Extracting edge flux/stress intensity functions with the Quasi-Dual Function Method along circular 3-D edges

Zohar Yosibash & Samuel Shannon

NCCM

Computational Mechanics Lab,

Mech. Eng. Dept., Ben-Gurion Univ., Beer-Sheva, Israel

In collaboration with Monique Dauge &Martin Costabel

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Dear Martin – A Happy Birthday and many returns on this joyful event!

Many thanks for the kind invitation to 7th Singular Days Conf.

Pablo Picasso, 1958
Bouquet of Flowers in the Hands

I met and started to collaborate with Martin about 18 years ago.

He visited Israel for ICOSAHOM in 1998….

Jerusalem,“Roman restaurant” - 1998

Le-Tronchet, 2001

Oberwolfach, 2002

Outline

- Motivation – the “real world” is 3-D and complex.
- Asymptotic solution of the elasticity system in the vicinity of a circular edge.
- Quasi-Dual Function Method for extracting GESIFs.
- Numerical Examples.
- Conclusions and future research.

Outline

Cracks in real 3-D world (engineering practice) are almost never straight lines….

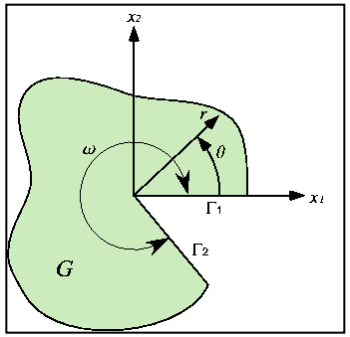
Outline

Past works on the topic - singular solutions

2-D Solution - singular point

The solution is of the form:

$$u(r,\theta)=\sum_{k=0}^{\infty}A_i r^{\alpha_k}\phi_k(\varphi)$$

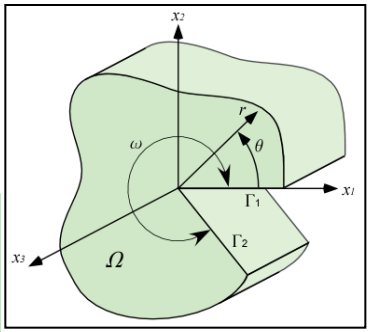


M.L. Williams. : Stress singularities resulting from various boundary condition in angular corners of plates in extension. *Trans. ASME, Jour. Appl. Mech*., **19**, 526-528, (1952).

3-D Solution - straight edge

The solution is of the form:

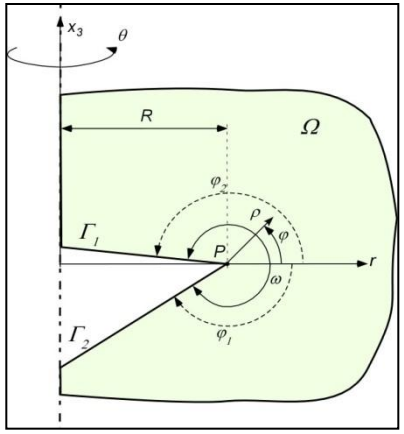
$$u(r,\theta,x_3)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\partial_{x_3}^{j}A_k(x_3)r^{\alpha_k+j}\phi_{k,j}(\theta)$$



Omer, N., Yosibash, Z., Costabel, M., Dauge, M.: Edge Flux intensity functions in polyhedral domains and their extraction by a quasidual function method. *Int. Jour. Fracture* **129**, 97-130 (2004).

Yosibash, Z., Omer, N., Costabel, M., Dauge, M.: Edge stress intensity functions in polyhedral domains and their extraction by a quasidual function method. *Int. Jour. Fracture* **129**, 97-130 (2005).

3-D Solution - circular edge



The Laplace equation

First three singular terms for the case of homogeneous Dirichlet boundary conditions.

T. Von Petersdorff, E.P. Stephan. Singularities of the solution of the Laplacian in domains with circular edges. *Applicable analysis*, **45**, 281-294, (1992).

The elasticity system

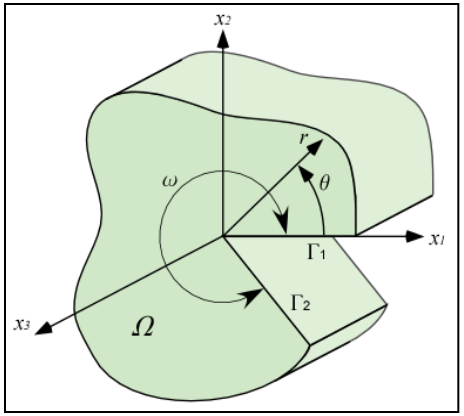
First two terms of the displacements series in the vicinity of a circular edge.

J.B. Leblond, O. Torlai.: The stress field near the front of arbitrarily shaped crack in a 3-dimensional elastic body. *Jour. Of Elasticity*, **29**, 97-131, (1992).

Axi-symmetric solution of a penny shaped crack.

A.Y.T. Leung, R.K.L. Su.: Eigenfunction expansion for penny-shaped and circumferential cracks. *Int. Jour. Fracture*, **89**, 205-222, (1998).

Past works on the topic - computation of the ESIF/EFIF



Computation of the ESIF - Straight edge

Extending of the Dual-Function Method to compute the ESIF along a straight edge.

Costabel, M., Dauge, M., Yosibash, Z.: A quasi-dual function method for extracting edge stress intensity functions, *SIAM Jour. Math. Anal*, **35**(5), 1177-1202 (2004).

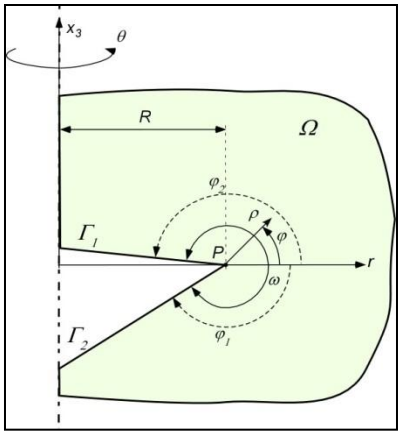
Computation of the ESIF - Circular edge

Penny-shaped crack with an Axisymmetric load

Extending of the Two Level Finite Element Method for compute the ESIF in the case of a penny-shaped crack with an axi-symmetric load.

A.Y.T. Leung, R.K.L. Su.: Two-level finite element study of axi-symmetric cracks. *Int. Jour. Fracture*, **89**, 193-203, (1998).

Penny-shaped crack in an axi-symmetric geometry with a non axisymmetric load



Computation of the ESIF along axi-symmetric body with a non-axi-symmetric load, by approximating of the boundary condition by a Fourier series.

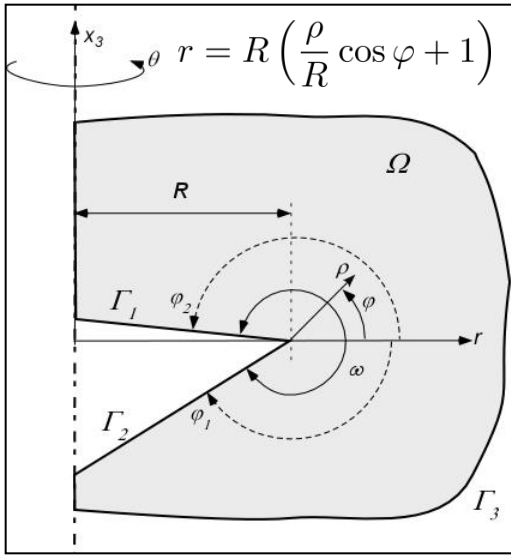
A.Y. Kou.: On the use of a path-independent line integral for axi-symmetric cracks with non axi-symmetric loading. *Trans. ASME, Jour. Appl. Mech.*, **54**, 833-837, (1987).

The elasticity system in the vicinity of a circular edge

Consider the system of 3 PDEs (equilibrium equations) in terms of the 6 stress tensor components, in (ρ, φ, θ) coordinates:

$$
\begin{aligned}
0 = \quad & \frac{\partial \sigma_{\rho\rho}}{\partial \rho} + \frac{1}{\rho}\frac{\partial \sigma_{\rho\varphi}}{\partial \varphi} + \frac{\sigma_{\rho\rho} - \sigma_{\varphi\varphi}}{\rho} + \frac{1}{r}\left(\frac{\partial \sigma_{\rho\theta}}{\partial \theta} + (\sigma_{\rho\rho} - \sigma_{\theta\theta})\cos\varphi - \sigma_{\rho\varphi}\sin\varphi\right) \\
0 = \quad & \frac{1}{\rho}\frac{\partial \sigma_{\varphi\varphi}}{\partial \varphi} + \frac{\partial \sigma_{\rho\varphi}}{\partial \rho} + \frac{2}{\rho}\sigma_{\rho\varphi} + \frac{1}{r}\left(\frac{\partial \sigma_{\varphi\theta}}{\partial \theta} + (\sigma_{\theta\theta} - \sigma_{\varphi\varphi})\sin\varphi + \sigma_{\rho\varphi}\cos\varphi\right) \\
0 = \quad & \frac{\partial \sigma_{\rho\theta}}{\partial \rho} + \frac{1}{\rho}\sigma_{\rho\theta} + \frac{1}{\rho}\frac{\partial \sigma_{\varphi\theta}}{\partial \varphi} + \frac{1}{r}\left(\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + 2\sigma_{\rho\theta}\cos\varphi - 2\sigma_{\varphi\theta}\sin\varphi\right)
\end{aligned}
$$

We use the Hooke constitutive law, and the kinematic connections between strains and displacements to finally obtain the complicated system of PDEs to be solved.



The elasticity system in the vicinity of a singular circular edge

The kinematic connection between the strains and the displacements:

$$
\begin{aligned}
\varepsilon_{\rho\rho} &= \frac{\partial u_\rho}{\partial \rho} \\
\varepsilon_{\varphi\varphi} &= \frac{1}{\rho}\frac{\partial u_\varphi}{\partial \varphi} + \frac{u_\rho}{\rho} \\
\varepsilon_{\theta\theta} &= \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_\rho \cos\varphi - u_\varphi \sin\varphi\right) \\
\varepsilon_{\rho\varphi} &= \frac{1}{2}\left[\frac{1}{\rho}\frac{\partial u_\rho}{\partial \varphi} + \frac{u_\varphi}{\rho} - \frac{u_\varphi}{\rho}\right] \\
\varepsilon_{\varphi\theta} &= \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_\varphi}{\partial \theta} + \frac{1}{\rho}\frac{\partial u_\theta}{\varphi} + \frac{1}{r}u_\theta \sin\varphi\right] \\
\varepsilon_{\rho\theta} &= \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_\rho}{\partial \theta} + \frac{\partial u_\theta}{\rho} - \frac{1}{r}u_\theta \cos\varphi\right]
\end{aligned}
$$

Hooke's constitutive law:

$$
\sigma_{ij} = \lambda\delta_{ij}\sum_{k=1}^{3}\varepsilon_{kk} + 2\mu\varepsilon_{ij}, \quad i,j = \rho,\varphi,\theta
$$

