{-i\lambda t}\,dt, \quad \forall \phi \in C_0^\infty(0,\infty)$$

$$\widehat{\phi^{(j)}}(\lambda) = (-1)^j(i\lambda-1)\cdots(i\lambda-j)\,\hat{\phi}(\lambda+ij) \quad (j \geq 1,\ \text{integer})$$
$$\widehat{x^\rho\phi}(\lambda) = \hat{\phi}(\lambda - i\rho) \quad (\rho \in \mathbb{R}).$$

Weighted Sobolev space

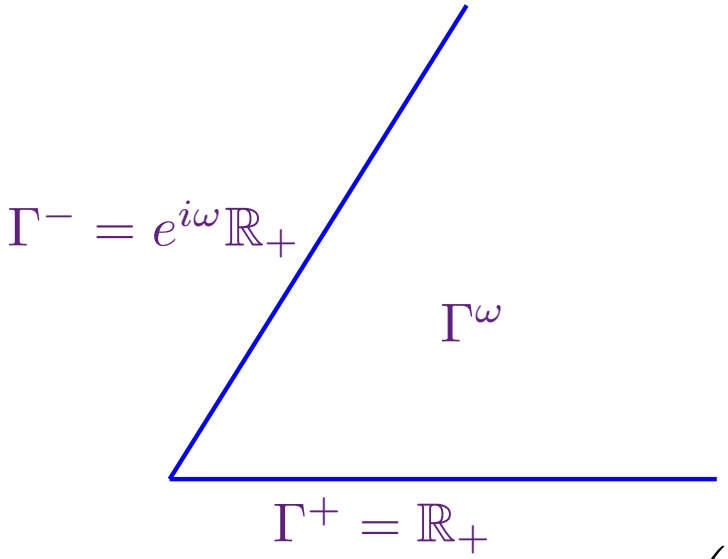
$$|\phi|^2_{H^{j,l}_\beta(I)} \simeq \int_{\Im m(\lambda)=l-1/2-\beta} |f_j(\lambda)|^2|\hat{\phi}(\lambda)|^2\,d\lambda \quad (j \geq l),$$

$$f_j(\lambda) := i\lambda\cdot(i\lambda+1)\cdots(i\lambda+j-1)$$

$(j = l)$:

$$\begin{aligned}
\|x^\beta \phi^{(j)}\|^2_{L^2(I)} &\simeq \int_{\Im m(\lambda)=-\frac{1}{2}} \left|\widehat{x^\beta \phi^{(j)}}(\lambda)\right|^2 d\lambda \\
&= \int_{\Im m(\lambda)=-\frac{1}{2}} \left|\widehat{\phi^{(j)}}(\lambda - i\beta)\right|^2 d\lambda \\
&= \int_{\Im m(\lambda)=-\frac{1}{2}-\beta} \left|\widehat{\phi^{(j)}}(\lambda)\right|^2 d\lambda \\
&= \int_{\Im m(\lambda)=j-\frac{1}{2}-\beta} |f_j(\lambda)|^2 |\hat{\phi}(\lambda)|^2 \, d\lambda.
\end{aligned}$$

N. Heuer, E.P. Stephan, *Boundary integral operators in countably normed spaces. Math. Nachr. 191 (1998), 123-151*



$$\phi \text{ on } \Gamma^\omega \;\hat{=}\; (\phi_-, \phi_+)$$

$$\phi_-(x) = \phi(xe^{i\omega}),\ \phi_+(x) = \phi(x) \quad (x > 0)$$

even: $\phi^e := \frac{1}{2}(\phi_- + \phi_+)$

odd: $\phi^o = \frac{1}{2}(\phi_- - \phi_+).$

$$V \hat{=} \begin{pmatrix} V_0 + V_\omega & 0 \\ 0 & V_0 - V_\omega \end{pmatrix}$$

$$V_\omega\phi(x) := -\frac{1}{\pi}\int_0^\infty \log|1 - \frac{x}{y}e^{-i\omega}|\phi(y)\,dy,$$

