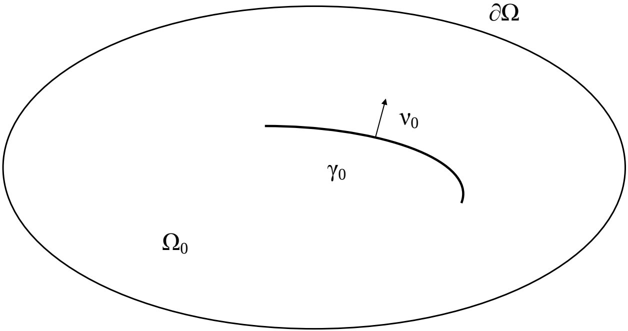
Shape sensitivity analysis for a body with a thin rigid inclusion

Rudoy E.M.

Lavrentyev Institute of Hydrodynamics of SB RAS
Novosibirsk State University
Novosibirsk, Russia

Equilibrium problem

Let $\Omega \subset \mathbb{R}^2$ be the bounded domain with the boundary $\partial\Omega \in C^{0,1}$.
Let γ_0 be the curve of the class $C^{1,1}$ without self-intersections;
$\Omega_0 = \Omega \setminus \overline{\gamma}_0$.



Let ν_0 be the unit normal vector.
Let $U = (u_1, u_2)^t$ be vector of displacements of points of the body Ω_0.

Linear Hooke's law in Ω_0:

$$\varepsilon_{ij}(U) = 1/2\,(u_{i,j} + u_{j,i})\,, \quad \sigma_{ij} = c_{ijkl}\varepsilon_{kl}(U),$$

where c_{ijkl} are components of symmetrical and positively defined tensor.

The set of infinitesimal rigid displacements:

$$R(\gamma_0) = \{\rho = (\rho_1, \rho_2)^t \mid \rho(x_1, x_2) = Bx + C,\ x = (x_1, x_2)^t \in \gamma_0\},$$

here B and C are arbitrary skew-symmetric matrix and constant vector, respectively:

$$B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \quad C = \begin{pmatrix} c^1 \\ c^2 \end{pmatrix},\ b, c^1, c^2 \in \mathbb{R}$$

The differential formulation of the problem

$$- \sigma_{ij,j}(U_0) = f_i \quad \text{in} \quad \Omega_0, \ (i = 1, 2), \tag{1}$$

$$U = (0,0)^t \quad \text{on} \quad \partial\Omega, \tag{2}$$

$$U_0 = B_0 x + C_0 \quad \text{on} \quad \gamma_0^-, \tag{3}$$

$$[U_0^t]\nu_0 \geq 0 \quad \text{on} \quad \gamma_0, \tag{4}$$

$$\sigma_\tau^+(U_0) = 0, \ \sigma_{\nu_0}^+(U_0) \leq 0 \quad \text{on} \quad \gamma_0, \tag{5}$$

$$\sigma_{\nu_0}^+(U_0)[U_0^t]\nu_0 = 0 \quad \text{on} \quad \gamma_0, \tag{6}$$

$$\int\limits_{\gamma_0} [\sigma(U_0)\nu_0]\rho ds_x = 0 \quad \forall \rho \in R(\gamma_0). \tag{7}$$

Here $\sigma_{\nu_0}(U_0) = \sigma_{ij}(U_0)\nu_{0i}\nu_{0j}$, $\sigma_{\tau i}(U_0) = \sigma_{ij}(U_0)\nu_{0j} - \sigma_{\nu_0}(U_0)\nu_{0i}$, $i = 1, 2$; $F^t = (f_1, f_2) \in \{C^1(\mathbb{R}^2)\}^2$ is specified vector; the signs $\pm$ in (3)-(6) denote the trace of a function taken on the corresponding face $\gamma_0^\pm$ of the crack γ_0.

The variational formulation of the problem

The functional space

$$H^{1,0}(\Omega_0) = \{v \in H^1(\Omega_0) \mid v = 0 \text{ a.e. on } \partial\Omega\}$$

$$H(\Omega_0) = H^{1,0}(\Omega_0) \times H^{1,0}(\Omega_0)$$

The set of admissible displacements

$$K_0(\Omega_0) = \{U \in H(\Omega_0) \mid [U^t]\nu_0 \geq 0 \text{ a.e. on } \gamma_0,\ U|_{\gamma_0^-} \in R(\gamma_0)\}$$