The stresses in terms of displacements are:

$$
\begin{pmatrix} \sigma_{\rho\rho} \\ \sigma_{\theta\theta} \\ \sigma_{\varphi\varphi} \\ \sigma_{\rho\theta} \\ \sigma_{\rho\varphi} \\ \sigma_{\theta\varphi} \end{pmatrix}
=
\begin{pmatrix}
\lambda\frac{1}{\rho} + (\lambda+2\mu)\partial_\rho & \lambda\frac{1}{\rho}\partial_\varphi & 0 \\
\lambda\left(\frac{1}{\rho} + \partial_\rho\right) & \lambda\frac{1}{\rho}\partial_\varphi & 0 \\
(\lambda+2\mu)\frac{1}{\rho} + \lambda\partial_\rho & (\lambda+2\mu)\frac{1}{\rho}\partial_\varphi & 0 \\
0 & 0 & \mu\partial_\rho \\
\mu\frac{1}{\rho}\partial_\varphi & \mu\left(-\frac{1}{\rho} + \partial_\rho\right) & 0 \\
0 & 0 & \mu\frac{1}{\rho}\partial_\varphi
\end{pmatrix}
\begin{pmatrix} u_\rho \\ u_\varphi \\ u_\theta \end{pmatrix}
$$

$$
+\frac{1}{R(1+\frac{\rho}{R}\cos\varphi)}
\begin{pmatrix}
\lambda\cos\varphi & -\lambda\sin\varphi & 0 \\
(\lambda+2\mu)\cos\varphi & -(\lambda+2\mu)\sin\varphi & 0 \\
\lambda\cos\varphi & -\lambda\sin\varphi & 0 \\
0 & 0 & -\mu\cos\varphi \\
0 & 0 & 0 \\
0 & 0 & \mu\sin\varphi
\end{pmatrix}
\begin{pmatrix} u_\rho \\ u_\varphi \\ u_\theta \end{pmatrix}
$$

$$
+\frac{1}{R(1+\frac{\rho}{R}\cos\varphi)}
\begin{pmatrix}
0 & 0 & \lambda \\
0 & 0 & \lambda+2\mu \\
0 & 0 & \lambda \\
\mu & 0 & 0 \\
0 & 0 & 0 \\
0 & \mu & 0
\end{pmatrix}
\partial_\theta
\begin{pmatrix} u_\rho \\ u_\varphi \\ u_\theta \end{pmatrix}
$$

The elasticity system in the vicinity of a circular edge

The Navier-Lamè system in the terms of $u_\rho, u_\varphi, u_\theta$ is multiplied by $r^2\left(\frac{\rho}{R}\right)^2$

$$
\begin{aligned}
0 \;=\; & \left(1+\frac{\rho}{R}\cos\varphi\right)^2 \left\{(\lambda+2\mu)\left[(\rho\partial_\rho)^2-1\right]u_\rho + \mu\,\partial_{\varphi\varphi}u_\rho - (\lambda+3\mu)\partial_\varphi u_\varphi + (\lambda+\mu)\partial_{\rho\varphi}u_\varphi\right\} \\
& + \left(1+\frac{\rho}{R}\cos\varphi\right)\frac{\rho}{R}\left[(\lambda+2\mu)\cos\varphi\,\rho\,\partial_\rho u_\rho - (\lambda+\mu)\sin\varphi\,\rho\,\partial_\rho u_\varphi + \mu\sin\varphi\,(u_\varphi - \partial_\varphi u_\rho)\right. \\
& \left. +(\lambda+\mu)\rho\,\partial_{\rho\theta}u_\theta\right] \\
& + \left(\frac{\rho}{R}\right)^2\left[(\lambda+2\mu)\cos\varphi(u_\varphi\sin\varphi - u_\rho\cos\varphi) + \mu\partial_{\theta\theta}u_\rho - (\lambda+3\mu)\cos\varphi\,\partial_\theta u_\theta\right]
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\; & \left(1+\frac{\rho}{R}\cos\varphi\right)^2 \left\{\mu\left[(\rho\partial_\rho)^2-1\right]u_\varphi + (\lambda+2\mu)\partial_{\varphi\varphi}u_\varphi + (\lambda+3\mu)\partial_\varphi u_\rho + (\lambda+\mu)\rho\,\partial_{\rho\varphi}u_\rho\right\} \\
& + \left(1+\frac{\rho}{R}\cos\varphi\right)\frac{\rho}{R}\left[(\lambda+\mu)\cos\varphi\,(\partial_\varphi u_\rho - u_\varphi) + \mu\cos\varphi\,\rho\partial_\rho u_\varphi\right. \\
& \left. -(\lambda+2\mu)\sin\varphi\,(\partial_\varphi u_\varphi + u_\rho) + (\lambda+\mu)\partial_{\varphi\theta}u_\theta\right] \\
& + \left(\frac{\rho}{R}\right)^2\left[(\lambda+2\mu)\sin\varphi(u_\rho\cos\varphi - u_\varphi\sin\varphi) + \mu\partial_{\theta\theta}u_\varphi + (\lambda+3\mu)\sin\varphi\partial_\theta u_\theta\right]
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\; & \left(1+\frac{\rho}{R}\cos\varphi\right)^2 \mu\left[(\rho\partial_\rho)^2 + \partial_{\varphi\varphi}\right]u_\theta \\
& + \left(1+\frac{\rho}{R}\cos\varphi\right)\frac{\rho}{R}\left[\mu\left(\cos\varphi\,\rho\partial_\rho u_\theta - \sin\varphi\,\partial_\varphi u_\theta\right) + (\lambda+\mu)\left(\partial_\theta u_\rho + \partial_{\varphi\theta}u_\varphi + \rho\partial_{\rho\theta}u_\rho\right)\right] \\
& + \left(\frac{\rho}{R}\right)^2\left[-\mu u_\theta + (\lambda+2\mu)\partial_{\theta\theta}u_\theta + (\lambda+3\mu)\left(\cos\varphi\,\partial_\theta u_\rho - \sin\varphi\,\partial_\theta u_\varphi\right)\right]
\end{aligned}
$$

The elasticity system in the vicinity of a circular edge

The Navier-Lamè equations are complemented by homogeneous boundary conditions on the faces intersecting at the singular edge:

$$\begin{aligned} u_\rho = u_\varphi = u_\theta = 0 \qquad & \text{on } \Gamma_1 \cup \Gamma_2 \qquad \text{Clamped BCs} \\ t_\varphi = t_\rho = t_\theta = 0 \qquad & \text{on } \Gamma_1 \cup \Gamma_2 \qquad \text{Traction-Free BCs} \end{aligned}$$

Similar to the Laplace equation, we herein consider a series expansion of the form:

$$\vec{u} = \sum_{\ell=0}^{\infty} \sum_{k\geq 0} \partial_\theta^\ell A_k(\theta)\, \rho^{\alpha_k} \sum_{i=0}^{\infty} \left(\frac{\rho}{R}\right)^{i+\ell} \begin{Bmatrix} \phi^\rho(\varphi) \\ \phi^\varphi(\varphi) \\ \phi^\theta(\varphi) \end{Bmatrix}_{\ell,k,i} = \sum_{\ell=0}^{\infty} \sum_{k\geq 0} \partial_\theta^\ell A_k(\theta)\, \rho^{\alpha_k} \sum_{i=0}^{\infty} \left(\frac{\rho}{R}\right)^{i+\ell} \vec{\phi}_{\ell,k,i}$$

Substituting the series expansion into the Navier-Lamè system results in a “messy” system of different orders of $\rho / R, \partial_\theta^\ell A_k(\theta)$.

The elasticity system in the vicinity of a circular edge

With either clamped or traction free BCs on Γ_1 and Γ_2:

Clamped BCs:

$$\boxed{\vec{\phi}_{\ell,k,i}(\varphi = \varphi_1, \varphi_2) = \mathbf{0}}$$

Traction free BCs:

$$[t_0]\vec{\phi}_{\ell,k,i} = -\left(\cos\varphi[t_0] + [t_{01}]\right)\vec{\phi}_{\ell,k,i-1} - [t_1]\vec{\phi}_{\ell-1,k,i}, \qquad \varphi = \varphi_1, \varphi_2$$

$$[t_0]\vec{\phi}_{\ell,k,i} = \begin{pmatrix} 2\mu + \lambda(\beta+1) & (\lambda+2\mu)\partial_\varphi & 0 \\ \mu\partial_\varphi & \mu(\beta-1) & 0 \\ 0 & 0 & \mu\partial_\varphi \end{pmatrix}\vec{\phi}_{\ell,k,i}$$

$$[t_{01}]\vec{\phi}_{\ell,k,i} = \begin{pmatrix} \lambda\cos\varphi & -\lambda\sin\varphi & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mu\sin\varphi \end{pmatrix}\vec{\phi}_{\ell,k,i}, \qquad [t_1]\vec{\phi}_{\ell,k,i} = \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 0 \\ 0 & \mu & 0 \end{pmatrix}\vec{\phi}_{\ell,k,i}$$

with $\boldsymbol{\phi}_{\ell,k,i} = \mathbf{0} \forall \ell, i < 0$

The elasticity system in the vicinity of a circular edge

$$[m_0]\vec{\phi}_{\ell,k,i} = -\left(2\cos\varphi[m_0] + [m_{01}]\right)\vec{\phi}_{\ell,k,i-1} - \left(\cos^2\varphi[m_0] + \cos\varphi[m_{01}] + [m_{02}]\right)\vec{\phi}_{\ell,k,i-2}$$
$$-[m_{10}]\vec{\phi}_{\ell-1,k,i} - \left(\cos\varphi[m_{10}] + [m_{11}]\right)\vec{\phi}_{\ell-1,k,i-1} - [m_2]\vec{\phi}_{\ell-2,k,i}, \qquad \ell \geq 0,\ i \geq 0$$

$$[m_0]\vec{\phi}_{\ell,k,i} = \begin{pmatrix} (\lambda+2\mu)\left(\beta^2-1\right) + \mu\partial_{\varphi\varphi} & \left((\lambda+\mu)\beta - (\lambda+3\mu)\right)\partial_\varphi & 0 \\ \left((\lambda+\mu)\beta + (\lambda+3\mu)\right)\partial_\varphi & \mu\left(\beta^2-1\right) + (\lambda+2\mu)\partial_{\varphi\varphi} & 0 \\ 0 & 0 & \mu\left(\beta^2 + \partial_{\varphi\varphi}\right) \end{pmatrix}\vec{\phi}_{\ell,k,i}$$

$$[m_{01}]\vec{\phi}_{\ell,k,i} = \begin{pmatrix} (\lambda+2\mu)\cos\varphi\beta - \mu\sin\varphi\partial_\varphi & \sin\varphi\left(\mu - (\lambda+\mu)\beta\right) & 0 \\ -(\lambda+2\mu)\sin\varphi + (\lambda+\mu)\cos\varphi\partial_\varphi & \cos\varphi\left(\mu\left(\beta-1\right) - \lambda\right) - (\lambda+2\mu)\sin\varphi\partial_\varphi & 0 \\ 0 & 0 & \mu\left(\beta\cos\varphi - \sin\varphi\partial_\varphi\right) \end{pmatrix}\vec{\phi}_{\ell,k,i}$$

$$[m_{02}]\vec{\phi}_{\ell,k,i} = \begin{pmatrix} -(\lambda+2\mu)\cos^2\varphi & (\lambda+2\mu)\cos\varphi\sin\varphi & 0 \\ (\lambda+2\mu)\sin\varphi\cos\varphi & -(\lambda+2\mu)\sin^2\varphi & 0 \\ 0 & 0 & -\mu \end{pmatrix}\vec{\phi}_{\ell,k,i} \quad [m_{10}]\vec{\phi}_{\ell,k,i} = \begin{pmatrix} 0 & 0 & (\lambda+\mu)\beta \\ 0 & 0 & (\lambda+\mu)\partial_\varphi \\ (\lambda+\mu)(\beta+1) & (\lambda+\mu)\partial_\varphi & 0 \end{pmatrix}\vec{\phi}_{\ell,k,i}$$

$$[m_{11}]\vec{\phi}_{\ell,k,i} = \begin{pmatrix} 0 & 0 & -(\lambda+3\mu)\cos\varphi \\ 0 & 0 & (\lambda+3\mu)\sin\varphi \\ (\lambda+3\mu)\cos\varphi & -(\lambda+3\mu)\sin\varphi & 0 \end{pmatrix}\vec{\phi}_{\ell,k,i} \qquad [m_2]\vec{\phi}_{\ell,k,i} = \begin{pmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & (\lambda+2\mu) \end{pmatrix}\vec{\phi}_{\ell,k,i}$$