Mellin symbol

Costabel,Stephan 1981

$$\widehat{V_\omega\phi}(\lambda) = \hat{V}_\omega(\lambda)\hat{\phi}(\lambda - i) := \frac{\cosh[(\pi - \omega)\lambda]}{\lambda \sinh \pi\lambda}\hat{\phi}(\lambda - i), \quad \Im m(\lambda) \in (0, 1),$$

Mellin symbols $\varphi \in C_0^\infty(0,\infty)$

Let $\phi \in C_0^\infty(0,\infty)$

$$\int_{-\infty}^{\infty} \frac{e^{-i\lambda t}}{e^{-t\pm i\omega}-1}\,dt = i\pi \frac{e^{\pm\lambda(\omega-\pi)}}{\sinh \pi\lambda}, \quad \omega \in (0,2\pi), \Im m(\lambda) \in (-1,0)$$

Hilbert Transform

$$N_\omega \varphi(x) := -\frac{1}{i\pi}\int_0^\infty \frac{\varphi(\xi)}{xe^{i\omega}-\xi}\,d\xi = -\frac{1}{i\pi}\int_0^\infty \frac{1}{\frac{x}{\xi}e^{i\omega}-1}\varphi(\xi)\frac{d\xi}{\xi}$$

$$k(\frac{x}{\xi}) := \frac{1}{\frac{x}{\xi}e^{i\omega}-1} \quad \text{homogeneous of degree 0 in } x,\ \xi\text{:} \quad k(\frac{tx}{t\xi}) = k(\frac{x}{\xi})$$

$$\widehat{N_{2\pi-\omega}\varphi}(\lambda) = -\frac{e^{\pm(\pi-\omega)\lambda}}{\sinh \pi\lambda}\widehat{\varphi}(\lambda) \quad 0<\omega<2\pi$$

convolution

Lemma

For $k, \phi \in C_0^\infty(0,\infty); \alpha, \beta \in \mathbb{C}$ *there holds*

$$u(x) := \int_0^\infty x^\alpha k\Big(\frac{x}{y}\Big) y^\beta \frac{\phi(y)}{y}\, dy$$

then

$$\widehat{u}(\lambda) = \widehat{k}(\lambda - i\alpha)\widehat{\phi}(\lambda - i(\alpha+\beta))$$

single layer potential

$$V_\omega \varphi(x) = -\frac{1}{\pi}\int_0^\infty \ln\Big|1 - \frac{x}{\xi}e^{i\omega}\Big|\varphi(\xi)\, d\xi = \frac{1}{\pi}\int_0^\infty \frac{M(\xi)}{\xi}\Re\left(\frac{xe^{i\omega}}{\xi - xe^{i\omega}}\right) d\xi$$

$$M(\xi) := \int_0^\xi \varphi(\eta)\, d\eta \qquad \widehat{M}(\lambda) = -\frac{1}{i\lambda}\widehat{\varphi}(\lambda - i), \quad \Im m(\lambda) \in (0,1)$$

M. Costabel, E. P. Stephan, *Banach Centre Publ. 15 (1985), 175-251*

$$V_\omega\varphi(x) = \frac{i}{2}\left\{e^{i\omega}TN_\omega T^{-1} + e^{-i\omega}TN_{2\pi-\omega}T^{-1}\right\}M(x)$$

$$T\varphi(x) := x\varphi(x), \quad T^{-1}\varphi(x) := \frac{1}{x}\varphi(x) \Rightarrow \widehat{T\varphi}(\lambda) = \widehat{\varphi}(\lambda - i), \quad \widehat{T^{-1}\varphi}(\lambda) = \widehat{\varphi}(\lambda + i)$$

$$\widehat{V_\omega\varphi}(\lambda) = \frac{\cosh(\pi-\omega)\lambda}{\lambda\sinh\pi\lambda}\widehat{\varphi}(\lambda - i), \quad \Im m(\lambda) \in (0,1)$$

single layer

$$K_\omega\varphi(x) = -\frac{1}{\pi}\int_0^\infty \frac{\partial}{\partial n_y}\ln\left|xe^{i\omega} - y\right|\varphi(y)\,dy = \frac{1}{\pi}\int_0^\infty \Im m\left(\frac{1}{xe^{i\omega} - y}\right)\varphi(y)\,dy$$

$$\widehat{K_\omega\varphi}(\lambda) = -\frac{\sinh(\pi-\omega)\lambda}{\lambda\sinh\pi\lambda}\widehat{\varphi}(\lambda), \quad \Im m(\lambda) \in (-1,1)$$

double layer

Integral operator round corner

Lemma 1

For $\frac{1}{2} < \beta < 1$, $\exists C > 0$ *indep. of* j*:*

$$|V\phi|_{H^{j,2}_{\beta}(\Gamma^{\omega})} \leq C|\phi|_{H^{j-1,1}_{\beta}(\Gamma^{\omega})} \quad (j \geq 2)$$

