Singular Solutions and Large Solutions to some Nonlinear Elliptic Equations in Polygonal Domains

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Augmented Singular Days, in honour of Martin Costabel
Rennes, August 27, 2013

Berlin 2010:

Plasma (Maxwellian) equilibrium near a conducting sharp end modelled by non-linear elliptic problem:

$$\Delta u = p(\boldsymbol{x})\, \mathrm{e}^u.$$

Physical problem, “usual” BC
$\rightsquigarrow$ singular solution

Asymptotic issues
$\rightsquigarrow$ auxiliary problem, “infinite” BC
$\rightsquigarrow$ large solution

Outline

The model: Boltzmann–Poisson

General properties (Singular solutions)

Interlude: Large solutions to $\Delta u = p\,\mathrm{e}^u$

Asymptotic Issues: Mass & Applied Voltage

Open problems

The Boltzmann–Poisson problem

$$-\Delta\phi = \kappa\, \mathrm{e}^{\phi_{\mathrm{e}}-\phi} := \rho \text{ in } \Omega \subset \mathbb{R}^d, \quad \int_\Omega \rho\, dx = M\,; \qquad (1)$$

$$\phi = 0 \text{ on } \Gamma_1 \cup \Gamma_2, \quad \partial_\nu \phi = 0 \text{ on } \Gamma_3. \qquad (2)$$

- $\rho(x)$: density of "moving" particles (electrons).
- $\phi(x)$: self-consistent potential.
- $\phi_{\mathrm{e}}(x)$: external (confining) potential;
  total potential $V = \phi - \phi_{\mathrm{e}}$; Maxwell distribution $\rho \propto \mathrm{e}^{-V}$.
- M: total mass of the particles (given).
- κ: normalization factor for the mass constraint (unknown).

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External potential $\phi_{\mathrm{e}} \in L^\infty(\Omega)$ solution to linear problem:

$$-\Delta\phi_{\mathrm{e}} = \rho_{\mathrm{e}} \text{ in } \Omega, \quad \partial_\nu \phi_{\mathrm{e}} = 0 \text{ on } \Gamma_3, \tag{3}$$

$$\phi_{\mathrm{e}} = 0 \text{ on } \Gamma_1, \quad \phi_{\mathrm{e}} = \phi_{\mathrm{in}} \text{ on } \Gamma_2, \quad \phi_{\mathrm{in}} \in H^{1/2}(\Gamma_2) \cap L^\infty(\Gamma_2).$$

- $\rho_{\mathrm{e}}(x)$: density of “neutralising background” (ions).
- $\phi_{\mathrm{in}}(x)$: applied voltage.
- $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, with Γ_2 and Γ_3 possibly empty.

Well-posedness: equivalent to convex optimisation pb. . .

Monotonicity Properties

Monotonicity of exponential & maximum principle [Lions 69]:

Theorem 1: Let ϕ_1 and ϕ_2 be two solutions to (1)–(2), corresponding to $(\phi_{\rm e}, \kappa, M) = (\phi_{\rm e}^1, \kappa_1, M_1)$ and $(\phi_{\rm e}^2, \kappa_2, M_2)$ respectively. There holds:

$$\phi_{\rm e}^1 + \ln \kappa_1 \geq \phi_{\rm e}^2 + \ln \kappa_2 \text{ in } \Omega \implies \begin{cases} \phi_1 & \geq & \phi_2 \text{ in } \Omega; \\ M_1 & \geq & M_2. \end{cases}$$

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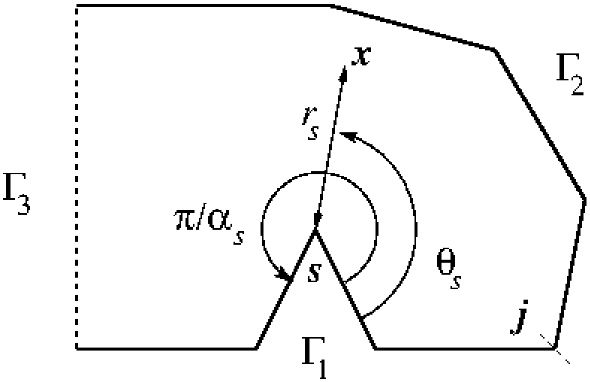
Corollary 3: Let $\phi_i,\ i = 1,\ 2$ be solutions to (1)–(2) with $\phi_{\rm e} = \phi_{\rm e}^i$ and the same mass $M > 0$. If $\phi_{\rm e}^1 \geq \phi_{\rm e}^2$, then $\kappa_1 \leq \kappa_2$.