The functional of the potential energy

$$\Pi(\Omega_0; U) = \frac{1}{2}\int\limits_{\Omega_0} \sigma_{ij}(U)\varepsilon_{ij}(U)dx - \int\limits_{\Omega_0} F^t U dx.$$

The problem (1)-(7) is equivalent to the following minimization problem: to find the function $U_0 \in K_0(\Omega_0)$ such that

$$\Pi(\Omega_0; U_0) = \inf_{U \in K_0(\Omega_0)} \Pi(\Omega_0; U). \tag{8}$$

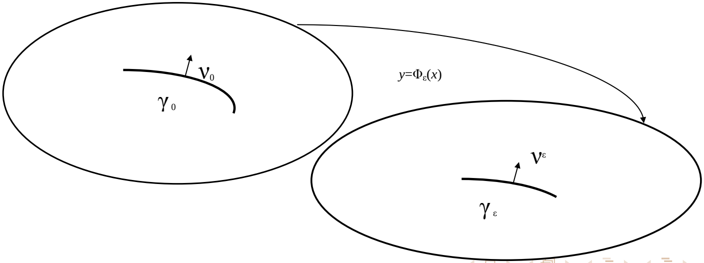
Shape change

The coordinate transformation:

$$y = \Phi_\varepsilon(x), \tag{9}$$

where $\varepsilon \in [0, \varepsilon_0)$, $(\varepsilon_0 = const)$,
$\Phi_{\varepsilon i} \in C^1([0, \varepsilon_0); W^{2,\infty}_{loc}(\mathbb{R}^2))$ $(i = 1, 2)$,
$\Phi_0(x) = x$.

The perturbed domain:
$\Omega_\varepsilon = \Phi_\varepsilon(\Omega_0)$ with the thin rigid inclusion $\gamma_\varepsilon = \Phi_\varepsilon(\gamma_0)$
ν^ε – the unit normal vector to the perturbed inclusion γ_ε



The perturbed problem

The set of admissible displacements for the perturbed problem:

$$K_\varepsilon(\Omega_\varepsilon) = \{U \in H(\Omega_\varepsilon) \mid [U^t]\nu^\varepsilon \geq 0 \text{ a.e. on } \gamma_\varepsilon,\ U \in R(\gamma_\varepsilon^-)\}$$

The energy functional:

$$\Pi(\Omega_\varepsilon; U) = \frac{1}{2}\int_{\Omega_\varepsilon} \sigma_{ij}(U)\varepsilon_{ij}(U)dy - \int_{\Omega_\varepsilon} F^t U dy$$

The perturbed problem:

$$\Pi(\Omega_\varepsilon; U^\varepsilon) = \inf_{U \in K_\varepsilon(\Omega_\varepsilon)} \Pi(\Omega_\varepsilon; U). \tag{10}$$

Transforming the perturbed problem back to the fixed domain

The inverse transformation

$$\Downarrow$$

$\Omega_\varepsilon \to \Omega_0, \quad H(\Omega_\varepsilon) \to H(\Omega_0)$

$\nu^\varepsilon \to \nu_\varepsilon$, where $\nu_\varepsilon(x) = \nu^\varepsilon(\Phi_\varepsilon(x))$

$R(\gamma_\varepsilon^-) \to R_\varepsilon(\gamma_0^-) = \{U \mid U(x) = B\Phi_\varepsilon(x) + C,\ x \in \gamma_0^-\}$

$$\Downarrow$$

$K_\varepsilon(\Omega_\varepsilon) \to K_\varepsilon(\Omega_0)$, where
$K_\varepsilon(\Omega_0) = \{U \in H(\Omega_0) \mid [U^t]\nu_\varepsilon \geq 0$ a.e. on $\gamma_0,\ U \in R_\varepsilon(\gamma_0^-),\}$

$$\nu_\varepsilon \neq \nu_0,\ R_\varepsilon(\gamma_0^-) \neq R(\gamma_0^-)$$
$$\Downarrow$$
$$K_\varepsilon(\Omega_0) \neq K(\Omega_0)$$

