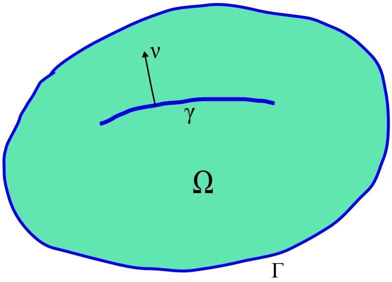
Thin inclusions in elastic bodies with possible delaminations

Alexander Khludnev

Lavrentiev Institute of Hydrodynamics, Novosibirsk
E-mail: khlud@hydro.nsc.ru

Crack in elastic body



Problem formulation

Find $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$, $\boldsymbol{\sigma} = \{\sigma_{ij}\}, i, j = 1, 2$, such that

$$-\operatorname{div}\boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega_\gamma\,, \qquad (1)$$

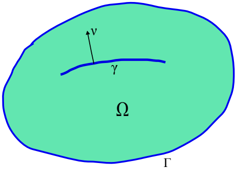
$$\boldsymbol{\sigma} = \mathbf{A}\varepsilon(\mathbf{u}) \quad \text{in} \quad \Omega_\gamma\,, \qquad (2)$$

$$\mathbf{u} = 0 \quad \text{on} \quad \Gamma\,, \qquad (3)$$

$$[\mathbf{u}]\nu \geq 0, \quad [\sigma_\nu] = 0, \quad [\mathbf{u}]\nu \cdot \sigma_\nu = 0 \quad \text{on} \quad \gamma\,, \qquad (4)$$

$$\sigma_\nu \leq 0, \; \sigma_\tau = 0 \quad \text{on} \quad \gamma^\pm\,, \qquad (5)$$

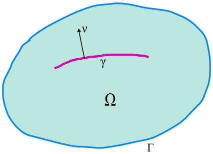
where $[\mathbf{u}] = \mathbf{u}^+ - \mathbf{u}^-$, $\sigma_\nu = \sigma_{ij}\nu_j\nu_i$, $\quad \sigma_\tau = \sigma\nu - \sigma_\nu \cdot \nu$



Directions of investigation

1. Solvability of boundary value problems, solution smoothness
2. Dependence on parameters, shape sensitivity analysis, differentiability of energy functionals, invariant integrals
3. Optimal control problems
4. Smooth domain method. Fictitious domain method
5. Contact of elastic bodies of different dimensions
6. Overlapping domain problems
7. Rigid inclusions in elastic bodies

Thin rigid inclusions (unti-cracks)



$$R(\gamma) = \{\rho = (\rho_1, \rho_2) \mid \rho(x) = b(-x_2, x_1) + (c^1, c^2),\ x \in \gamma\}$$

Khludnev, Leugering (2010, 1012); Itou, Khludnev, Rudoy, Tani(2012)

Find $\mathbf{u} = (\mathbf{u_1}, \mathbf{u_2}), \rho_0 \in \mathbf{R}(\gamma)$, $\boldsymbol{\sigma} = \{\sigma_{\mathbf{ij}}\}, \mathbf{i}, \mathbf{j} = \mathbf{1}, \mathbf{2}$, such that

$$-\mathrm{div}\boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega_\gamma, \tag{6}$$

$$\boldsymbol{\sigma} - \mathbf{A}\varepsilon(\mathbf{u}) = \mathbf{0} \quad \text{in} \quad \Omega_\gamma, \tag{7}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma, \tag{8}$$

