Interactions between moderately close inclusions. The Neumann case and the Dirichlet case in the plane.

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V. Bonnaillie-Noël, S. Tordeux et G. Vial

dépasser les frontières

UNIVERSITÉ DE PAU ET DES PAYS DE L'ADOUR

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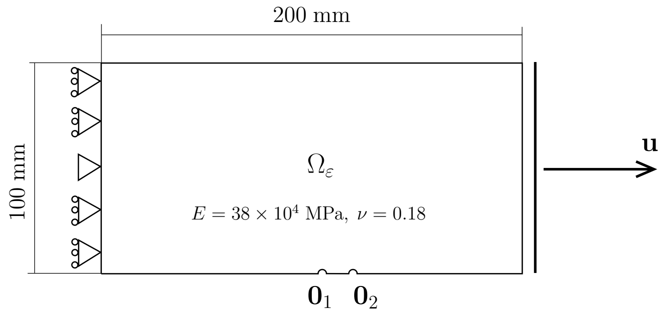
Initial Motivations

A toy problem in civil engeneering



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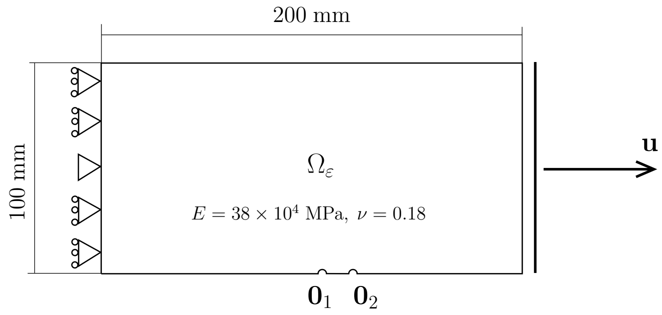

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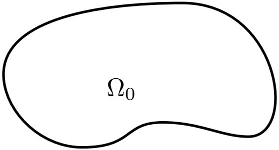
Our idea : take the micro-defects into account by multiscale asymptotic analysis and propose a dedicated XFEM method to make the numerical simulations avaible on a laptop.

The basic case : a single inclusion

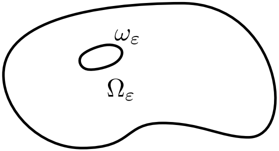
- $O \in \Omega_0$, $O \in \omega$
- $f \in C_0^\infty(\Omega_0)$
- $\Omega_\varepsilon = \Omega_0 \setminus \omega_\varepsilon$, $\omega_\varepsilon = \varepsilon\omega$
- $\Gamma = \partial\Omega_0$

Unperturbed domain



$$\left\{ \begin{array}{rcll} -\Delta u_0 & = & f & \text{dans } \Omega_0 \\ u_0 & = & 0 & \text{sur } \Gamma \end{array} \right.$$

Perturbed domain



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Compare u_ε et u_0

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  $\Rightarrow$ correctors in fast variable to *compensate* Taylor expansion of u_0 at 0

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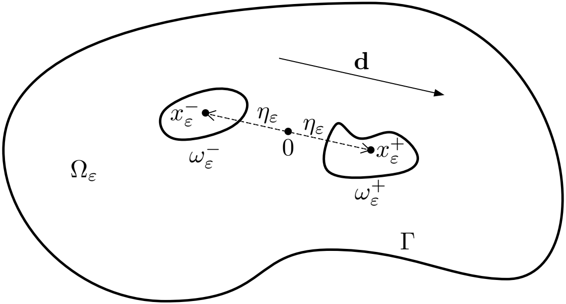
  - ◇ *slow variable* x at the scale of the domain
  - ◇ *fast variable* x/ε at the scale of the perturbation
- compare u_ε and the limit u_0
  $\Rightarrow$ correctors in fast variable to *compensate* Taylor expansion of u_0 at 0
- plug the correctors on Ω_ε
  $\Rightarrow$ generation of correctors in slow variable