where: $\beta = (\alpha_k + \ell + i).$

The elasticity system: penny shaped crack with traction free BCs

$$
\begin{aligned}
\begin{Bmatrix} u^\rho \\ u^\varphi \\ u^\theta \end{Bmatrix} = \; & A_1(\theta)\rho^{\frac{1}{2}} \Bigg[\begin{pmatrix} -\frac{\lambda+5\mu}{\lambda+\mu}\cos\frac{\varphi}{2} + \cos\frac{3\varphi}{2} \\ \frac{3\lambda+7\mu}{\lambda+\mu}\sin\frac{\varphi}{2} - \sin\frac{3\varphi}{2} \\ 0 \end{pmatrix} + \left(\frac{\rho}{R}\right) \begin{pmatrix} \frac{3\lambda^2+22\lambda\mu-13\mu^2}{12(\lambda+\mu)^2}\cos\frac{\varphi}{2} + \left(\frac{3}{4} - \frac{\lambda}{\lambda+\mu}\right)\cos\frac{3\varphi}{2} \\ \frac{9\lambda^2-14\lambda\mu-55\mu^2}{12(\lambda+\mu)^2}\sin\frac{\varphi}{2} + \left(-\frac{5}{4} + \frac{\lambda}{\lambda+\mu}\right)\sin\frac{3\varphi}{2} \\ 0 \end{pmatrix} \\
& + \left(\frac{\rho}{R}\right)^2 \begin{pmatrix} -\frac{3\lambda^2+134\lambda\mu+67\mu^2}{96(\lambda+\mu)^2}\cos\frac{\varphi}{2} - \frac{45\lambda^3+255\lambda^2\mu+199\lambda\mu^2+53\mu^3}{180(\lambda+\mu)^3}\cos\frac{3\varphi}{2} + \frac{3(3\lambda-\mu)}{32(\lambda+\mu)}\cos\frac{5\varphi}{2} \\ \frac{69\lambda^2+170\lambda\mu+37\mu^2}{96(\lambda+\mu)^2}\sin\frac{\varphi}{2} + \frac{-15\lambda^3+155\lambda^2\mu+371\lambda\mu^2+137\mu^3}{180(\lambda+\mu)^3}\sin\frac{3\varphi}{2} - \frac{3(\lambda-3\mu)}{32(\lambda+\mu)}\sin\frac{5\varphi}{2} \\ 0 \end{pmatrix} + \cdots \Bigg] + A_1'(\theta)\rho^{\frac{1}{2}} \left(\frac{\rho}{R}\right) \Bigg[\begin{pmatrix} 0 \\ 0 \\ 2\cos\frac{\varphi}{2} - \frac{2(3\lambda+7\mu)}{3(\lambda+\mu)}\cos\frac{3\varphi}{2} \end{pmatrix} \\
& + \left(\frac{\rho}{R}\right) \begin{pmatrix} 0 \\ 0 \\ \frac{3\lambda+23\mu}{6(\lambda+\mu)}\cos\frac{\varphi}{2} - \frac{3}{2}\cos\frac{3\varphi}{2} + \frac{15\lambda^2+34\lambda\mu+3\mu^2}{15(\lambda+\mu)^2}\cos\frac{5\varphi}{2} \end{pmatrix} + \cdots \Bigg] + A_1''(\theta)\rho^{\frac{1}{2}} \left(\frac{\rho}{R}\right)^2 \begin{pmatrix} -\frac{3\lambda-\mu}{6(\lambda+\mu)}\cos\frac{\varphi}{2} + \frac{45\lambda^2+138\lambda\mu+61\mu^2}{90(\lambda+\mu)^2}\cos\frac{3\varphi}{2} \\ -\frac{3\lambda+7\mu}{6(\lambda+\mu)}\sin\frac{\varphi}{2} + \frac{15\lambda^2-2\lambda\mu-49\mu^2}{90(\lambda+\mu)^2}\sin\frac{3\varphi}{2} \\ 0 \end{pmatrix} + \cdots \\
& + A_2(\theta)\rho^{\frac{1}{2}} \Bigg[\begin{pmatrix} -\frac{\lambda+5\mu}{3(\lambda+\mu)}\sin\frac{\varphi}{2} + \sin\frac{3\varphi}{2} \\ -\frac{3\lambda+7\mu}{3(\lambda+\mu)}\cos\frac{\varphi}{2} + \cos\frac{3\varphi}{2} \\ 0 \end{pmatrix} + \left(\frac{\rho}{R}\right) \begin{pmatrix} -\frac{123\lambda^2+262\lambda\mu+107\mu^2}{180(\lambda+\mu)^2}\sin\frac{\varphi}{2} - \frac{\lambda-3\mu}{12(\lambda+\mu)}\sin\frac{3\varphi}{2} \\ -\frac{15\lambda^2+62\lambda\mu+79\mu^2}{180(\lambda+\mu)^2}\cos\frac{\varphi}{2} + \frac{(\lambda+5\mu)}{12(\lambda+\mu)}\cos\frac{3\varphi}{2} \\ 0 \end{pmatrix} \\
& + \left(\frac{\rho}{R}\right)^2 \times \begin{pmatrix} -\frac{483\lambda^2+1350\lambda\mu+803\mu^2}{1440(\lambda+\mu)^2}\sin\frac{\varphi}{2} + \frac{2625\lambda^3+8779\lambda^2\mu+9251\lambda\mu^2+3161\mu^3}{6300(\lambda+\mu)^3}\sin\frac{3\varphi}{2} + \frac{3\lambda-\mu}{32(\lambda+\mu)}\sin\frac{5\varphi}{2} \\ -\frac{357\lambda^2+1002\lambda\mu+581\mu^2}{1440(\lambda+\mu)^2}\cos\frac{\varphi}{2} + \frac{1365\lambda^3+5271\lambda^2\mu+6719\lambda\mu^2+2749\mu^3}{6300(\lambda+\mu)^3}\cos\frac{3\varphi}{2} + \frac{\lambda-3\mu}{32(\lambda+\mu)}\cos\frac{5\varphi}{2} \\ 0 \end{pmatrix} + \cdots \Bigg] + A_2'(\theta)\rho^{\frac{1}{2}} \left(\frac{\rho}{R}\right) \Bigg[\begin{pmatrix} 0 \\ 0 \\ \frac{2}{3}\sin\frac{\varphi}{2} \end{pmatrix} \\
& + \left(\frac{\rho}{R}\right) \begin{pmatrix} 0 \\ 0 \\ -\frac{8(\lambda+2\mu)}{15(\lambda+\mu)}\sin\frac{\varphi}{2} - \frac{1}{2}\sin\frac{3\varphi}{2} \end{pmatrix} + \cdots \Bigg] + A_2''(\theta)\rho^{\frac{1}{2}} \left(\frac{\rho}{R}\right)^2 \begin{pmatrix} \left(\frac{1}{18} - \frac{2\lambda}{9(\lambda+\mu)}\right)\sin\frac{\varphi}{2} - \frac{45\lambda^2+90\lambda\mu+77\mu^2}{630(\lambda+\mu)^2}\sin\frac{3\varphi}{2} \\ \left(\frac{7}{18} - \frac{2\lambda}{9(\lambda+\mu)}\right)\cos\frac{\varphi}{2} + \frac{-105\lambda^2-130\lambda\mu+7\mu^2}{630(\lambda+\mu)^2}\cos\frac{3\varphi}{2} \\ 0 \end{pmatrix} + \cdots \\
& + A_3(\theta)\rho^{\frac{1}{2}} \Bigg[\begin{pmatrix} 0 \\ 0 \\ \sin\frac{\varphi}{2} \end{pmatrix} + \left(\frac{\rho}{R}\right) \begin{pmatrix} 0 \\ 0 \\ \frac{1}{4}\sin\frac{\varphi}{2} \end{pmatrix} + \left(\frac{\rho}{R}\right)^2 \begin{pmatrix} 0 \\ 0 \\ \frac{1}{4}\sin\frac{\varphi}{2} - \frac{3}{32}\sin\frac{3\varphi}{2} \end{pmatrix} + \cdots \Bigg] + A_3'(\theta)\rho^{\frac{1}{2}} \left(\frac{\rho}{R}\right) \Bigg[\begin{pmatrix} -\frac{2(3\lambda+\mu)}{15(\lambda+\mu)}\sin\frac{\varphi}{2} \\ -\frac{4\mu}{15(\lambda+\mu)}\cos\frac{\varphi}{2} \\ 0 \end{pmatrix} \\
& + \left(\frac{\rho}{R}\right) \begin{pmatrix} \left(-\frac{7}{30} - \frac{\lambda}{15(\lambda+\mu)}\right)\sin\frac{\varphi}{2} + \frac{2(75\lambda^2+124\lambda\mu+53\mu^2)}{525(\lambda+\mu)^2}\sin\frac{3\varphi}{2} \\ -\frac{3\lambda+2\mu}{15(\lambda+\mu)}\cos\frac{\varphi}{2} + \frac{105\lambda^2+242\lambda\mu+129\mu^2}{525(\lambda+\mu)^2}\cos\frac{3\varphi}{2} \\ 0 \end{pmatrix} + \cdots \Bigg] + A_3''(\theta)\rho^{\frac{1}{2}} \left(\frac{\rho}{R}\right)^2 \begin{pmatrix} 0 \\ 0 \\ -\frac{3}{10}\sin\frac{\varphi}{2} \end{pmatrix} + \cdots
\end{aligned}
$$

The elasticity system: penny shaped crack with traction free BCs

$$
\begin{Bmatrix} \sigma_{\rho\rho} \\ \sigma_{\theta\theta} \\ \sigma_{\varphi\varphi} \\ \sigma_{\rho\theta} \\ \sigma_{\rho\varphi} \\ \sigma_{\theta\varphi} \end{Bmatrix}
=
K_{0_1}(\theta)
\begin{pmatrix} -1-\cos 2\varphi \\ 2 \\ -1+\cos 2\varphi \\ 0 \\ \sin 2\varphi \\ 0 \end{pmatrix}
\left[\left(\frac{\rho}{R}\right) - 2\cos\varphi\left(\frac{\rho}{R}\right)^2 + \cdots\right]
+ K'_{0_1}(\theta)\left(\frac{\rho}{R}\right)
\left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ \cos\varphi \\ 0 \\ -\sin\varphi \end{pmatrix} + \left(\frac{\rho}{R}\right)\begin{pmatrix} 0 \\ 0 \\ 0 \\ -4\cos^2\varphi \\ 0 \\ 2\sin 2\varphi \end{pmatrix} + \cdots\right]
$$

$$
+ K''_{0_1}(\theta)\left(\frac{\rho}{R}\right)^2
\left[\begin{pmatrix} -\frac{2}{3}\cos\varphi \\ -\frac{\lambda}{3(\lambda+\mu)}\cos\varphi \\ 0 \\ 0 \\ \frac{1}{3}\sin\varphi \\ 0 \end{pmatrix} + \cdots\right]
+ K'_{0_2}(\theta)\left(\frac{\rho}{R}\right)
\left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}\left(\frac{\rho}{R}\right)\right]
+ K''_{0_2}(\theta)\left(\frac{\rho}{R}\right)^2
\left[\begin{pmatrix} 0 \\ -\frac{3\lambda+2\mu}{\lambda+\mu}\sin\varphi \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \cdots\right]
$$

$$
+ K'_{0_3}(\theta)\left(\frac{\rho}{R}\right)
\left[\begin{pmatrix} 0 \\ \frac{3\lambda+2\mu}{\lambda+\mu} \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{\lambda^2-5\lambda\mu-4\mu^2}{3\mu(\lambda+\mu)}\cos\varphi \\ -\frac{\lambda\left(4\lambda^2+7\lambda\mu+2\mu^2\right)}{6\mu(\lambda+\mu)^2}\cos\varphi \\ -\frac{\lambda}{\mu}\cos[\varphi] \\ 0 \\ -\frac{\lambda^2+4\lambda\mu+2\mu^2}{3\mu(\lambda+\mu)}\sin\varphi \\ 0 \end{pmatrix}\left(\frac{\rho}{R}\right) + \cdots\right]
+ K''_{0_3}(\theta)\left(\frac{\rho}{R}\right)^2
\left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1-\frac{\lambda}{2(\lambda+\mu)} \\ 0 \\ 0 \end{pmatrix} + \cdots\right]
$$