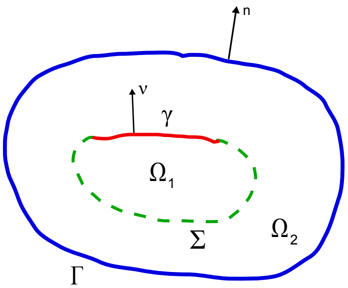
$$[\mathbf{u}]\nu \geq \mathbf{0},\ \mathbf{u}^- = \rho_0,\ \sigma_\nu^+ \leq \mathbf{0},\ \sigma_\tau^+ = \mathbf{0} \quad \text{on} \quad \gamma, \tag{9}$$

$$\sigma_\nu^+ \cdot [\mathbf{u}]\nu = \mathbf{0} \quad \text{on} \quad \gamma, \tag{10}$$

$$\int_\gamma [\sigma\nu]\rho = \mathbf{0} \quad \forall \rho \in \mathbf{R}(\gamma). \tag{11}$$

Thin rigid inclusions in plates

Khludnev (2012)



Notations Spaces of infinitesimal rigid displacements

$$L(\gamma) = \{l \mid l(x) = a_0 + a_1 x_1 + a_2 x_2,$$
$$a_i = \text{const}, \;\; i = 0, 1, 2; \; x = (x_1, x_2) \in \gamma\},$$
$$R(\gamma) = \{\rho = (\rho_1, \rho_2) \mid \rho(x) = b(-x_2, x_1) + (c^1, c^2), \; x \in \gamma\}$$

Set of admissible displacements

$$K = \{(v, \varphi) \in H \mid [v]\nu \geq |[\varphi_\nu]| \text{ on } \;\gamma; \tag{12}$$
$$(v, \varphi)|_{\gamma^-} \in R(\gamma) \times L(\gamma)\}.$$

Find functions $\mathbf{u} = (\mathbf{u_1}, \mathbf{u_2}), \mathbf{w},\ \rho_0 \in \mathbf{R}(\gamma), \mathbf{l_0} \in \mathbf{L}(\gamma)$, $\boldsymbol{\sigma} = \{\sigma_{\mathbf{ij}}\}, \mathbf{m} = \{\mathbf{m_{ij}}\}, \mathbf{i}, \mathbf{j} = \mathbf{1}, \mathbf{2}$, such that

$$-\,\mathrm{div}\boldsymbol{\sigma} = \mathbf{F},\ -\nabla\nabla\mathbf{m} = \mathbf{f} \quad \text{in} \quad \Omega_\gamma, \quad (13)$$

$$\boldsymbol{\sigma} - \mathbf{A}\varepsilon(\mathbf{u}) = \mathbf{0},\ \mathbf{m} + \mathbf{D}\nabla\nabla\mathbf{w} = \mathbf{0} \quad \text{in} \quad \Omega_\gamma, \quad (14)$$

$$\mathbf{u} = \mathbf{w} = \mathbf{w_n} = \mathbf{0} \quad \text{on} \quad \Gamma, \quad (15)$$

$$[\mathbf{u}]\nu \geq |[\mathbf{w}_\nu]| \quad \text{on} \quad \gamma, \quad (16)$$

$$\mathbf{u} = \rho_0,\ \mathbf{w} = \mathbf{l_0} \quad \text{on} \quad \gamma^-, \quad (17)$$

$$-\int_\gamma [\sigma\nu \cdot \mathbf{u}] + \int_\gamma [\mathbf{t}^\nu \mathbf{w}] - \int_\gamma [\mathbf{m}_\nu \mathbf{w}_\nu] = \mathbf{0}, \quad (18)$$

$$-\int_\gamma [\sigma\nu \cdot \mathbf{v}] + \int_\gamma [\mathbf{t}^\nu \varphi] - \int_\gamma [\mathbf{m}_\nu \varphi_\nu] \geq \mathbf{0} \quad \forall(\mathbf{v}, \varphi) \in \mathbf{K}. \quad (19)$$

Thin elastic inclusion without delamination

Find $\mathbf{u} = (\mathbf{u_1}, \mathbf{u_2})$, $\boldsymbol{\sigma} = \{\sigma_{\mathbf{ij}}\}, \mathbf{i}, \mathbf{j} = 1, 2$, $\mathbf{w}$, such that

$$-\text{div}\,\boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega_\gamma, \tag{20}$$