Corner behaviour

We assume that Ω is a bounded polygonal domain in $\mathbb{R}^2$, with one re-entrant corner of opening π/α $(1/2 < \alpha < 1)$, located on Γ_1.



[Grisvard 85, 92. . .]: for all $p \in \left(\frac{2}{2-\alpha}, \frac{1}{1-\alpha}\right)$ we have:

$$\phi = \phi_R + \lambda\,\chi(r)\,r^{\alpha}\,\sin(\alpha\theta)$$

where $\phi_R \in W^{2,p}(\Omega)$ is the regular part of ϕ, $\lambda = -\int_\Omega \Delta\phi\, P_s$ is the singularity coefficient and P_s is the dual singularity given by:

$$-\Delta P_s = 0 \quad \text{in } \Omega, \quad P_s = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2, \quad \partial_\nu P_s = 0 \quad \text{on } \Gamma_3\,;$$
$$P_s = \tfrac{1}{\pi}\, r^{-\alpha}\,\sin(\alpha\theta) + \text{l.s.t.} \quad \text{near the reentrant corner.}$$

Singular term **dominant** near the corner, hence by Thm 1:

Proposition 4: Let $\phi_1,\ \phi_2$ be solutions to (1)–(2) with $M = M_1$, resp. M_2, and the same external potential ϕ_{e}. If $M_1 \geq M_2$, then $\lambda_1 \geq \lambda_2$.

What is a "Large Solution"? (aka: Boundary blow-up problem)

We consider the problem, set in a bounded Lipschitz domain Ω:

$$\Delta u = f(\boldsymbol{x}, u) \text{ in } \Omega, \quad (4)$$

$$u \to +\infty \text{ near } \Gamma_B, \quad u = g_D \text{ on } \Gamma_D, \quad \partial_\nu u = g_N \text{ on } \Gamma_N. \quad (5)$$

First part of (5): Bieberbach condition. Γ_D and Γ_N can be empty, but: $\Gamma_D \neq \emptyset \implies \Gamma_N \neq \emptyset$ and $\overline{\Gamma}_B \cap \overline{\Gamma}_D = \emptyset$.

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Refs. for **pure Bieberbach problem** ($\Gamma_B = \partial\Omega$):
Bieberbach 1916, J.B. Keller 1957, Bandle & Marcus (80s–90s), Lazer & McKenna (93–94), J. Rossi et al. (00s), ...

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Theorem 5: Assume that: Ω is a plane polygon if mixed BC;

$$g_D \in H^{\frac{1}{2}+\epsilon}(\Gamma_D), \quad g_N \in H^{-\frac{1}{2}+\epsilon}(\Gamma_D), \qquad \text{if } \Gamma_D \neq \emptyset \text{ resp. } \Gamma_N \neq \emptyset,$$

$$0 < p_* \leq p(\boldsymbol{x}) \leq p^* < +\infty \qquad \text{a.e. in } \Omega.$$

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Remark: setting $u = u_\circ + v$, where $v \in H^{1+\epsilon}(\Omega)$ solves

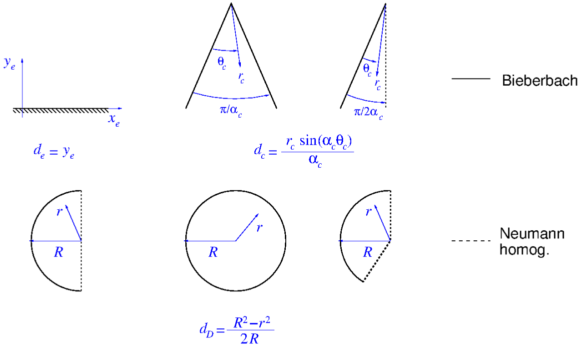
$$\Delta v = 0 \text{ in } \Omega, \quad v = 0 \text{ on } \Gamma_B, \quad v = g_D \text{ on } \Gamma_D, \quad \partial_\nu v = g_N \text{ on } \Gamma_N,$$

one arrives at the homogeneous problem (if Γ_D or $\Gamma_N \neq \emptyset$):

$$\Delta u_\circ = p_\circ(\boldsymbol{x})\,\mathrm{e}^{u_\circ} \text{ in } \Omega, \quad \text{with } p_\circ := p\,\mathrm{e}^{v} \text{ bounded above \& below};$$
$$u_\circ \to +\infty \text{ near } \Gamma_B, \quad u_\circ = 0 \text{ on } \Gamma_D, \quad \partial_\nu u_\circ = 0 \text{ on } \Gamma_N.$$