$$
- \frac{1}{4}\frac{K_I(\theta)}{\sqrt{2\pi\rho}}
\left[\begin{pmatrix} -5\cos\frac{\varphi}{2} + \cos\frac{3\varphi}{2} \\ -\frac{4\lambda}{\lambda+\mu}\cos\frac{\varphi}{2} \\ -3\cos\frac{\varphi}{2} - \cos\frac{3\varphi}{2} \\ 0 \\ -\sin\frac{\varphi}{2} - \sin\frac{3\varphi}{2} \\ 0 \end{pmatrix} + \left(\frac{\rho}{R}\right)\begin{pmatrix} -\frac{5\lambda+13\mu}{4(\lambda+\mu)}\cos\frac{\varphi}{2} + \frac{\lambda+9\mu}{4(\lambda+\mu)}\cos\frac{3\varphi}{2} \\ -\frac{2(2\lambda+\mu)(\lambda+5\mu)}{(\lambda+\mu)^2}\cos\frac{\varphi}{2} + \frac{3\lambda+2\mu}{\lambda+\mu}\cos\frac{3\varphi}{2} \\ -\frac{3(\lambda+9\mu)}{4(\lambda+\mu)}\cos\frac{\varphi}{2} - \frac{\lambda+9\mu}{4(\lambda+\mu)}\cos\frac{3\varphi}{2} \\ 0 \\ \frac{\lambda-7\mu}{4(\lambda+\mu)}\sin\frac{\varphi}{2} + \frac{\lambda-7\mu}{4(\lambda+\mu)}\sin\frac{3\varphi}{2} \\ 0 \end{pmatrix}\right.
$$

$$
\left. + \left(\frac{\rho}{R}\right)^2\begin{pmatrix} -\frac{-33\lambda^2+430\lambda\mu+335\mu^2}{96(\lambda+\mu)^2}\cos\frac{\varphi}{2} - \frac{21\lambda^2+154\lambda\mu+53\mu^2}{36(\lambda+\mu)^2}\cos\frac{3\varphi}{2} + \frac{311\lambda-5\mu}{32(\lambda+\mu)}\cos\frac{5\varphi}{2} \\ \frac{\mu(23\lambda+19\mu)}{2(\lambda+\mu)^2}\cos\frac{\varphi}{2} - \frac{2(4\lambda+3\mu)\left(-3\lambda^2-10\lambda\mu+\mu^2\right)}{9(\lambda+\mu)^3}\cos\frac{3\varphi}{2} - \frac{3(5\lambda+4\mu)}{8(\lambda+\mu)}\cos\frac{5\varphi}{2} \\ \frac{111\lambda^2+142\lambda\mu-97\mu^2}{96(\lambda+\mu)^2}\cos\frac{\varphi}{2} + \frac{-3\lambda^2+74\lambda\mu+61\mu^2}{36(\lambda+\mu)^2}\cos\frac{3\varphi}{2} - \frac{3(3\lambda-13\mu)}{32(\lambda+\mu)}\cos\frac{5\varphi}{2} \\ 0 \\ \frac{105\lambda^2+322\lambda\mu+89\mu^2}{96(\lambda+\mu)^2}\sin\frac{\varphi}{2} + \frac{3\lambda^2+38\lambda\mu+19\mu^2}{12(\lambda+\mu)^2}\sin\frac{3\varphi}{2} + \frac{3(-9\lambda+7\mu)}{32(\lambda+\mu)}\sin\frac{5\varphi}{2} \\ 0 \end{pmatrix} + \cdots\right]
$$

$$
- \frac{1}{4}\frac{K'_I(\theta)}{\sqrt{2\pi\rho}}\left(\frac{\rho}{R}\right)
\left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{2(\lambda-\mu)}{\lambda+\mu}\cos\frac{\varphi}{2} - \frac{2(\lambda+3\mu)}{\lambda+\mu}\cos\frac{3\varphi}{2} \\ 0 \\ \frac{2(\lambda+3\mu)}{\lambda+\mu}\sin\frac{\varphi}{2} + \frac{2(\lambda+3\mu)}{\lambda+\mu}\sin\frac{3\varphi}{2} \end{pmatrix} + \left(\frac{\rho}{R}\right)\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{9\lambda^2+96\lambda\mu+71\mu^2}{6(\lambda+\mu)^2}\cos\frac{\varphi}{2} - \frac{3(3\lambda+\mu)}{2(\lambda+\mu)}\cos\frac{3\varphi}{2} + \frac{9\lambda^2+24\lambda\mu+7\mu^2}{3(\lambda+\mu)^2}\cos\frac{5\varphi}{2} \\ 0 \\ \frac{(9\lambda+\mu)(3\lambda+5\mu)}{6(\lambda+\mu)^2}\sin\frac{\varphi}{2} + \frac{3(\lambda-\mu)}{2(\lambda+\mu)}\sin\frac{3\varphi}{2} - \frac{9\lambda^2+24\lambda\mu+7\mu^2}{3(\lambda+\mu)^2}\sin\frac{5\varphi}{2} \end{pmatrix} + \cdots\right]
$$

The elasticity system: penny shaped crack with traction free BCs

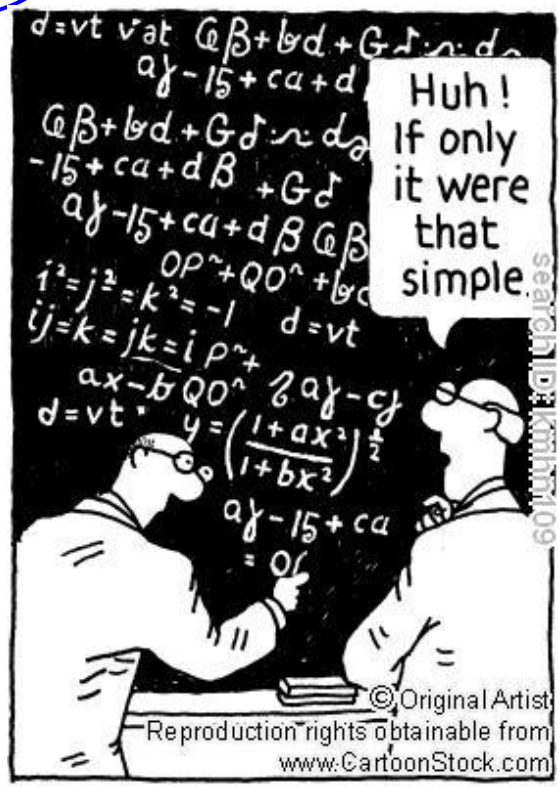
$$
\begin{aligned}
&- \frac{1}{4}\frac{K_I''(\theta)}{\sqrt{2\pi\rho}}\left(\frac{\rho}{R}\right)^2 \begin{pmatrix} \frac{-3\lambda+5\mu}{6(\lambda+\mu)}\cos\frac{\varphi}{2} + \frac{21\lambda+61\mu}{18(\lambda+\mu)}\cos\frac{3\varphi}{2} \\ \frac{2(3\lambda+2\mu)}{\lambda+\mu}\cos\frac{\varphi}{2} - \frac{4(4\lambda+3\mu)(3\lambda+7\mu)}{9(\lambda+\mu)^2}\cos\frac{3\varphi}{2} \\ \frac{3\lambda-5\mu}{6(\lambda+\mu)}\cos\frac{\varphi}{2} + \frac{3\lambda-5\mu}{18(\lambda+\mu)}\cos\frac{3\varphi}{2} \\ 0 \\ -\frac{3\lambda+11\mu}{6(\lambda+\mu)}\sin\frac{\varphi}{2} - \frac{3\lambda+11\mu}{6(\lambda+\mu)}\sin\frac{3\varphi}{2} \\ 0 \end{pmatrix} + \cdots \\
&+ \frac{3}{4}\frac{K_{II}(\theta)}{\sqrt{2\pi\rho}}\left[\begin{pmatrix} -\frac{5}{3}\sin\frac{\varphi}{2} + \sin\frac{3\varphi}{2} \\ -\frac{4\lambda}{3(\lambda+\mu)}\sin\frac{\varphi}{2} \\ -\sin\frac{\varphi}{2} - \sin\frac{3\varphi}{2} \\ 0 \\ \frac{1}{3}\left(\cos\frac{\varphi}{2} + 3\cos\frac{3\varphi}{2}\right) \\ 0 \end{pmatrix} + \left(\frac{\rho}{R}\right)\begin{pmatrix} -\frac{51\lambda+107\mu}{60(\lambda+\mu)}\sin\frac{\varphi}{2} + \frac{\lambda+9\mu}{12(\lambda+\mu)}\sin\frac{3\varphi}{2} \\ \frac{2\left(34\lambda^2+83\lambda\mu+45\mu^2\right)}{15(\lambda+\mu)^2}\sin\frac{\varphi}{2} + \frac{3\lambda+2\mu}{3(\lambda+\mu)}\sin\frac{3\varphi}{2} \\ -\frac{\lambda+9\mu}{12(\lambda+\mu)}\sin\frac{\varphi}{2} - \frac{\lambda+9\mu}{12(\lambda+\mu)}\sin\frac{3\varphi}{2} \\ 0 \\ -\frac{23\lambda+31\mu}{60(\lambda+\mu)}cos\frac{\varphi}{2} + \frac{-\lambda+7\mu}{12(\lambda+\mu)}\cos\frac{3\varphi}{2} \\ 0 \end{pmatrix} + \left(\frac{\rho}{R}\right)^2\begin{pmatrix} -\frac{1983\lambda^2+6126\lambda\mu+4015\mu^2}{1440(\lambda+\mu)^2}\sin\frac{\varphi}{2} + \frac{2121\lambda^2+5490\lambda\mu+3161\mu^2}{1260(\lambda+\mu)^2}\sin\frac{3\varphi}{2} + \frac{11\lambda-5\mu}{32(\lambda+\mu)}\sin\frac{5\varphi}{2} \\ \frac{72\lambda^2+207\lambda\mu+131\mu^2}{30(\lambda+\mu)^2}\sin\frac{\varphi}{2} - \frac{2\left(420\lambda^3+1413\lambda^2\mu+1482\lambda\mu^2+497\mu^3\right)}{315(\lambda+\mu)^3}\sin\frac{3\varphi}{2} - \frac{5\lambda+4\mu}{8(\lambda+\mu)}\sin\frac{5\varphi}{2} \\ -\frac{177\lambda^2+1074\lambda\mu+1025\mu^2}{1440(\lambda+\mu)^2}\sin\frac{\varphi}{2} - \frac{39\lambda^2+78\lambda\mu+55\mu^2}{180(\lambda+\mu)^2}\sin\frac{3\varphi}{2} - \frac{3\lambda-13\mu}{32(\lambda+\mu)}\sin\frac{5\varphi}{2} \\ 0 \\ -\frac{777\lambda^2+2178\lambda\mu+1273\mu^2}{1440(\lambda+\mu)^2}\cos\frac{\varphi}{2} + \frac{399\lambda^2+1006\lambda\mu+591\mu^2}{420(\lambda+\mu)^2}\cos\frac{3\varphi}{2} + \frac{9\lambda-7\mu}{32(\lambda+\mu)}\cos\frac{5\varphi}{2} \end{pmatrix} + \cdots\right] \\
&+ \frac{3}{4}\frac{K_{II}'(\theta)}{\sqrt{2\pi\rho}}\left(\frac{\rho}{R}\right)\left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{2(\lambda-\mu)}{3(\lambda+\mu)}\sin\frac{\varphi}{2} + \sin\frac{3\varphi}{2} \\ 0 \\ -\frac{2(\lambda+3\mu)}{3(\lambda+\mu)}\cos\frac{\varphi}{2} + \cos\frac{3\varphi}{2} \end{pmatrix} + \left(\frac{\rho}{R}\right)\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{423\lambda^2+1222\lambda\mu+767\mu^2}{180(\lambda+\mu)^2}\sin\frac{\varphi}{2} - \frac{3\lambda+\mu}{2(\lambda+\mu)}\sin\frac{3\varphi}{2} - \frac{1}{2}\sin\frac{5\varphi}{2} \\ 0 \\ \frac{-3\lambda^2+34\lambda\mu+5\mu^2}{180(\lambda+\mu)^2}\cos\frac{\varphi}{2} + \frac{-\lambda+\mu}{2(\lambda+\mu)}\cos\frac{3\varphi}{2} - \frac{1}{2}\cos\frac{5\varphi}{2} \end{pmatrix} + \cdots\right] + \frac{3}{4}\frac{K_{II}''(\theta)}{\sqrt{2\pi\rho}}\left(\frac{\rho}{R}\right)^2\begin{pmatrix} \frac{-3\lambda+5\mu}{18(\lambda+\mu)}\sin\frac{\varphi}{2}\frac{69\lambda+77\mu}{126(\lambda+\mu)}\sin\frac{3\varphi}{2} \\ \frac{2(3\lambda+2\mu)}{3(\lambda+\mu)}\sin\frac{\varphi}{2} - \frac{4\lambda(3\lambda+7\mu)}{63(\lambda+\mu)^2}\sin\frac{3\varphi}{2} \\ \frac{3\lambda-5\mu}{18(\lambda+\mu)}\sin\frac{\varphi}{2} + \frac{3\lambda-5\mu}{18(\lambda+\mu)}\sin\frac{3\varphi}{2} \\ 0 \\ \frac{3\lambda+11\mu}{18(\lambda+\mu)}\cos\frac{\varphi}{2} - \frac{15\lambda+7\mu}{42(\lambda+\mu)}\cos\frac{3\varphi}{2} \\ 0 \end{pmatrix} + \cdots \\
&+ 2\frac{K_{III}(\theta)}{\sqrt{2\pi\rho}}\left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{2}\sin\frac{\varphi}{2} \\ 0 \\ \frac{1}{2}\cos\frac{\varphi}{2} \end{pmatrix} + \left(\frac{\rho}{R}\right)\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{7}{8}\sin\frac{\varphi}{2} - \frac{1}{2}\sin\frac{3\varphi}{2} \\ 0 \\ \frac{5}{8}\cos\frac{\varphi}{2} - \frac{1}{2}\cos\frac{3\varphi}{2} \end{pmatrix} + \left(\frac{\rho}{R}\right)^2\begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{5}{4}\sin\frac{\varphi}{2} - \frac{39}{64}\sin\frac{3\varphi}{2} + \frac{1}{4}\sin\frac{5\varphi}{2} \\ 0 \\ \frac{1}{4}\cos\frac{\varphi}{2} - \frac{33}{64}\cos\frac{3\varphi}{2} + \frac{1}{4}\cos\frac{5\varphi}{2} \end{pmatrix} + \cdots\right] \\
&+ 2\frac{K_{III}'(\theta)}{\sqrt{2\pi\rho}}\left(\frac{\rho}{R}\right)\left[\begin{pmatrix} \frac{\lambda}{\mu}\sin\frac{\varphi}{2} - \frac{5\lambda+2\mu}{5\mu}\sin\frac{\varphi}{2} \\ -\frac{\lambda(5\lambda+\mu)}{5\mu(\lambda+\mu)}\sin\frac{\varphi}{2} + \frac{(\lambda+2\mu)}{\mu}\sin\frac{\varphi}{2} \\ 0 \\ 0 \\ -\frac{1}{5}\cos\frac{\varphi}{2} \\ 0 \end{pmatrix} + \left(\frac{\rho}{R}\right)\begin{pmatrix} -\frac{39\lambda+35\mu}{30(\lambda+\mu)}\sin\frac{\varphi}{2} + \frac{2(66\lambda+53\mu)}{105(\lambda+\mu)}\sin\frac{3\varphi}{2} \\ \frac{21\lambda+19\mu}{10(\lambda+\mu)}\sin\frac{\varphi}{2} - \frac{(11\lambda+7\mu)(15\lambda+13\mu)}{105(\lambda+\mu)^2}\sin\frac{3\varphi}{2} \\ -\frac{3\lambda+5\mu}{15(\lambda+\mu)}\sin\frac{\varphi}{2} - \frac{3\lambda+5\mu}{15(\lambda+\mu)}\sin\frac{3\varphi}{2} \\ 0 \\ -\frac{27\lambda+19\mu}{60(\lambda+\mu)}\cos\frac{\varphi}{2} + \frac{51\lambda+47\mu}{70(\lambda+\mu)}\cos\frac{3\varphi}{2} \\ 0 \end{pmatrix} + \cdots\right] + 2\frac{K_{III}''(\theta)}{\sqrt{2\pi\rho}}\left(\frac{\rho}{R}\right)^2\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{69\lambda+53\mu}{60(\lambda+\mu)}\sin\frac{\varphi}{2} \\ 0 \\ -\frac{9\lambda+25\mu}{60(\lambda+\mu)}\cos\frac{\varphi}{2} \end{pmatrix} + \cdots
\end{aligned}
$$