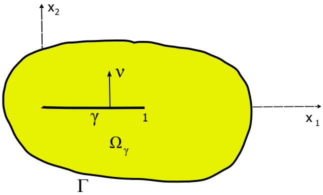
$$\boldsymbol{\sigma} - \mathbf{A}\varepsilon(\mathbf{u}) = \mathbf{0} \quad \text{in} \quad \Omega, \tag{21}$$

$$\mathbf{w_{xxxx}} - \mathbf{g} = [\sigma_\nu] \quad \text{on} \quad \gamma, \tag{22}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma, \tag{23}$$

$$\mathbf{w_{xx}} = \mathbf{w_{xxx}} = \mathbf{0} \quad \text{for} \quad \mathbf{x} = \mathbf{0}, \mathbf{1}, \tag{24}$$

$$\mathbf{w} = \mathbf{u}_\nu, \ [\sigma_\tau] = \mathbf{0} \quad \text{on} \quad \gamma. \tag{25}$$



Thin elastic inclusion with delamination

Khludnev, Negri (2012)

Find $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$, $\boldsymbol{\sigma} = \{\sigma_{ij}\}, \mathbf{i}, \mathbf{j} = 1, 2$, $\mathbf{w}$, such that

$$-\mathrm{div}\, \boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega_\gamma, \tag{26}$$

$$\boldsymbol{\sigma} - \mathbf{A}\varepsilon(\mathbf{u}) = 0 \quad \text{in} \quad \Omega_\gamma, \tag{27}$$

$$\mathbf{w}_{xxxx} - \mathbf{g} = [\sigma_\nu] \quad \text{on} \quad \gamma, \tag{28}$$

$$\mathbf{u} = 0 \quad \text{on} \quad \Gamma, \tag{29}$$

$$\mathbf{w}_{xx} = \mathbf{w}_{xxx} = 0 \quad \text{for} \quad \mathbf{x} = 0, 1, \tag{30}$$

$$[\mathbf{u}_\nu] \geq 0, \; \mathbf{w} = \mathbf{u}_\nu^- \quad \text{on} \quad \gamma, \tag{31}$$

$$\sigma_\nu^+ \leq 0, \; \sigma_\nu^+[\mathbf{u}_\nu] = 0, \; \sigma_\tau^\pm = 0 \quad \text{на} \quad \gamma. \tag{32}$$

$$\mathbf{K} = \{(\mathbf{u}, \mathbf{w}) \in \mathbf{H}^1_\Gamma(\Omega_\gamma)^2 \times \mathbf{H}^2(\gamma) \mid [\mathbf{u}_\nu] \geq \mathbf{0},\ \mathbf{w} = \mathbf{u}^-_\nu \text{ on } \gamma\},$$

Variational inequality

$$(\mathbf{u}, \mathbf{w}) \in \mathbf{K},\ \int\limits_{\Omega_\gamma} \sigma(\mathbf{u})\varepsilon(\bar{\mathbf{u}} - \mathbf{u}) - \int\limits_{\Omega_\gamma} \mathbf{f}(\bar{\mathbf{u}} - \mathbf{u}) + \tag{33}$$

$$+ \int\limits_\gamma \mathbf{w}_{xx}(\bar{\mathbf{w}}_{xx} - \mathbf{w}_{xx}) - \int\limits_\gamma \mathbf{g}(\bar{\mathbf{w}} - \mathbf{w}) \geq \mathbf{0} \quad \forall\, (\bar{\mathbf{u}}, \bar{\mathbf{w}}) \in \mathbf{K}.$$