Asymptotic formulas

$$\Phi_\varepsilon(x) = x + \varepsilon V(x) + r_1(\varepsilon, x), \quad \|r_1(\varepsilon, x)\|_{[W^{2,\infty}_{loc}(\mathbb{R}^2)]^2} = o(\varepsilon),$$
$$\frac{\partial \Phi_\varepsilon(x)}{\partial x} = I + \varepsilon \frac{\partial V(x)}{\partial x} + r_2(\varepsilon, x), \quad \|r_2(\varepsilon, x)\|_{[W^{1,\infty}_{loc}(\mathbb{R}^2)]^4} = o(\varepsilon), \tag{11}$$

where

$$V(x) = (V_1(x), V_2(x))^t = \left.\frac{\partial \Phi_\varepsilon(x)}{\partial \varepsilon}\right|_{\varepsilon=0},$$
$$\frac{\partial V(x)}{\partial x} = \begin{pmatrix} V_{1,1}(x) & V_{2,1}(x) \\ V_{1,2}(x) & V_{2,2}(x) \end{pmatrix}.$$

$$\Downarrow$$

$$K_\varepsilon(\Omega_0) = \{U \in H(\Omega_0) \mid U \text{ holds (12) and (13)}\},$$

where

$$[U^t]\nu_0 - \varepsilon[U^t]\left(\frac{\partial V}{\partial x} - \frac{r_4(\varepsilon, x)}{\varepsilon}\right)\nu_0 \geq 0 \text{ a.e. on } \gamma_0 \tag{12}$$

$$U|_{\gamma_0^-} = Bx + C + \varepsilon B\left(V + \frac{r_1(\varepsilon, x)}{\varepsilon}\right). \tag{13}$$

Auxiliary statements

Theorem

Let $U_0 \in K_0(\Omega_0)$ *be the solution of unperturbed problem* (8) *and let* $U_\varepsilon \in K_\varepsilon(\Omega_0)$ *be the solution of the perturbed problem, transformed onto the domain* Ω_0. *Then for any* $\varepsilon \in (0, \varepsilon_0)$ *there exist vector-functions* W_ε^1 *and* W_ε^2 *such that*

$$U_0 + \varepsilon W_\varepsilon^1 \in K_\varepsilon(\Omega_0), \quad U_\varepsilon - \varepsilon W_\varepsilon^2 \in K_0(\Omega_0),$$

The functions W_ε^i, $i = 1, 2$ have the following form: $W_\varepsilon^i = P_\varepsilon^i + Q_\varepsilon^i$, where

P_ε^i correspond to the non-penetration condition and is constructed using the extension operator from the boundary of nonsmooth domain into the whole domain,

Q_ε^i provide for inclusion to the set of rigid displacements and is constructed explicitly, using a cut-off function θ such that $\theta = 1$ in some neighborhood of γ_0.

Theorem

$$\|U_\varepsilon - U_0\|_{H(\Omega_0)} \leq c\sqrt{\varepsilon}. \tag{14}$$

Theorem

For $i = 1, 2$, as $\varepsilon \to 0$, we have

$$W_\varepsilon^i \to W_0 = P_0 + Q_0 \quad \textit{strongly in} \quad H(\Omega_0)$$

with the following properties

$$P_0|_{\gamma_0^+} = \left(\frac{\partial V}{\partial x}\right)^t [U_0],$$

$$P_0|_{\gamma_0^-} = (0, 0)^t.$$

$$Q_0 = \theta B_0 V.$$

Shape derivative of the energy functional

Applying the transformation of coordinates (9) to integrals in $\Pi(\Omega_\varepsilon, U)$, we obtain a transformed functional

$$\Pi_\varepsilon(\Omega_0; U) = \frac{1}{2}\int\limits_{\Omega_0} \sigma_{ij}(U)\varepsilon_{ij}(U)dx - \int\limits_{\Omega_0} F^t U dx +$$

$$+\frac{1}{2}\varepsilon\int\limits_{\Omega_0} A_1(V; U, U)dx - \varepsilon\int\limits_{\Omega_0} \mathrm{div}\,(V f_i)u_i dx + o(\varepsilon)R_3(\varepsilon, U),$$

where $|R_3(\varepsilon, U)| \leq c_1\|U\|^2_{H(\Omega_0)} + c_2$.

Shape derivative

$$\Pi'(\Omega_0; U_0) = \lim_{\varepsilon\to 0+} \frac{\Pi(\Omega_\varepsilon; U^\varepsilon) - \Pi(\Omega_0; U_0)}{\varepsilon} =$$

$$= \lim_{\varepsilon\to 0+} \frac{\Pi_\varepsilon(\Omega_0; U_\varepsilon) - \Pi(\Omega_0; U_0)}{\varepsilon}.$$