Two inclusions and three scales



$$\Omega_\varepsilon = \Omega_0 \setminus \overline{(\omega_\varepsilon^+ \cup \omega_\varepsilon^-)} \quad \text{with} \quad \omega_\varepsilon^\pm = x_\varepsilon^\pm + \varepsilon\omega^\pm, \quad x_\varepsilon^\pm = \pm\eta_\varepsilon \mathbf{d}$$

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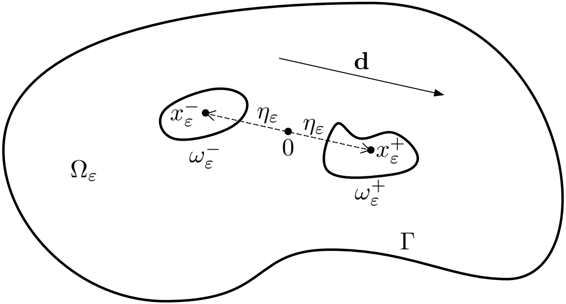

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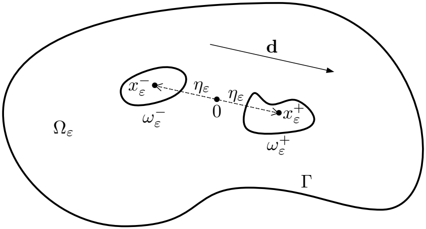
$$\eta_\varepsilon = \varepsilon^\alpha \text{ with } 0 < \alpha < 1$$

The nice case: the Neuman case and Dirichlet case in d=3

[Bonnaillie-Noël, Dambrine, Tordeux, Vial, M3AS, 2008]



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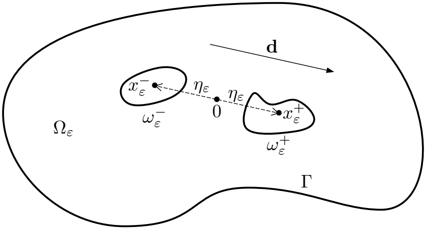

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One can play the usual two steps iterative strategy to build the ansatz:

- corrector in fast variable x/ε
- corrector in slow variable x.

Two inclusions: the nice case

The result

Theorem

The solution u_ε has the asymptotic expension at order N

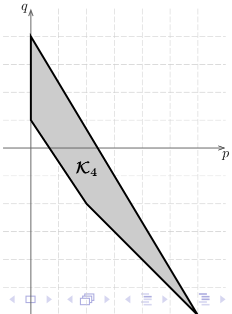
$$u_\varepsilon(x) = u_0(x) + \varepsilon\left[V_0^-\left(\tfrac{x-x_\varepsilon^-}{\varepsilon}\right) + V_0^+\left(\tfrac{x-x_\varepsilon^+}{\varepsilon}\right)\right]$$

$$+ \sum_{(p,q)\in\mathcal{K}_N} \varepsilon^{p+\alpha q}\left(v_{p+\alpha q}(x) + \varepsilon\left[V_{p+\alpha q}^-\left(\tfrac{x-x_\varepsilon^-}{\varepsilon}\right) + V_{p+\alpha q}^+\left(\tfrac{x-x_\varepsilon^+}{\varepsilon}\right)\right]\right) + r_\varepsilon^N(x)$$

with

$$\mathcal{K}_N = \{(p,q)\in\mathbb{Z}^2 \mid p\geq 0,$$
$$q \geq -\tfrac{3}{2}p+1,\ q\geq -p \text{ et } p+\alpha q \leq N\}$$

$$\|r_\varepsilon^N\|_{\mathrm{H}^1(\Omega_\varepsilon)} = o(\varepsilon^N)$$



Two inclusions

Interpretation of leading terms

$$u_\varepsilon(x) = u_0(x) + \varepsilon \left[V_0^- \left(\frac{x - x_\varepsilon^-}{\varepsilon}\right) + V_0^+ \left(\frac{x - x_\varepsilon^+}{\varepsilon}\right) \right] + r_\varepsilon^1(x)$$

with

$$\|r_\varepsilon^1\|_{\mathrm{H}^1(\Omega_\varepsilon)} = \mathcal{O}\big(\varepsilon^{\min(1+\alpha, 3-2\alpha)}\big)$$

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The rest r_ε^1 contains knowledge of high order terms.