Convergence of parameter to infinity

Find $\mathbf{u}^\delta = (\mathbf{u}_1^\delta, \mathbf{u}_2^\delta)$, $\boldsymbol{\sigma}^\delta = \{\sigma_{\mathbf{ij}}^\delta\}, \mathbf{i}, \mathbf{j} = 1, 2$, $\mathbf{w}^\delta$, such that

$$-\mathrm{div}\, \boldsymbol{\sigma}^\delta = \mathbf{f} \quad \text{in} \quad \Omega_\gamma, \tag{34}$$

$$\boldsymbol{\sigma}^\delta - \mathbf{A}\varepsilon(\mathbf{u}^\delta) = \mathbf{0} \quad \text{in} \quad \Omega_\gamma, \tag{35}$$

$$\delta \mathbf{w}^\delta_{\mathbf{xxxx}} - \mathbf{g} = [\sigma_\nu^\delta] \quad \text{on} \quad \gamma, \tag{36}$$

$$\mathbf{u}^\delta = \mathbf{0} \quad \text{on} \quad \Gamma, \tag{37}$$

$$\mathbf{w}^\delta_{\mathbf{xx}} = \mathbf{w}^\delta_{\mathbf{xxx}} = \mathbf{0} \quad \text{for} \quad \mathbf{x} = \mathbf{0}, \mathbf{1}, \tag{38}$$

$$[\mathbf{u}_\nu^\delta] \geq \mathbf{0}, \ \mathbf{w}^\delta = \mathbf{u}_\nu^{\delta -} \quad \text{on} \quad \gamma, \tag{39}$$

$$\sigma_\nu^{\delta +} \leq \mathbf{0}, \ \sigma_\nu^{\delta +}[\mathbf{u}_\nu^\delta] = \mathbf{0}, \ \sigma_\tau^{\delta \pm} = \mathbf{0} \quad \text{на} \quad \gamma. \tag{40}$$

Limit problem

$$R_s(\gamma) = \{l(x) \mid l(x) = c_0 + c_1 x,\ x \in [0,1]\},$$

$$K_r = \{(u, l) \in H^1_\Gamma(\Omega_\gamma)^2 \times R_s(\gamma) \mid [u_\nu] \geq 0,\ l = u_\nu^- \text{ on } \gamma\}.$$

$$(u, l_0) \in K_r,\ \int\limits_{\Omega_\gamma} \sigma(u)\varepsilon(\bar{u} - u) - \int\limits_{\Omega_\gamma} f(\bar{u} - u) - \tag{41}$$

$$- \int\limits_{\gamma} g(l - l_0) \geq 0 \quad \forall\, (\bar{u}, l) \in K_r.$$

Differential formulation I

Find $\mathbf{u} = (\mathbf{u_1}, \mathbf{u_2})$, $\boldsymbol{\sigma} = \{\sigma_{\mathbf{ij}}\}, \mathbf{i}, \mathbf{j} = 1, 2$, $\mathbf{l_0} \in \mathbf{R_s}(\gamma)$, such that

$$-\mathrm{div}\,\boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \boldsymbol{\Omega}_\gamma, \tag{42}$$

$$\boldsymbol{\sigma} - \mathbf{A}\varepsilon(\mathbf{u}) = \mathbf{0} \quad \text{in} \quad \boldsymbol{\Omega}_\gamma, \tag{43}$$

$$\mathbf{u} = \mathbf{0} \quad \text{in} \quad \boldsymbol{\Gamma}, \tag{44}$$

$$\mathbf{l_0} = \mathbf{u}_\nu^- \quad \text{in} \quad \gamma, \tag{45}$$

$$[\mathbf{u}_\nu] \geq \mathbf{0},\ \sigma_\tau^\pm = \mathbf{0},\ \sigma_\nu^+ \leq \mathbf{0},\ \sigma_\nu^+[\mathbf{u}_\nu] = \mathbf{0} \quad \text{on} \quad \gamma, \tag{46}$$

$$-\int_\gamma [\sigma_\nu]\mathbf{l} = \int_\gamma \mathbf{g}\mathbf{l} \quad \forall \mathbf{l} \in \mathbf{R_s}(\gamma). \tag{47}$$