Proof.

$$V\phi = \begin{pmatrix} V_{ee} & V_{0e} \\ V_{eo} & V_{oo} \end{pmatrix} \begin{pmatrix} \phi^{e} \\ \phi^{o} \end{pmatrix}$$

Integral operator round corner

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Proof.

$$V\phi = \begin{pmatrix} V_{ee} + V_0 + V_\omega & V_{oe} + 0 \\ V_{eo} + 0 & V_{oo} + V_0 - V_\omega \end{pmatrix} \begin{pmatrix} \phi^e \\ \phi^o \end{pmatrix}$$

Integral operator round corner

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$$V\phi = \begin{pmatrix} V_{ee} & V_{oe} \\ V_{eo} & V_{oo} \end{pmatrix} \begin{pmatrix} \phi^e \\ \phi^o \end{pmatrix}$$

$V_0 + V_\omega$, 0, 0, $V_0 - V_\omega$

$$|V_{ee}\phi^e|^2_{H^{j,2}_\beta(\mathbb{R}_+)} \simeq \int_{\Im m(\lambda) = \frac{3}{2}-\beta} |f_j(\lambda)|^2 \Big| \underbrace{\frac{\cosh \pi\lambda + \cosh[(\pi-\omega)\lambda]}{\lambda \sinh \pi\lambda}}_{1/\lambda} \hat{\phi}^e(\lambda - i) \Big|^2 d\lambda.$$

$$\simeq \int_{\Im m(\lambda) = \frac{1}{2}-\beta} |f_{j-1}(\lambda)|^2 |\hat{\phi}^e(\lambda)|^2 \, d\lambda$$

Integral operator round corner

Lemma 1

For $\frac{1}{2} < \beta < 1$, $\exists C > 0$ *indep. of* j*:*

$$|V\phi|_{H^{j,2}_\beta(\Gamma^\omega)} \leq C|\phi|_{H^{j-1,1}_\beta(\Gamma^\omega)} \quad (j \geq 2)$$

Proof.

$$V\phi = \begin{pmatrix} V_{ee} & V_{eo} \\ V_{oe} & V_{oo} \end{pmatrix} \begin{pmatrix} \phi^e \\ \phi^o \end{pmatrix}$$

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$$\simeq \int_{\Im m(\lambda) = \frac{1}{2}-\beta} |f_{j-1}(\lambda)|^2 |\hat{\phi}^e(\lambda)|^2 \, d\lambda$$

Same way for $V_{oo}\phi^0$

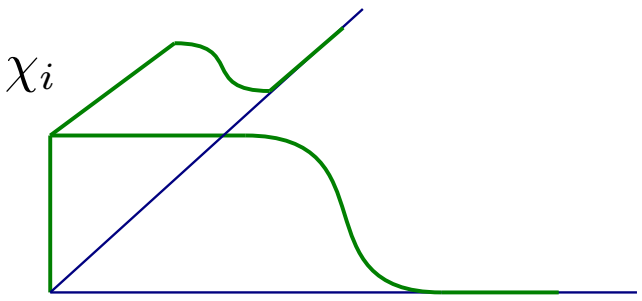
mapping property

Using partitions of unity we can carry over this result onto polygon Γ.