Yosibash, Z., Shannon, S., Dauge, M. Costabel, M.: Circular edge singularities for the Laplace equation and the elasticity system in 3-D domains. *Int. Jour. Fracture* 168, 31-52 (2011)

Computation of the Generalized Edge Flux Intensity Functions (GEFIFs) along a circular edge

$$\vec{u} = \sum_{\ell=0}^{\infty} \sum_{k\geq 0} \partial_\theta^\ell A_k(\theta) \rho^{\alpha_k} \sum_{i=0}^{\infty} \left(\frac{\rho}{R}\right)^{i+\ell} \begin{Bmatrix} \phi^\rho(\varphi) \\ \phi^\varphi(\varphi) \\ \phi^\theta(\varphi) \end{Bmatrix}_{\ell,k,i}$$

Note that negative eigenvalues $-\alpha_k$ with corresponding "dual" eigenfunctions and dual shadow $\vec{\psi}(\varphi)_{\ell,k,i}$ satisfy the Navier-Lamè equation also.



Computation of the Generalized Edge Flux Intensity Functions (GEFIFs) along a circular edge

Multiply the solution by a test function K, integrate over the volume and then apply the Green theorem:

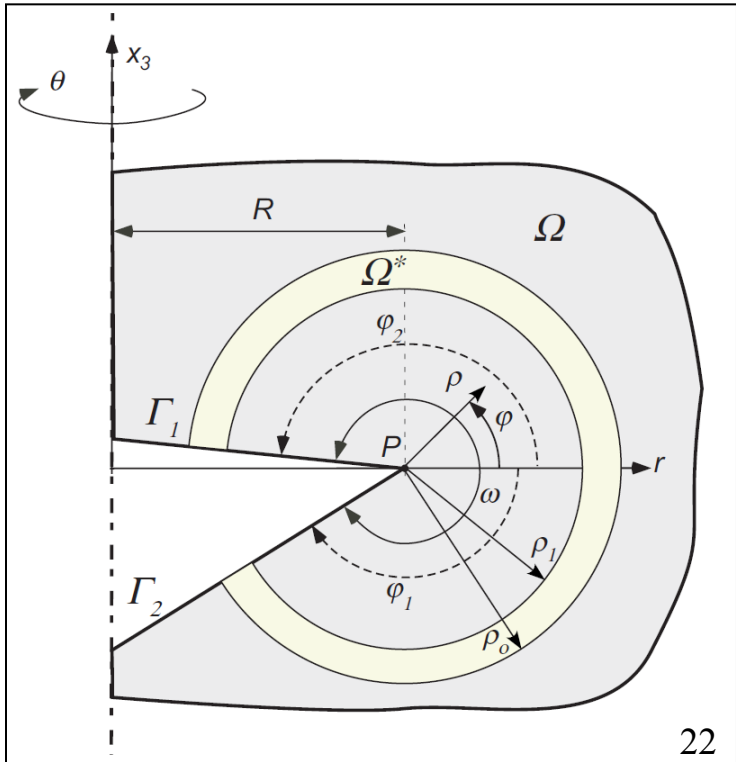
$$0 = \int_{\partial\Omega^*} (\boldsymbol{K}\cdot([T]\boldsymbol{u}) - ([T]\boldsymbol{K})\cdot\boldsymbol{u})\,d\Gamma + \int_{\Omega^*} \cancelto{0}{\boldsymbol{K}\cdot(\operatorname{div}\underset{\approx}{\boldsymbol{\sigma}}^{(\boldsymbol{u})})}\,d\Omega^* - \int_{\Omega^*} \left(\operatorname{div}\underset{\approx}{\boldsymbol{\sigma}}^{(\boldsymbol{K})}\right)\cdot\boldsymbol{u}\,d\Omega$$

Based on the (DFM) 2-D method choose $K_{n,m}^{\alpha_j}(\rho,\varphi,\theta)$ as follows:

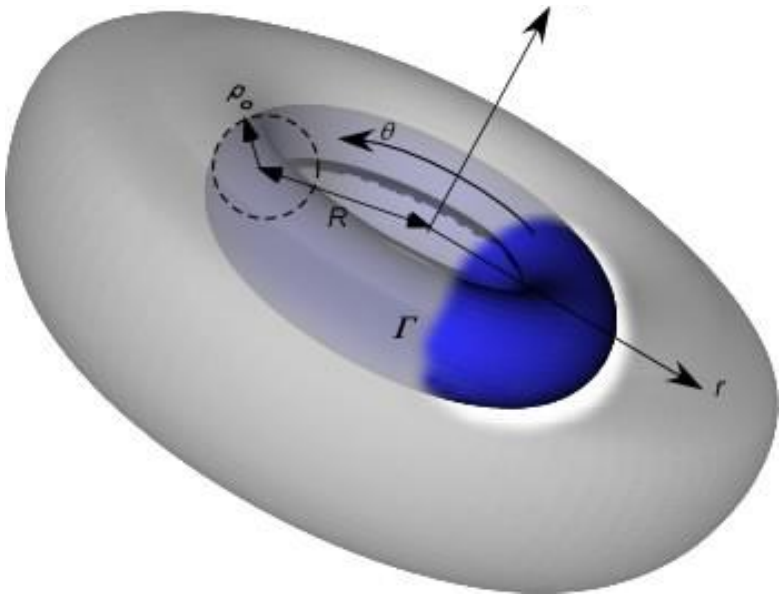
$$\boldsymbol{K}_{n,m}^{\alpha_j} \triangleq \sum_{h=0,1,2\cdots}^{m} \partial_\theta^h B_j(\theta)\rho^{-\alpha_j} \sum_{f=0}^{n} \left(\frac{\rho}{R}\right)^{h+f} \boldsymbol{\psi}_{h,j,f}(\varphi,\alpha_j)$$

$K_{\infty,\infty}^{\alpha_j}$ is a harmonic function which satisfies the elasticity system and homogeneous boundary conditions on the surfaces $\Gamma_1 \cup \Gamma_2$.

$\boldsymbol{\psi}_{h,j,f}$ is obtain by replacing $(\boldsymbol{\phi},\alpha_j)$ by $(\boldsymbol{\psi},-\alpha_j)$ in the ODEs set.



$$YCD_\rho\left[\boldsymbol{u},\boldsymbol{K}_{n,m}^{(\alpha_j)}\right]\stackrel{\text{def}}{=}\int_0^{2\pi}\int_{\varphi_1}^{\varphi_2}\left(\boldsymbol{K}_{n,m}^{(\alpha_j)}\cdot[T]\boldsymbol{u}-[T]\boldsymbol{K}_{n,m}^{(\alpha_j)}\cdot\boldsymbol{u}\right)_\rho(R+\rho\cos\varphi)\rho\,d\varphi d\theta.$$



$$YCD_{\rho_0}\left[\boldsymbol{u},\boldsymbol{K}_{n,m}^{(\alpha_j)}\right]=YCD_{\rho_1}\left[\boldsymbol{u},\boldsymbol{K}_{n,m}^{(\alpha_j)}\right]+\int_{\Omega^*}\operatorname{div}\underset{\approx}{\boldsymbol{\sigma}}^{\boldsymbol{K}_{n,m}^{(\alpha_j)}}\cdot\boldsymbol{u}\,d\Omega.$$

$$YCD_{\rho_0}\left[\boldsymbol{u},\boldsymbol{K}_{n,n}^{(\alpha_j)}\right]=Const+\mathcal{O}\left(\rho_0^{\alpha_0-\alpha_j}\left(\frac{\rho_0}{R}\right)^{1+n}\right)$$

Consider first axi-symmetric case – A_j independent of θ .