Differential formulation II

Find $\mathbf{u} = (\mathbf{u_1}, \mathbf{u_2})$, $\boldsymbol{\sigma} = \{\sigma_{ij}\}, \mathbf{i}, \mathbf{j} = 1, 2$, $\mathbf{l_0} \in \mathbf{R_s}(\gamma)$, such that

$$-\mathrm{div}\,\boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega_\gamma, \tag{48}$$

$$\boldsymbol{\sigma} - \mathbf{A}\varepsilon(\mathbf{u}) = \mathbf{0} \quad \text{in} \quad \Omega_\gamma, \tag{49}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma, \tag{50}$$

$$[\mathbf{u}_\nu] \geq \mathbf{0}, \;\; \mathbf{l_0} = \mathbf{u}_\nu^- \quad \text{on} \quad \gamma, \tag{51}$$

$$-\int_\gamma [\sigma\nu \cdot \mathbf{u}] = \int_\gamma \mathbf{g}\mathbf{l_0}, \tag{52}$$

$$-\int_\gamma [\sigma\nu \cdot \bar{\mathbf{u}}] \geq \int_\gamma \mathbf{g}\mathbf{l} \quad \forall(\bar{\mathbf{u}}, \mathbf{l}) \in \mathbf{K_r}. \tag{53}$$

Convergence of parameter to zero

Find $\mathbf{u}^\delta = (\mathbf{u}_1^\delta, \mathbf{u}_2^\delta)$, $\boldsymbol{\sigma}^\delta = \{\sigma_{\mathbf{ij}}^\delta\}, \mathbf{i}, \mathbf{j} = \mathbf{1}, \mathbf{2}$, $\mathbf{w}^\delta$, such that

$$-\text{div}\,\boldsymbol{\sigma}^\delta = \mathbf{f} \quad \text{in} \quad \boldsymbol{\Omega}_\gamma, \tag{54}$$

$$\boldsymbol{\sigma}^\delta - \mathbf{A}\varepsilon(\mathbf{u}^\delta) = \mathbf{0} \quad \text{in} \quad \boldsymbol{\Omega}_\gamma, \tag{55}$$

$$\delta \mathbf{w}^\delta_{\mathbf{xxxx}} = [\sigma^\delta_\nu] \quad \text{on} \quad \gamma, \tag{56}$$

$$\mathbf{u}^\delta = \mathbf{0} \quad \text{on} \quad \boldsymbol{\Gamma}, \tag{57}$$

$$\mathbf{w}^\delta_{\mathbf{xx}} = \mathbf{w}^\delta_{\mathbf{xxx}} = \mathbf{0} \quad \text{for} \quad \mathbf{x} = \mathbf{0}, \mathbf{1}, \tag{58}$$

$$[\mathbf{u}^\delta_\nu] \geq \mathbf{0},\ \mathbf{w}^\delta = \mathbf{u}^{\delta -}_\nu \quad \text{on} \quad \gamma, \tag{59}$$

$$\sigma^{\delta +}_\nu \leq \mathbf{0},\ \sigma^{\delta +}_\nu [\mathbf{u}^\delta_\nu] = \mathbf{0},\ \sigma^{\delta \pm}_\tau = \mathbf{0} \quad \text{on} \quad \gamma. \tag{60}$$

Limit problem

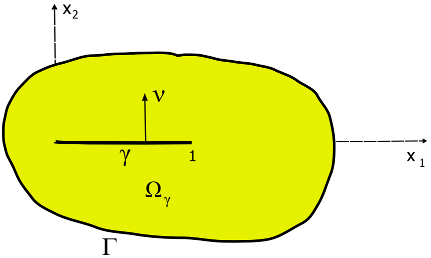
Find $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$, $\boldsymbol{\sigma} = \{\sigma_{\mathbf{ij}}\}$, $\mathbf{i}, \mathbf{j} = 1, 2$, such that

$$-\text{div}\, \boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \Omega_\gamma,$$

$$\boldsymbol{\sigma} - \mathbf{A}\varepsilon(\mathbf{u}) = \mathbf{0} \quad \text{in} \quad \Omega_\gamma,$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma,$$

$$[\mathbf{u}_\nu] \geq 0,\ \sigma_\nu^\pm \leq 0,\ [\sigma_\nu] = 0,\ \boldsymbol{\sigma}_{\boldsymbol{\tau}}^\pm = 0,\ \sigma_\nu[\mathbf{u}_\nu] = 0 \text{ on } \gamma.$$