- ◇ if $\alpha < 2/3$: inclusions well separated
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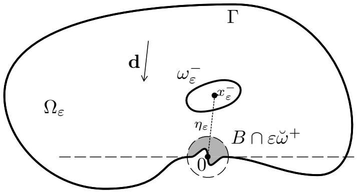
  main error comes from Taylor's expension of u_0 at 0
- ♢ if $2/3 < \alpha < 1$: close inclusions
  main error comes from the *interaction* between V_0^- et V_0^+
- ♢ if $\alpha = 2/3$:
  both errors are of equivalent size

An extension: one defect lays on the boundary



Theorem

$$u_\varepsilon(x) = \zeta(|\tfrac{x}{\varepsilon}|)u_0(x) + \varepsilon\left[V_0^-\left(\tfrac{x-x_\varepsilon^-}{\varepsilon}\right) + \chi(|x|)V_0^+\left(\tfrac{x}{\varepsilon}\right)\right]$$

$$+ \sum_{(p,q)\in\mathcal{K}_N} \varepsilon^{p+\alpha q}\left(\zeta(|\tfrac{x}{\varepsilon}|)v_{p+\alpha q}(x) + \varepsilon\left[V_{p+\alpha q}^-\left(\tfrac{x-x_\varepsilon^-}{\varepsilon}\right) + \chi(|x|)V_{p+\alpha q}^+\left(\tfrac{x}{\varepsilon}\right)\right]\right)$$

$$+ r_\varepsilon^N(x)$$

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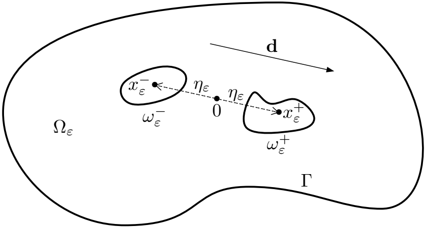
The not so nice case: Dirichlet case in d=2

[Bonnaillie-Noël, Dambrine, Asymptotic Analysis, 2013]



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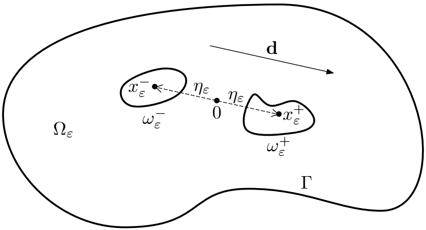

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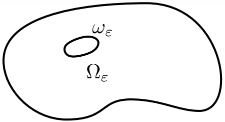
One cannot play the usual two steps iterative strategy to build the ansatz because of the logarithmic term.

The ideas to build the ansatz



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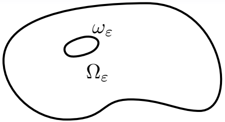

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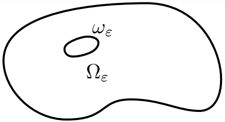

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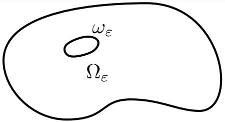

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$$a(\varepsilon) = \frac{1}{w(0) - \ln\varepsilon} u_0(0) \qquad \text{and} \qquad b(\varepsilon) = \frac{-1}{w(0) - \ln\varepsilon} u_0(0).$$

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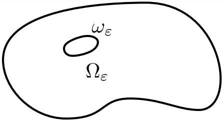

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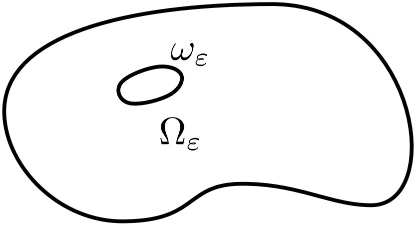
- New *scale* $h_\varepsilon = \frac{1}{w(0)-\ln\varepsilon}$ and corrector $\mathfrak{w}(x) = \ln|x| - w(x)$ appear !