Since the ESIFs are θ-independent, so would be the coefficient in the quasi-dual function (one sum on derivatives of ESIFs vanishes):

$$\boldsymbol{u}(\rho,\varphi)=\sum_{k=0} A_k\rho^{\alpha_k}\sum_{i=0}^{\infty}\left(\frac{\rho}{R}\right)^i \boldsymbol{\phi}_{0,k,i}(\varphi) \qquad \boldsymbol{K}_{n,0}^{(\alpha_j)}(\rho,\varphi)\triangleq B_j\rho^{-\alpha_j}\sum_{i=0}^{n}\left(\frac{\rho}{R}\right)^i \boldsymbol{\psi}_{0,j,i}(\varphi)$$

Computing YCD on a torus of small radius ρ →0 :

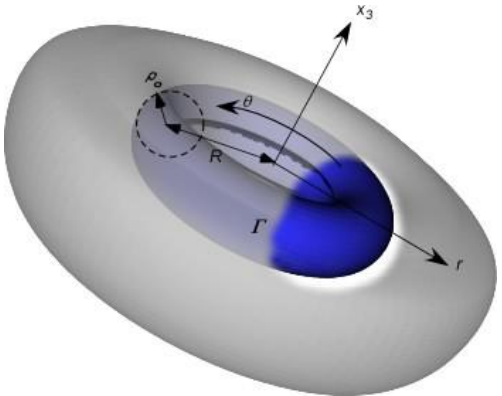
$$YCD_\rho\left[\boldsymbol{u},\boldsymbol{K}_{n,0}^{(\alpha_j)}\right]\overset{\rho\to 0}{=}\begin{cases}\frac{32\pi^2\alpha_j\mu(\lambda+2\mu)R}{(\alpha_j^2-1)(\lambda+\mu)}A_jB_j & j=1,2,7,8,13,14\cdots\\ 4\pi^2\alpha_j\mu R A_j B_j & j=3,9,15\end{cases}$$

Thus we choose:

$$B_j=\frac{(\alpha_j^2-1)(\lambda+\mu)}{32\pi^2\alpha_j\mu(\lambda+2\mu)R}, \qquad j=1,4,7,8,13,14\cdots$$

$$B_j=\frac{1}{4\pi^2\alpha_j\mu R}, \qquad j=3,9,15\cdots$$

$$\boxed{J_{\rho_0}[\boldsymbol{u},\boldsymbol{K}_{n,0}^{\alpha_j}]=A_j+\mathcal{O}\left\{\rho_0^{\alpha_0-\alpha_j}\left(\frac{\rho_0}{R}\right)^{n+1}\right\}}$$



Computation of A_j : axi-symmetric solution

We compute A_j by the QDFM with increasing orders of dual functions:

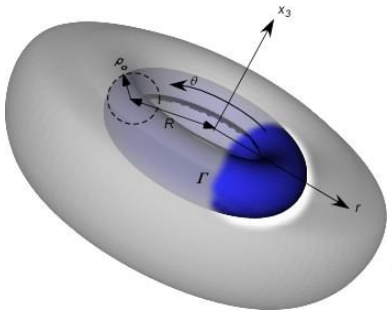
$$
\begin{aligned}
\boldsymbol{K}_{0,0}^{\alpha_j} &= B_j \rho^{-\alpha_j} \left[\boldsymbol{\psi}_{0,j,0}(\varphi)\right] \\
\boldsymbol{K}_{1,0}^{\alpha_j} &= B_j \rho^{-\alpha_j} \left[\boldsymbol{\psi}_{0,j,0}(\varphi) + \left(\frac{\rho}{R}\right) \boldsymbol{\psi}_{0,j,1}(\varphi)\right] \\
\boldsymbol{K}_{2,0}^{\alpha_j} &= B_j \rho^{-\alpha_j} \left[\boldsymbol{\psi}_{0,j,0}(\varphi) + \left(\frac{\rho}{R}\right) \boldsymbol{\psi}_{0,j,1}(\varphi) + \left(\frac{\rho}{R}\right)^2 \boldsymbol{\psi}_{0,j,2}(\varphi)\right] \\
\boldsymbol{K}_{3,0}^{\alpha_j} &= B_j \rho^{-\alpha_j} \left[\boldsymbol{\psi}_{0,j,0}(\varphi) + \left(\frac{\rho}{R}\right) \boldsymbol{\psi}_{0,j,1}(\varphi) + \left(\frac{\rho}{R}\right)^2 \boldsymbol{\psi}_{0,j,2}(\varphi) + \left(\frac{\rho}{R}\right)^3 \boldsymbol{\psi}_{0,j,3}(\varphi)\right]
\end{aligned}
$$

Computation of K_I for a crack

$$
\begin{aligned}
K_{3,0}^{(1/2)} \;=\; & B_1 \rho^{-1/2} \left\{ \begin{pmatrix} \cos\frac{\varphi}{2} - \frac{(3\lambda+7\mu)}{3(\lambda+\mu)}\cos\frac{3\varphi}{2} \\ -\sin\frac{\varphi}{2} + \frac{(\lambda+5\mu)}{3(\lambda+\mu)}\sin\frac{3\varphi}{2} \\ 0 \end{pmatrix} + \left(\frac{\rho}{R}\right) \begin{pmatrix} -\frac{\left(\lambda^2-14\lambda\mu-47\mu^2\right)}{12(\lambda+\mu)^2}\cos\frac{\varphi}{2} + \frac{\lambda+5\mu}{12(\lambda+\mu)}\cos\frac{5\varphi}{2} \\ -\frac{\left(5\lambda^2+42\lambda\mu+69\mu^2\right)}{12(\lambda+\mu)^2}\sin\frac{\varphi}{2} - \frac{-\lambda+3\mu}{12(\lambda+\mu)}\sin\frac{5\varphi}{2} \\ 0 \end{pmatrix} \right. \\
& + \left(\frac{\rho}{R}\right)^2 \begin{pmatrix} -\frac{4(3\lambda-\mu)\mu(\lambda+2\mu)}{9(\lambda+\mu)^3}\cos\frac{\varphi}{2} - \frac{(\lambda+9\mu)(3\lambda+11\mu)}{96(\lambda+\mu)^2}\cos\frac{3\varphi}{2} + \frac{(\lambda-3\mu)}{32(\lambda+\mu)}\cos\frac{7\varphi}{2} \\ \frac{4\mu(\lambda+2\mu)(3\lambda+7\mu)}{9(\lambda+\mu)^3}\sin\frac{\varphi}{2} + \frac{\left(21\lambda^2+106\lambda\mu+149\mu^2\right)}{96(\lambda+\mu)^2}\sin\frac{3\varphi}{2} + \frac{(-3\lambda+\mu)}{32(\lambda+\mu)}\sin\frac{7\varphi}{2} \\ 0 \end{pmatrix} \\
& \left. + \left(\frac{\rho}{R}\right)^3 \begin{pmatrix} \frac{\mu(\lambda+2\mu)(9\lambda+5\mu)}{9(\lambda+\mu)^3}\cos\frac{\varphi}{2} + \frac{2\mu(\lambda+2\mu)\left(45\lambda^2+42\lambda\mu+13\mu^2\right)}{135(\lambda+\mu)^4}\cos\frac{3\varphi}{2} + \frac{5\lambda^2+26\lambda\mu+53\mu^2}{128(\lambda+\mu)^2}\cos\frac{5\varphi}{2} - \frac{5(3\lambda-\mu)}{384(\lambda+\mu)}\cos\frac{9\varphi}{2} \\ -\frac{(3\lambda-\mu)\mu(\lambda+2\mu)}{9(\lambda+\mu)^3}\sin\frac{\varphi}{2} - \frac{2\mu(\lambda+2\mu)\left(45\lambda^2+98\lambda\mu+37\mu^2\right)}{135(\lambda+\mu)^4}\sin\frac{3\varphi}{2} + \frac{-15\lambda^2-62\lambda\mu-79\mu^2}{128(\lambda+\mu)^2}\sin\frac{5\varphi}{2} + \frac{5(5\lambda+\mu)}{384(\lambda+\mu)}\sin\frac{9\varphi}{2} \\ 0 \end{pmatrix} \right\}
\end{aligned}
$$

with: $$B_1 = \frac{\left(\frac{1}{2}^2 - 1\right)(\lambda+\mu)}{32\pi^2 \frac{1}{2}\mu(\lambda+2\mu)R} = -\frac{3(\lambda+\mu)}{64\pi^2 \mu R(\lambda+2\mu)}$$