Vertical and tangential displacements of inclusion

Khludnev, Leugering (2013)

Find $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2)$, $\boldsymbol{\sigma} = \{\sigma_{ij}\}, i, j = 1, 2$, $\mathbf{v}$, $\mathbf{w}$, such that

$$
\begin{aligned}
-\mathrm{div}\, \boldsymbol{\sigma} = \mathbf{f} \quad &\text{in} \quad \Omega_\gamma, \\
\boldsymbol{\sigma} - \mathbf{A}\varepsilon(\mathbf{u}) = 0 \quad &\text{in} \quad \Omega_\gamma, \\
\delta_1 \mathbf{w}_{xxxx} = [\sigma_\nu] \quad &\text{on} \quad \gamma, \\
-\delta_2 \mathbf{v}_{xx} = [\sigma_\tau] \quad &\text{on} \quad \gamma, \\
\mathbf{u} = 0 \quad &\text{на} \quad \Gamma, \\
\mathbf{w}_{xx} = \mathbf{w}_{xxx} = \mathbf{v}_x = 0 \quad &\text{for} \quad x = 0, 1, \\
[\mathbf{u}_\nu] \geq 0,\ \mathbf{w} = \mathbf{u}_\nu^-,\ \mathbf{v} = \mathbf{u}_\tau^-,\ \sigma_\nu^+ [\mathbf{u}_\nu] = 0 \quad &\text{on} \quad \gamma, \\
\sigma_\nu^+ \leq 0,\ \sigma_\tau^+ = 0 \quad &\text{on} \quad \gamma.
\end{aligned}
$$

Find $\mathbf{u} = (\mathbf{u_1}, \mathbf{u_2})$, $\boldsymbol{\sigma} = \{\sigma_{\mathbf{ij}}\}, \mathbf{i}, \mathbf{j} = \mathbf{1}, \mathbf{2}$, and $\mathbf{l_0} \in \mathbf{R_s}(\gamma), \mathbf{q_0} \in \mathbf{R}$ such that

$$-\text{div}\, \boldsymbol{\sigma} = \mathbf{f} \quad \text{in} \quad \boldsymbol{\Omega}_\gamma, \tag{61}$$

$$\boldsymbol{\sigma} - \mathbf{A}\varepsilon(\mathbf{u}) = \mathbf{0} \quad \text{in} \quad \boldsymbol{\Omega}_\gamma, \tag{62}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on} \quad \boldsymbol{\Gamma}, \tag{63}$$

$$[\mathbf{u}_\nu] \geq \mathbf{0},\ \mathbf{l_0} = \mathbf{u}_\nu^-,\ \mathbf{q_0} = \mathbf{u}_\tau^- \quad \text{on} \quad \gamma, \tag{64}$$

$$\sigma_\tau^+ = \mathbf{0},\ \sigma_\nu^+ \leq \mathbf{0},\ \sigma_\nu^+[\mathbf{u}_\nu] = \mathbf{0} \quad \text{on} \quad \gamma, \tag{65}$$

$$\int\limits_\gamma \sigma_\tau^- = \mathbf{0},\ \int\limits_\gamma [\sigma_\nu]\mathbf{l} = \mathbf{0}\ \ \forall \mathbf{l} \in \mathbf{R_s}(\gamma) \tag{66}$$

$$\mathbf{R_s}(\gamma) = \{\mathbf{l}(\mathbf{x}) \mid \mathbf{l}(\mathbf{x}) = \mathbf{c_0} + \mathbf{c_1}\mathbf{x},\ \mathbf{x} \in (\mathbf{0}, \mathbf{1})\}$$