One inclusion: Dirichlet case in d=2

A typical result



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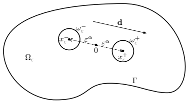
We get

$$u_\varepsilon(x) = u_0(x) + u_0(0)\mathfrak{w}(x)\ h_\varepsilon + \sum_{j\geq 1, k\geq 2} \alpha_{j,k}\mathfrak{w}(x)\ h_\varepsilon^j \varepsilon^k + \sum_{j\geq 0, k\geq 1} V_{j,k}(\tfrac{x}{\varepsilon})\ h_\varepsilon^j \varepsilon^k + \sum_{j\geq 0, k\geq 2} v_{j,k}(x)\ h_\varepsilon^j \varepsilon^k,$$

where any coefficient $\alpha_{j,k}$ and any functions $V_{j,k}$ and $v_{j,k}$ are obtained in a iterative way.

The setting and a technical result



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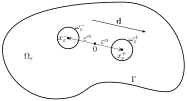

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Let $x \in \partial\omega_\varepsilon^\pm$, there exists $X \in \mathcal{B}(0,1)$ such that $x = \pm(\varepsilon^\alpha \mathbf{d} + \varepsilon X)$

$$\begin{array}{rcl} |x - x_\varepsilon^\mp|^2 & = & 4\varepsilon^{2\alpha}\left(1 + \varepsilon^{1-\alpha}\langle \mathbf{d}, X\rangle + \dfrac{\varepsilon^{2-2\alpha}}{4}\right), \\ \ln|x - x_\varepsilon^\mp| & = & \ln(2\varepsilon^\alpha) + \displaystyle\sum_{k\geq 1} \tilde{a}_k(X)\varepsilon^{(1-\alpha)k}. \end{array}$$

with

$$\tilde{a}_k(X) = \frac{(-1)^{k+1}}{k}\frac{\cos(k\theta)}{2^{k-1}} \text{ and } \int_{\partial\mathcal{B}(0,1)} \tilde{a}_k = 0!!$$

Applying the strategy 1/2

- Look for $a_\pm(\varepsilon)$ and $b(\varepsilon)$ such that r^1_ε defined by

$$u_\varepsilon(x) = u_0(x) + a_+(\varepsilon)\ \ln|x - x^+_\varepsilon| + a_-(\varepsilon)\ \ln|x - x^-_\varepsilon| + b(\varepsilon)\ w(x) + r^1_\varepsilon(x),$$

is reduced with respect to $r^0_\varepsilon = u_\varepsilon - u_0$.

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  is reduced with respect to $r_\varepsilon^0 = u_\varepsilon - u_0$.
- Get a linear system in $(a_\pm(\varepsilon), b(\varepsilon))$:

$$\begin{array}{rcrcrcl} a_+(\varepsilon) & + & a_-(\varepsilon) & + & b(\varepsilon) & = & 0, \\ \ln\varepsilon\; a_+(\varepsilon) & + & (\ln 2 + \alpha\ln\varepsilon)\; a_-(\varepsilon) & + & w(0)\; b(\varepsilon) & = & -u_0(0), \\ (\ln 2 + \alpha\ln\varepsilon)\; a_+(\varepsilon) & + & \ln\varepsilon\; a_-(\varepsilon) & + & w(0)\; b(\varepsilon) & = & -u_0(0), \end{array}$$

  to set

$$a_+(\varepsilon) = a_-(\varepsilon) = \frac{u_0(0)}{2w(0) - \ln 2 - (1+\alpha)\ln\varepsilon} \text{ and } b(\varepsilon) = -2a_\pm(\varepsilon).$$

Applying the strategy 2/2

- A new scale appears:

$$h_{\varepsilon,\alpha} = \frac{1}{2w(0) - \ln 2 - (1+\alpha)\ln\varepsilon} = \frac{-1}{(1+\alpha)\ln\varepsilon} + \mathcal{O}\left(\frac{1}{\ln^2\varepsilon}\right).$$

α appears as a multiplicative factor in the leading term of the asymptotic !