Computation of A_1 : axisymmetric solution



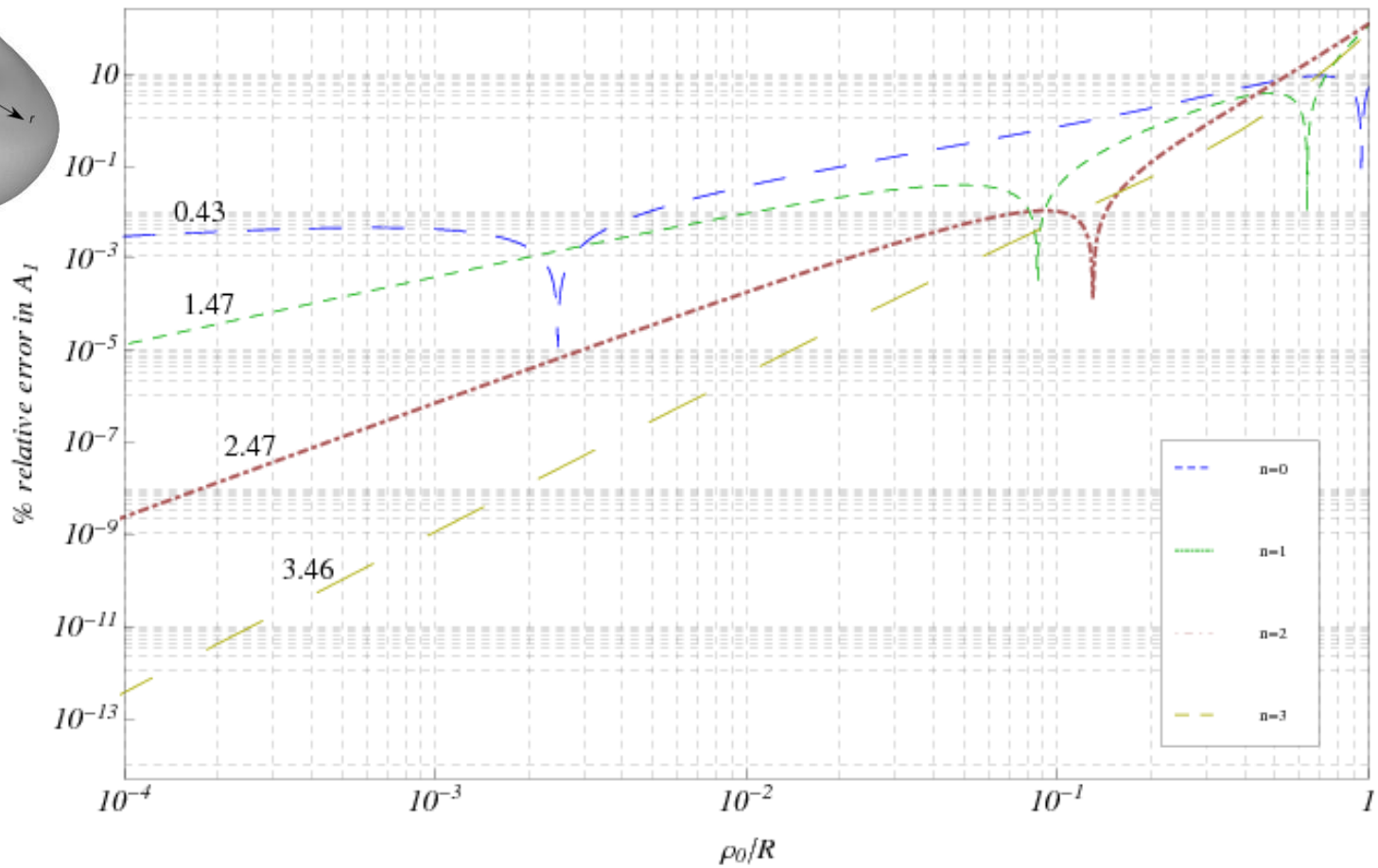
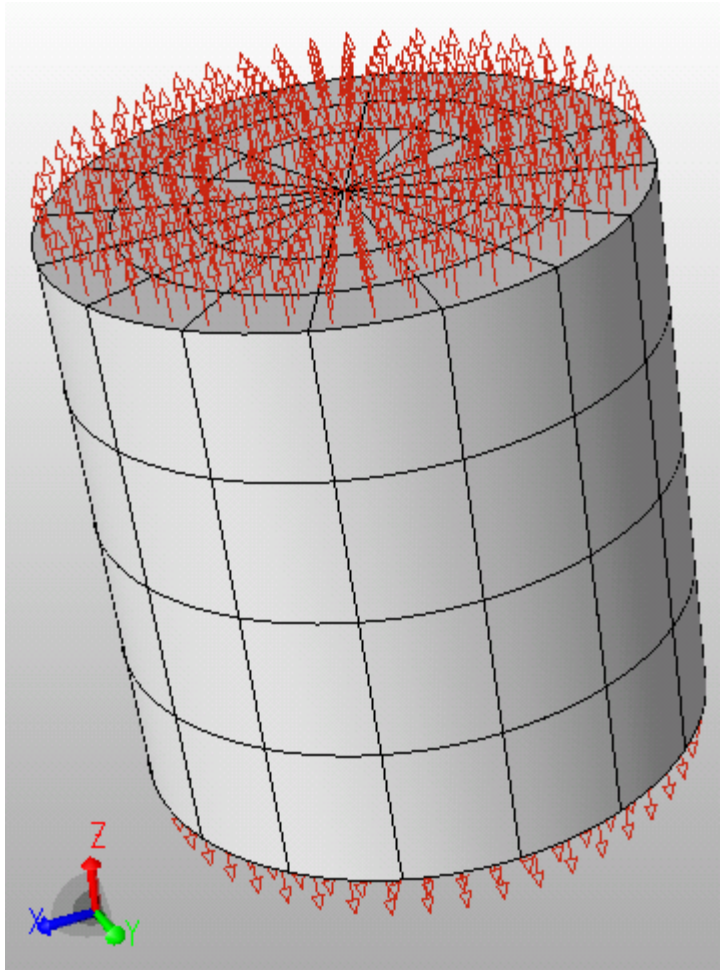
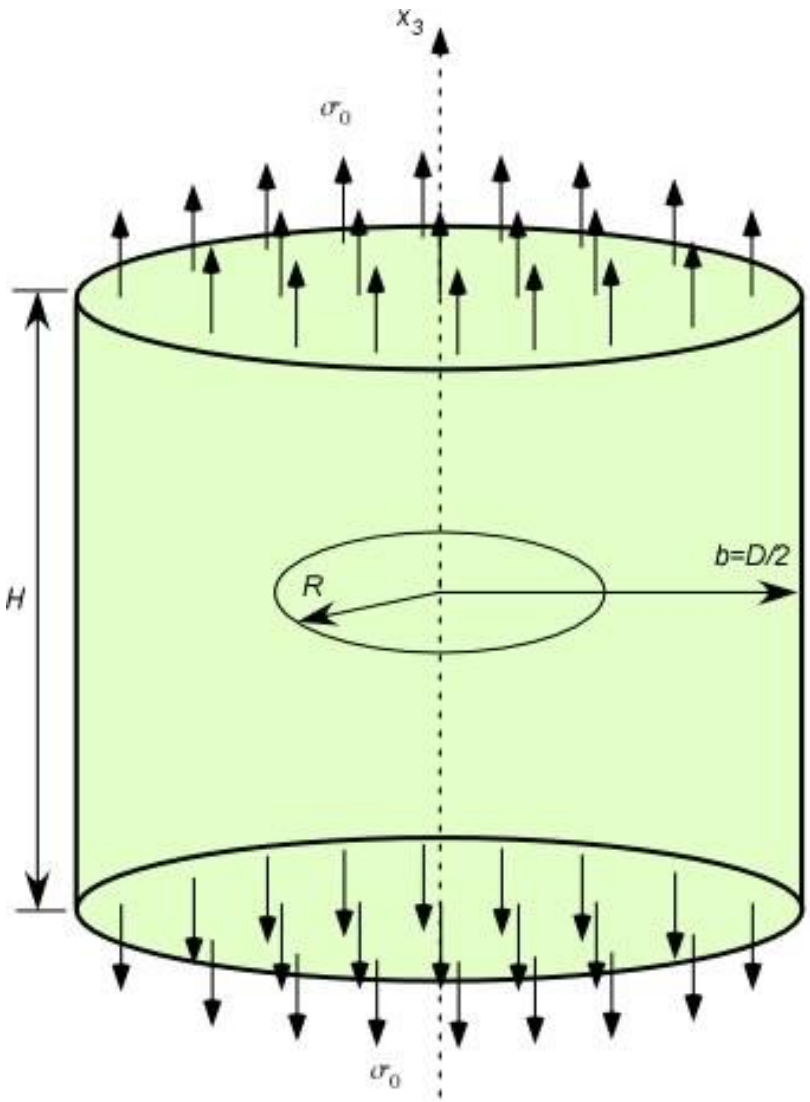


Figure 2: Convergence of the extracted A_1 with different QDFs. The numbers above the lines are the asymptotic rates of convergence that are very close to the theoretical estimates.

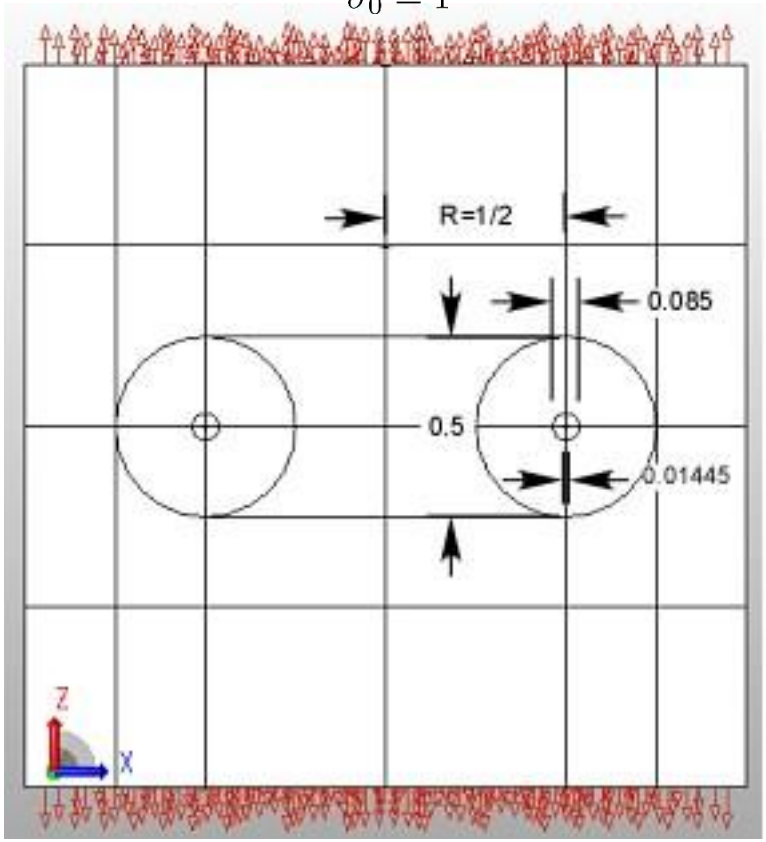
Numerical example: An infinite cylinder with a penny-shaped crack under uniform tension





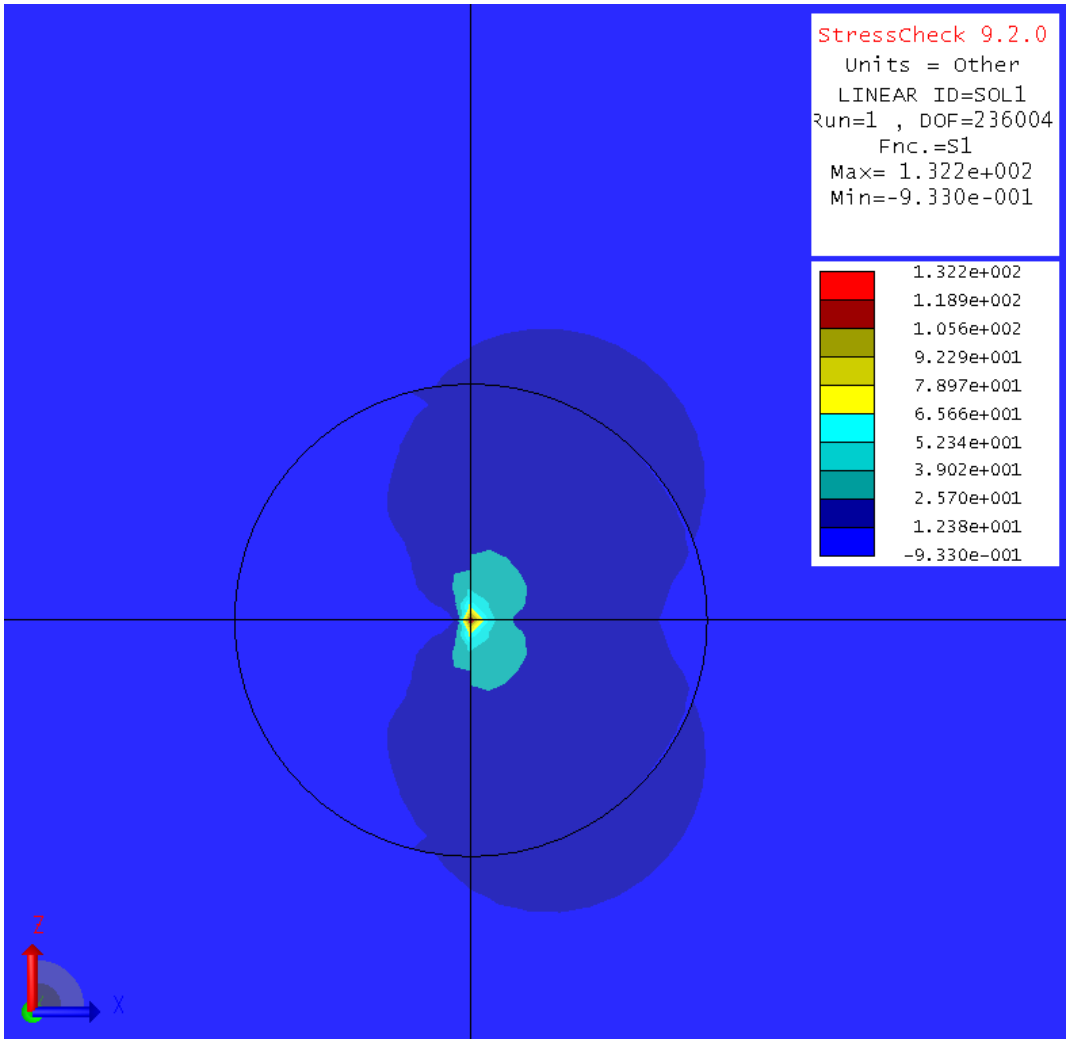
Numerical exapmle: infinite cylinder with a penny-shaped crack under an uniform tension