Theorem

For $1/2 < \beta < 1$. *There holds* $\phi \in B^1_\beta(\Gamma) \Rightarrow V\phi \in B^2_\beta(\Gamma)$

Proof. $V\phi \in H^1(\Gamma)$. Consider $|V\phi|_{H^{j,2}_\beta(\Gamma^i_k)}$ $(j \geq 2)$



$$\chi_i \in C^\infty(\Gamma)$$

$$\text{supp } \chi_i \subset \Gamma_i \cup \{t_i\} \cup \Gamma_{i+1}$$

$$\text{and } \sum_{i=1}^{J} \chi_i = 1$$

From lemma 1 local regularity

$$|V\chi_i\phi|_{H^{j,2}_\beta(\Gamma_i \cup \{t_i\} \cup \Gamma_{i+1})} \leq c|\chi_i\phi|_{H^{j-1,1}_\beta(\Gamma_i \cup \{t_i\} \cup \Gamma_{i+1})}$$

$$\leq c\widetilde{C}\widetilde{d}^{L-1}(j-2)!,$$

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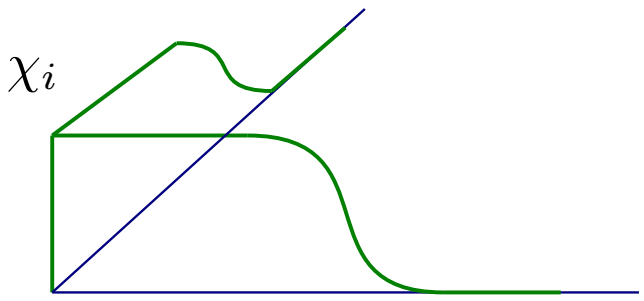

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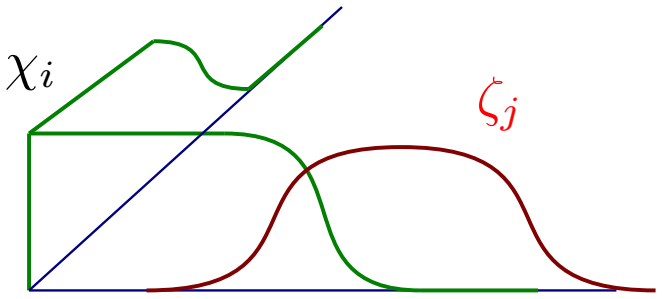

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$$\Rightarrow V\chi_i\phi \in B^{2,L+1}_\beta(\Gamma_i \cup \{t_i\} \cup \Gamma_{i+1}).$$

$$\Rightarrow V\phi = \sum_{j=1}^{J}\sum_{i=1}^{J} \zeta_j V\chi_i\phi \in B^{2,L+1}_\beta(\Gamma)$$

Lemma 2 (local regularity)

Let $\rho = 3/2 - \min\{\frac{\pi}{2\pi-\omega}, \frac{\pi}{\omega}\}$. *Then* $1/2 \le \rho < 1$ *and there holds:*
Let $\phi \in L^2(\Gamma^\omega)$ *with* $V\phi \in H^{j+1,2}_\beta(\Gamma^\omega)$ *for* $j \ge 1$ *then* $\phi \in H^{j,1}_\beta(\Gamma^\omega)$ *for* $\rho < \beta < 1$ *and*

$$|\phi|_{H^{j,1}_\beta(\Gamma^\omega)} \le C|V\phi|_{H^{j+1,2}_\beta(\Gamma^\omega)}.$$

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$$|\phi^e|^2_{H^{j,1}_\beta(\mathbb{R}_+)} \simeq \int_{\Im m(\lambda)=\frac{1}{2}-\beta} |f_j(\lambda)|^2 \left|(\lambda+i)\frac{\sinh[\pi(\lambda+i)]}{\cosh[\pi(\lambda+i)]+\cosh[(\pi-\omega)(\lambda+i)]}\right|^2$$

$$\cdot \left|(\widehat{V_0}+\widehat{V_\omega}))(\lambda+i)\right|^2 \left|\widehat{\phi^e}(\lambda)\right|^2 d\lambda$$

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solution of integral equation in countably normed space

Partition of unity $\Rightarrow$ local $\longrightarrow$ global

Theorem

Let $\rho = 3/2 - \min\{\frac{\pi}{2\pi-\omega_j}, \frac{\pi}{\omega_j};\ j = 1, \ldots, J\}$ *and* $\rho < \beta < 1$. *Then there holds:*

(i) $V\phi \in B^2_\beta(\Gamma) \Rightarrow \phi \in B^1_\beta(\Gamma)$

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Partition of unity $\Rightarrow$ local $\rightarrow$ global