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- Set $\mathfrak{w}_\varepsilon(x) = \ln|x - x_\varepsilon^+| + \ln|x - x_\varepsilon^-| - 2w(x)$ and define the first approximation

$$u_\varepsilon(x) = u_0(x) + u_0(0)\mathfrak{w}_\varepsilon(x)\ h_{\varepsilon,\alpha} + \tilde{r}_\varepsilon^1(x).$$

- Inject $\tilde{r}_\varepsilon^1$ in the equations.

$$\begin{cases} -\Delta\tilde{r}_\varepsilon^1 = 0 & \text{in } \Omega_\varepsilon, \\ \tilde{r}_\varepsilon^1 = -u_0(0)\mathfrak{w}_\varepsilon\ h_{\varepsilon,\alpha} & \text{on } \partial\Omega_0, \\ \tilde{r}_\varepsilon^1 = -u_0 - u_0(0)\mathfrak{w}_\varepsilon\ h_{\varepsilon,\alpha} & \text{on } \partial\omega_\varepsilon^\pm. \end{cases}$$

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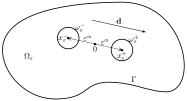
- examine its traces on Ω_ε, that is on $\partial\Omega_0$ and on $\partial\omega_\varepsilon^\pm$ and iterate ...

Moderately close inclusions: Dirichlet case in d=2



The result

$$\left\{\begin{array}{rcll} -\Delta u_\varepsilon & = & f & \text{in } \Omega_\varepsilon \\ u_\varepsilon & = & 0 & \text{on } \Gamma \\ u_\varepsilon & = & 0 & \text{on } \partial\Omega_\varepsilon^\pm \end{array}\right.$$

We get

$$\begin{aligned} u_\varepsilon(x) &= u_0(x) + u_0(0)\mathfrak{w}_\varepsilon(x)\ h_{\varepsilon,\alpha} + \sum_{i\geq 0, j\geq 1, k\geq 0} c_{i,j,k}\mathfrak{w}_\varepsilon(x) h^i_{\varepsilon,\alpha}\varepsilon^{j+\alpha k} \\ &+ \sum_{i\geq 1, k\geq 2} \beta_{i,k}(b_k^+(x)+b_k^-(x))\ h^i_{\varepsilon,\alpha}\varepsilon^{\alpha k} + \sum_{i\geq 1, k\geq 2} \beta_{i,k}(b_k^+(0)+b_k^-(0))\mathfrak{w}_\varepsilon(x)\ h^{i+1}_{\varepsilon,\alpha}\varepsilon^{\alpha k} \\ &+ \sum_{i\geq 0, j\geq 0, k\geq 1} \left(A_{i,j,k}\left(\tfrac{x-x_\varepsilon^+}{\varepsilon}\right) + A_{i,j,k}\left(\tfrac{x-x_\varepsilon^-}{\varepsilon}\right)\right)\ h^i_{\varepsilon,\alpha}\varepsilon^{j+k(1-\alpha)} \\ &+ \sum_{i\geq 0, j\geq 0, k\geq 1} \hat{a}_{i,j,k}(x)\ h^i_{\varepsilon,\alpha}\varepsilon^{j+1+k(1-\alpha)} \\ &+ \sum_{i\geq 0, j+k>0} \left(V^+_{i,j,k}\left(\tfrac{x-x_\varepsilon^+}{\varepsilon}\right) + V^-_{i,j,k}\left(\tfrac{x-x_\varepsilon^-}{\varepsilon}\right)\right) h^i_{\varepsilon,\alpha}\varepsilon^{j+\alpha k} \\ &+ \sum_{i\geq 0, j+k>0} (\hat{v}^+_{i,j,k}(x) + \hat{v}^-_{i,j,k}(x)) h^i_{\varepsilon,\alpha}\varepsilon^{j+1+\alpha k}. \end{aligned}$$