The Magic Formula

Solution to $\Delta u = p_* \, e^u$, with $p_* =$ const., is $u = \ln \dfrac{2}{p_* \, d^2}$:



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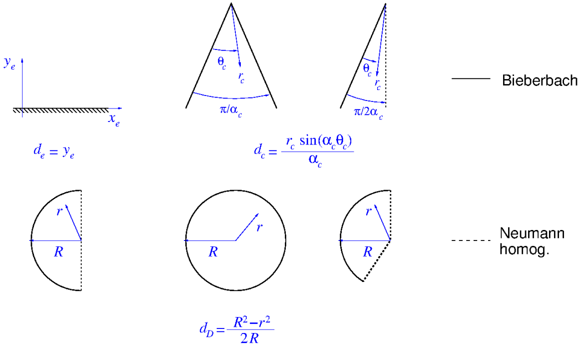

Also, supersolution to $\Delta u = p(\boldsymbol{x}) \, e^u$ with "usual" Dirichlet / homogeneous Neumann, if $p(\boldsymbol{x}) \geq p_*$ (and R small enough).

Proof of Theorem 5 (sketched)

Introduce u_K solution to usual problem:

$$\Delta u_K = p(\boldsymbol{x})\, \mathrm{e}^{u_K} \text{ in } \Omega\,;$$

$$u_K = K \text{ on } \Gamma_B, \quad u_K = 0 \text{ on } \Gamma_D, \quad \partial_\nu u_K = 0 \text{ on } \Gamma_N.$$

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Maximum principle: $u_K \geq u_k$ if $K \geq k$.

For any $\boldsymbol{x}_0 \in \overline{\Omega} \setminus \overline{\Gamma}_B$, u_K bounded uniformly in K by "disk solution" on $\overline{\Omega \cap B(\boldsymbol{x}_0, R)}$, for R small enough. Thus:

$$u_K(\boldsymbol{x}) \nearrow u(\boldsymbol{x}) \quad \text{and} \quad \Delta u_K(\boldsymbol{x}) = p(\boldsymbol{x})\, \mathrm{e}^{u_K(\boldsymbol{x})} \nearrow p(\boldsymbol{x})\, \mathrm{e}^{u(\boldsymbol{x})}.$$

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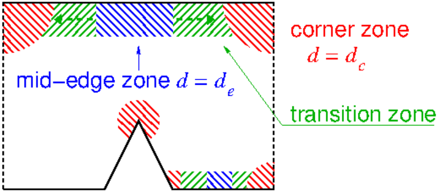
What about uniqueness?

Adapt the ideas of Lazer–McKenna, Rossi, Bandle & Marcus, Matero. . . to our case.

Theorem 6: Assume the hypotheses of Theorem 5. The solution to (4)–(5) is unique. Furthermore, there exists $C > 0$ such that

$$|u - \ln(d^{-2})| \leq C \tag{6}$$

where the pseudo-distance d to Γ_B is defined as follows:



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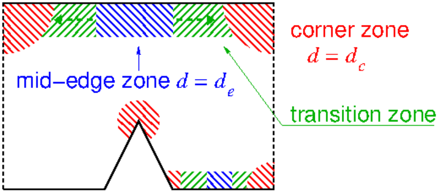

Remark: The r.h.s $p\,e^u \notin L^1(\Omega)$.

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Proposition 7: As $M \to 0$, we have

$$\kappa_M \sim M \Big(\int_\Omega \mathrm{e}^{\phi_{\mathrm{e}}}\, dx \Big)^{-1} \quad \text{and} \quad \lambda_M \sim \kappa_M \int_\Omega \mathrm{e}^{\phi_{\mathrm{e}}}\, P_s\, dx.$$

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Theorem 8: As $M \to +\infty$, there holds:

$$\frac{M}{\kappa_M} \to 0 \quad \text{and} \quad \frac{\lambda_M}{\kappa_M} \to 0 \quad \text{and} \quad \lambda_M \to +\infty. \tag{8}$$

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- Dominated convergence:

$$\frac{M}{\kappa_M} = \frac{1}{\kappa_M} \int_\Omega \rho_M \leq \mathrm{e}^{\|\phi_\mathrm{e}\|_\infty} \int_\Omega \mathrm{e}^{-\phi_M} \to 0,$$
$$\frac{\lambda_M}{\kappa_M} = \frac{1}{\kappa_M} \int_\Omega \rho_M P_s \leq \mathrm{e}^{\|\phi_\mathrm{e}\|_\infty} \int_\Omega P_s \,\mathrm{e}^{-\phi_M} \to 0.$$

Proof of Theorem 8 (2)