By variational properties of the solutions U_0 and U_ε , the following chain of inequalities is true:

$$\frac{\Pi_\varepsilon(\Omega_0; U_\varepsilon) - \Pi(\Omega_0; U_\varepsilon - \varepsilon W_\varepsilon^2)}{\varepsilon} \leq \\ \leq \frac{\Pi_\varepsilon(\Omega_0; U_\varepsilon) - \Pi(\Omega_0; U_0)}{\varepsilon} \leq \\ \leq \frac{\Pi_\varepsilon(\Omega_0; U_0 + \varepsilon W_\varepsilon^1) - \Pi(\Omega_0; U_0)}{\varepsilon}. \quad (15)$$

And we get

$$\Pi'(\Omega_0;U_0) = \frac{1}{2}\int\limits_{\Omega_0} A_1(V;U_0,U_0)dx - \int\limits_{\Omega_0} \operatorname{div}(Vf_i)u_{0i}dx + \\ + \int\limits_{\Omega_0} \sigma_{ij}(U_0)\varepsilon_{ij}(W_0)dx - \int\limits_{\Omega_0} F^t W_0 dx \quad (16)$$

with

$$A_1(V;U_0,U_0) = \operatorname{div} V \sigma_{ij}(U_0)\varepsilon_{ij}(U_0) - 2\sigma_{ij}(U_0)E_{ij}\left(\frac{\partial V}{\partial x};U_0\right),$$

$$E_{ij}\left(\frac{\partial V}{\partial x};U_0\right) = \frac{1}{2}\left(\frac{\partial u_{0i}}{\partial x_k}V_{k,j} + \frac{\partial u_{0j}}{\partial x_k}V_{k,i}\right).$$

The derived formula contains the function W_0 which is not unique. Using properties of W_0, the formula for shape derivative can be rewritten as follows:

$$\Pi'(\Omega_0;U_0)=\frac{1}{2}\int\limits_{\Omega_0}\mathrm{div}V\sigma_{ij}(U_0)\varepsilon_{ij}(U_0)dx-\int\limits_{\Omega_0}\sigma_{ij}(U_0)E_{ij}\left(\frac{\partial V}{\partial x};U_0\right)dx-$$

$$-\int\limits_{\Omega_0}\mathrm{div}\,(Vf_i)u_{0i}dx-\left\langle\sigma_{\nu_0}^{+}(U_0),[U_0^t]\frac{\partial V}{\partial x}\nu_0\right\rangle_{1/2,\gamma_0}^{00}-\langle[\sigma(U_0)\nu_0],B_0V\rangle_{1/2,\partial\Sigma_0}, \tag{17}$$

where Σ_0 is an arbitrary closed curve such that $\gamma_0\subset\Sigma_0$.

Invariant integrals

For special types of perturbations the derivative of the energy functional can be represented in the form of invariant integral, i.e., curvilinear integral, the value of which is independent of the path of integration.

Example – perturbation of a rigid inclusion tip

Let γ_0 be a rectilinear rigid inclusion, i.e., it belongs to some line $x^t q = a$, where q is a specified unit vector and a is specified constant. Let C_1 and C_2 be the tips of γ_0. Let $F = (0,0)^t$ in some vicinity $\mathcal{U}$ of the inclusion γ_0. Choose the compactly supported in Ω function $\eta \in W^{2,\infty}_{loc}(\mathbb{R}^2)$; $\eta = 1$ in some vicinity $\mathcal{O} \subset \mathcal{U}$ of tip C_1 of γ_0;

$$\operatorname{supp} \eta \cap C_2 = \emptyset.$$

Let the unit vector $p = (p_1, p_2)^t$ be orthogonal to q, i.e., $p = (p_1, p_2)^t = (q_2, -q_1)^t$. Consider a shift perturbation in the direction of vector p:

$$\begin{aligned} y_1 &= x_1 + \varepsilon p_1 \eta(x_1, x_2), \\ y_2 &= x_2 + \varepsilon p_2 \eta(x_1, x_2). \end{aligned} \tag{18}$$

By using additional H^2-smoothness of the solution, we obtain

$$\Pi'(\Omega_0;\eta p) = -\int\limits_S \Big((p\cdot n)W(U_0) - t_i(U_0)(p\cdot\nabla u_{0i}) + \\ + b_0 p_2 t_1(U_0) - b_0 p_1 t_2(U_0) \Big) ds_x. \quad (19)$$

Here $W(U_0) = \frac{1}{2}\sigma_{ij}(U_0)\varepsilon_{ij}(U_0)$ is the density of elastic energy of body, $t_i(U_0) = \sigma_{ij}(U_0)n_j$, $i = 1, 2$, are components of the stress vector, S is the arbitrary contour around the tip of the rigid inclusion.