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Let $\rho = 3/2 - \min\{\frac{\pi}{2\pi-\omega_j}, \frac{\pi}{\omega_j};\ j = 1, \ldots, J\}$ *and* $\rho < \beta < 1$. *Then there holds:*

(i) $V\phi \in B^2_\beta(\Gamma) \Rightarrow \phi \in B^1_\beta(\Gamma)$

(ii) $D\phi \in B^1_\beta(\Gamma) \Rightarrow \phi \in B^2_\beta(\Gamma)$

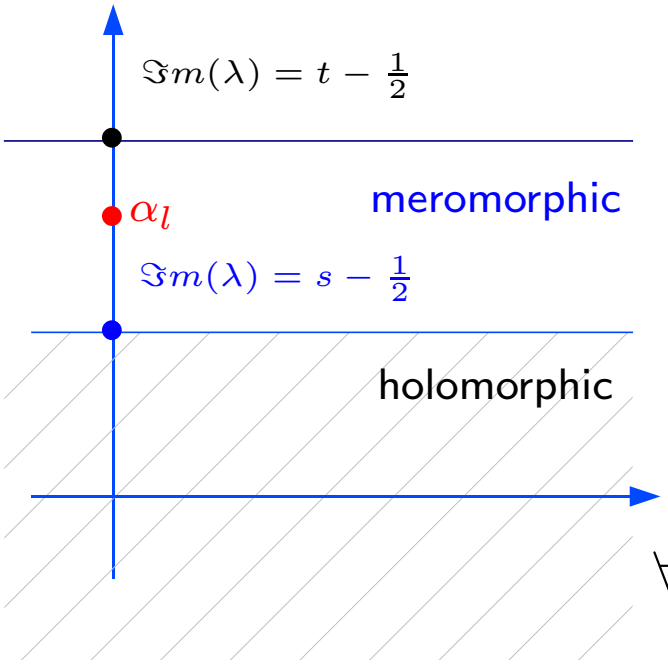
(iii) $(I \pm K)\phi \in B^2_\beta(\Gamma) \Rightarrow \phi \in B^2_\beta(\Gamma)$

(iv) $(I \pm K')\phi \in B^1_\beta(\Gamma) \Rightarrow \phi \in B^1_\beta(\Gamma)$

Double layer: $$K\phi(x) = -\frac{1}{\pi}\int_\Gamma \phi(y)\frac{\partial}{\partial n_y}\log|x-y|\,ds_y \quad (x \in \Gamma),$$

Adjoint of K : $$K'\phi(x) = -\frac{1}{\pi}\int_\Gamma \phi(y)\frac{\partial}{\partial n_x}\log|x-y|\,ds_y \quad (x \in \Gamma),$$

Hypersingular operator: $$D\phi(x) = -\frac{\partial}{\partial n_x}K\phi(x) \quad (x \in \Gamma).$$



Inverse Mellin Tranformation

$$u_h(x) = \frac{1}{2\pi} \int_{\Im m(\lambda)=h} x^{-i\lambda} \widehat{u}(\lambda)\, d\lambda$$

$\forall u \in C_0^\infty(0,\infty)$ $\widehat{u}(\lambda)$ entire analytic, $u_h = u \ \forall h \in \mathbb{R}$

$\forall u \in \widetilde{H}^s_{comp}(\mathbb{R}_+)), s < t$ If $\widehat{u}(\lambda)$ meromorphic for $\Im m(\lambda) < t - \frac{1}{2} = k$

with poles of order $m_l + 1$ at $\lambda_l = i\alpha_l$, $\quad s - \frac{1}{2} < \alpha_l < k$

$$\Rightarrow u(x) = \sum_{l=1}^{L} \sum_{m=0}^{m_l} c_{lm} x^{\alpha_l} \log^m x \ \chi(x) + u_k(x), \quad \chi \equiv 1 \text{ near } x = 0$$

$c_{lm} = \frac{i^{m+1}}{m!} \operatorname*{Res}_{\lambda=\lambda_l} \{(\lambda - \lambda_l)^m \widehat{u}(\lambda)\}$ expansion with singular functions