- $P_s\,\rho_M$ converges monotonically toward $P_s\,\rho_\infty$ ($P_s \geq 0$):

$$\lambda_M = \int_\Omega P_s\,\rho_M \to \int_\Omega P_s\,\rho_\infty := \lambda_\infty \in [0, +\infty].$$

- Asymptotic behaviour of u_∞ (Theorem 6) and P_s near the reentrant corner $\boldsymbol{s}$:

$$P_s\,\rho_\infty \geq C\,\frac{r_s^{-\alpha_s}\ \sin(\alpha_s\theta_s)}{(r_s\ \sin(\alpha_s\theta_s))^2} = \frac{C}{r_s^{\alpha_s+2}\ \sin(\alpha_s\theta_s)} \notin L^1(\Omega_s),$$

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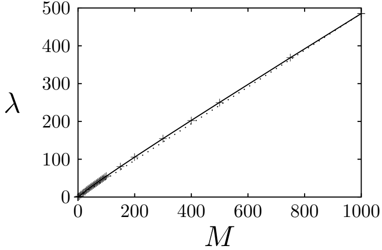
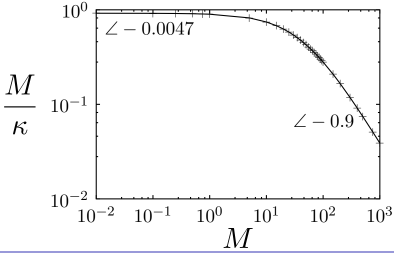
Behavior with respect to total mass

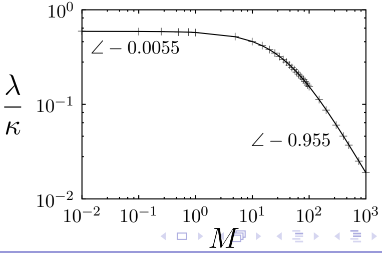


$$\phi_{\mathrm{e}} \equiv 0, \quad \Gamma_1 = \partial\Omega$$

$$\lambda_M \propto \kappa_M \propto M \quad \text{as} \quad M \to 0$$

$$\left.\begin{array}{l} \lambda_M \propto M^{1-\epsilon} \\ \kappa_M \propto M^{2-\epsilon} \end{array}\right\} \quad \text{as} \quad M \to +\infty$$





Behaviour w.r.t. applied voltage: Cloud problem

$$-\Delta\phi = \kappa\, \mathrm{e}^{\phi_{\mathrm{e}}-\phi} := \rho, \quad \int_\Omega \rho\, dx = M\,;$$
$$-\Delta\phi_{\mathrm{e}} = \rho_{\mathrm{e}} \text{ in } \Omega, \quad \partial_\nu \phi_{\mathrm{e}} = 0 \text{ on } \Gamma_3,$$
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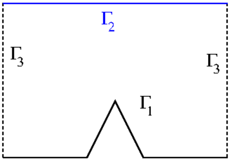

Above m is a variable real parameter and

$$\bar\phi_{\mathrm{in}} \in H^s(\Gamma_2) \text{ for some } s > \tfrac{1}{2}, \quad 1 \le \bar\phi_{\mathrm{in}} \le \bar\phi_{\mathrm{in}}^* < +\infty.$$

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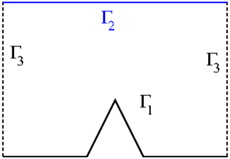

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⇝ notations: $\kappa_m,\ \rho_m,\ \phi_m,\ \lambda_m$ = its singularity coefficient.

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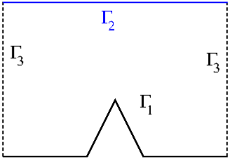

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$\rightsquigarrow$ notations: $\kappa_m,\ \rho_m,\ \phi_m,\ \lambda_m =$ its singularity coefficient.

We decompose $\phi_e := m\,H_{in} + \psi_e$, where H_{in} and ψ_e solve:

$$\Delta H_{in} = 0 \text{ in } \Omega, \quad H_{in} = 0 \text{ resp. } \bar\phi_{in} \text{ on } \Gamma_1 \text{ resp. } \Gamma_2, \quad \partial_\nu H_{in} = 0 \text{ on } \Gamma_3\,;$$

$$-\Delta\psi_e = \rho_e \text{ in } \Omega, \quad \psi_e = 0 \text{ on } \Gamma_1 \cup \Gamma_2, \quad \partial_\nu \psi_e = 0 \text{ on } \Gamma_3.$$

Proposition 9: As $m \to +\infty$, there holds:

$$\kappa_m \to 0 \quad \text{and} \quad \kappa_m \exp(m\,\bar{\phi}^*_{\text{in}}) \to +\infty. \tag{9}$$

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$w_m = \ln \kappa_m$ on Γ_1, $w_m = \ln \kappa_m + m\,\bar{\phi}_{\text{in}}$ on Γ_2, $\partial_\nu w_m = 0$ on Γ_3.

Assume $\kappa_\infty > 0$. Comparison argument: $w_m \ge \underline{w}_m$, where:

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As $m \to +\infty$, $\underline{w}_m$ converges monotonically toward $\underline{w}_\infty$ s.t.:

$$\Delta \underline{w}_\infty = \exp(\psi_{\rm e} + \underline{w}_\infty) := \underline{\rho}_\infty \text{ in } \Omega,$$

$$\underline{w}_\infty = \ln \kappa_\infty \text{ on } \Gamma_1, \quad \underline{w}_\infty = +\infty \text{ on } \Gamma_2, \quad \partial_\nu \underline{w}_\infty = 0 \text{ on } \Gamma_3\,;$$

and $\underline{\rho}_m \nearrow \underline{\rho}_\infty$. By Thm 6, $\underline{\rho}_\infty$ is not integrable near Γ_2.

We have just seen $\underline{\rho}_\infty \notin L^1(\Omega)$. Monotone convergence:

$$M = \int_\Omega \rho_m \geq \int_\Omega \underline{\rho}_m \to \int_\Omega \underline{\rho}_\infty = +\infty,$$

a contradiction which proves $\kappa_\infty = \lim \kappa_m = 0$.

2. Assume there is a sequence $m_n \to +\infty$ such that:

$\kappa_{m_n} \exp(m_n \bar{\phi}^*_{\text{in}}) \leq C$, thus $\kappa_{m_n} \exp(\beta\, m_n) \to 0,\ \forall \beta < \bar{\phi}^*_{\text{in}}$.

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As a harmonic function, $H_{\rm in} < \bar{\phi}^*_{\rm in}$ in Ω. So, for all $\boldsymbol{x} \in \Omega$:
$\rho_{m_n}(\boldsymbol{x}) = \kappa_{m_n} \exp(m_n\, H_{\rm in} + \psi_{\rm e} - \phi_{m_n}) \to 0.$

Dominated convergence: $M = \int_\Omega \rho_{m_n} \to 0$, a contradiction.

Proposition 10: Assume there exists $\beta > 0$ and $C > 0$ such that

$$\kappa_m \, \mathrm{e}^{\beta\, m} \leq C \quad \text{as } m \to +\infty. \tag{10}$$

Then the singularity coefficient λ_m remains bounded as $m \to +\infty$. If (10) holds for all $\beta < \bar{\phi}^*_{\mathrm{in}}$, then $\lambda_m \to 0$.

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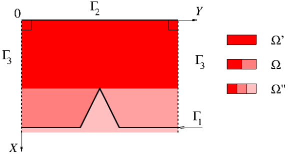
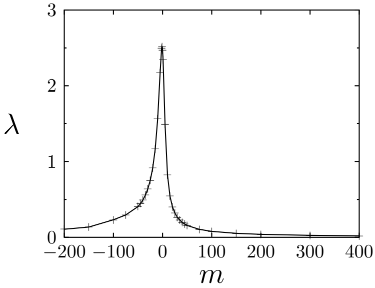
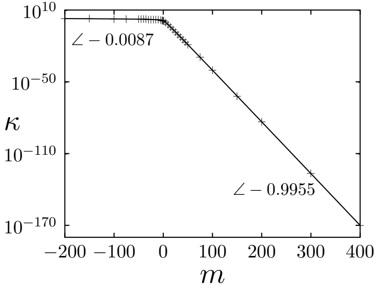
Proposition 11: If Ω is the domain represented below, the bound (10) holds for all $\beta < 1$.



Proof: Use 1D model to construct a solution on Ω', extended to a subsolution on $\Omega'' \supset \Omega$; use Prop 9 together with $M = \text{const}$.

Behavior with respect to applied voltage: Cloud problem





$\bar{\phi}_{\rm in} =$ const. $(\bar{\phi}^*_{\rm in} = 1)$. Electrical neutrality $(M = \int_\Omega \rho_{\rm e})$.
$\kappa_m \propto {\rm e}^{-(1-\epsilon)\, m}$ as $m \to +\infty$,
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  - Full Vlasov–Poisson... Vlasov–Maxwell... Mere **existence of solutions** unknown.
- Realistic and mathematically complete/tractable modelling of lightning???