$\sigma_0 = 1$



$\sigma_0 = 1$

e=0.3%



Numerical exapmle: infinite cylinder with a penny-shaped crack under an uniform tension

Table 4: The extracted $K_{I_{FE}}$ by $YCD_{\rho_0=0.1}[\boldsymbol{u}_{FE}, \boldsymbol{K}_{4,0}^{(1/2)}]$

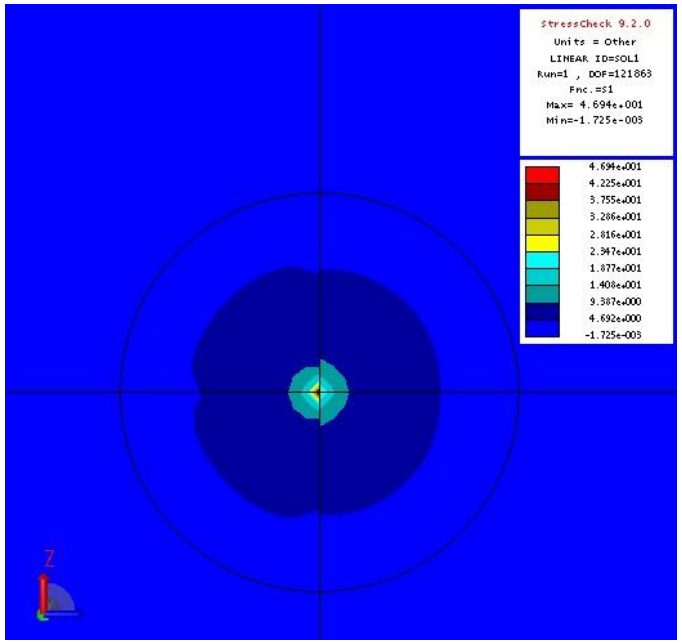
	$b=2.5$	$b=5$	$b=10$	$b=15$	$b=20$	$b=25$
$K_{I_{FE}}$	0.804678	0.798714	0.797928	0.797786	0.797574	0.797204
$\%\ relative\ error$	0.851319	0.103892	0.005331	0.012452	0.039020	0.085314

$$K_{I_{Sneddon}} = 0.797885$$

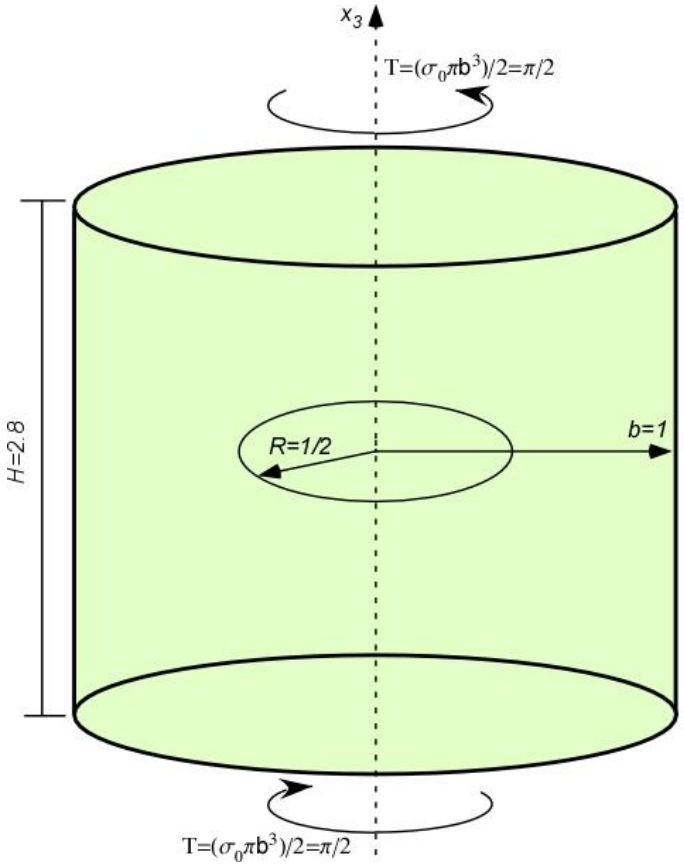
b is cylinder's radius, *R=1/2* is crack radius, *H=b*

Sneddon, I.N.: The distribution of stress in the neighbourhood of a crack in an elastic solid. *Proceeding of the Royal Society of London* A **187**, 229-260 (1946).

Numerical example: a finite cylinder with a penny-shaped crack under torsion



e=0.1%



Numerical example: a finite cylinder with a penny-shaped crack under torsion

Table 7: Extracted $K_{III_{FE}}$ by $YCD_{\rho_0=0.1}[\boldsymbol{u}, \boldsymbol{K}_{n,0}^{(1/2)}]$ and the % estimated relative error

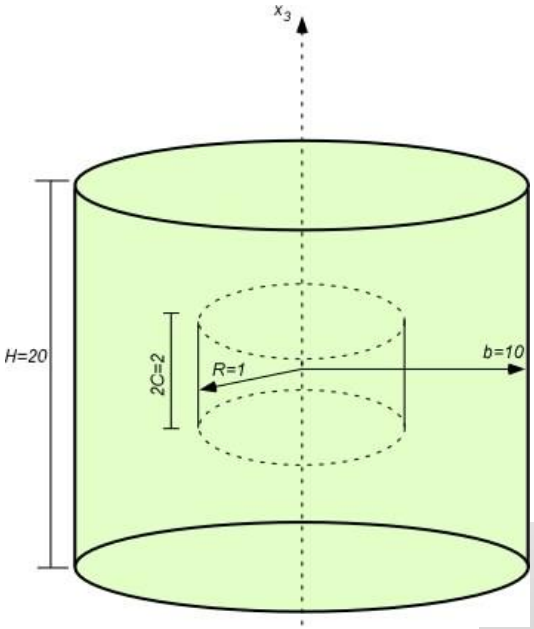
	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	Benthem &Koiter [2]	Kuo [10]
$K_{III_{FE}}$	0.278360	0.274324	0.269480	0.268880	0.268540	0.268488	0.268276
$\%\ estimated\ rel.\ error$	3.714362	2.210723	0.405942	0.182345	0.055586		

$$K_{III_{FE}} = \mu\sqrt{\pi/2}A_{3_{FE}}$$

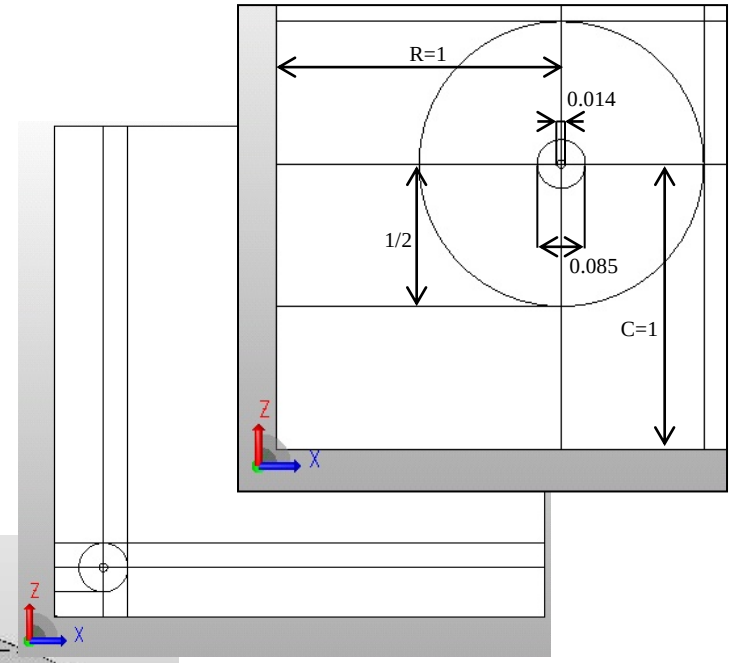
Benthem, J.P., Koiter, W.T. (Edited by G.C. Sih): Asymptotic approximations to crack problems in mechanics of fracture. *Noordhoff, Leyden*, 131-178 (1973).

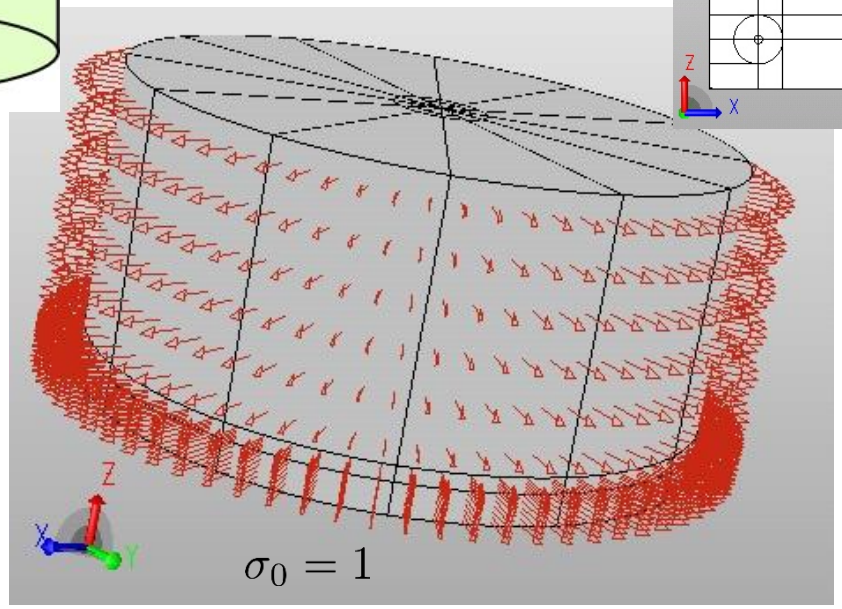
Kuo A.Y.: On the use of a Path-Independent line integral for Axisymmetric cracks with Nonaxisymmetric loading. *Journal of Applied Mechanics*, **54**, 833-837 (1987).

Numerical exapmle: a finite cylinder with a cylindrical crack



e=0.2%





Numerical exapmle: a finite cylinder with a cylindrical crack

Table 8: Extracted $K_{I_{FE}}$ & $K_{II_{FE}}$ by $YCD_{\rho_0}[\boldsymbol{u}, \boldsymbol{K}_{4,0}^{(1/2)}]$ and the references values.

	$\rho_0/R =$					Demir et al.	Chen & Farris	StressCheck
	1/2	1/4	1/10	1/20	1/40	[7]	[3]	at $\rho = 0.0125$
K_I	0.8723	0.9568	0.9609	0.9606	0.9605	0.95	0.95	0.9391
K_{II}	0.1397	0.1484	0.1489	0.1489	0.1489	0.15	0.15	0.1486

Demir, I., Hirth, J.P., Zbib, H.M.: The extended stress field around a cylindrical crack using the theory of dislocation pile-ups. *Int. Jour Eng. Sci.* **30**, 829-845 (1992).

Chen, S.Y., Farris, T.N.: Boundary Element crack closure calculation of axisymmetric stress intensity factors. *Computers & Structures* **50**, 491-497 (1994).

Non axisymmetric case

The GEFIFs now are functions along the edge. For a circular closed edge, we approximate $A_j(\theta)$ by Fourier series:

$$\tilde{A}_{k_P}(\theta) = a_{k_0} + \sum_{p=1}^{P} a_{k_{2p-1}} \cos(p\theta) + \sum_{p=1}^{P} a_{k_{2p}} \sin(p\theta)$$

The extraction function $B_j(\theta)$ is selected each time as one term in the Fourier series:

$$B_{j_{2q-1}}(\theta) = b_{j_{2q-1}} \cos(q\theta) \quad \text{or} \quad B_{j_{2q}}(\theta) = b_{j_{2q}} \sin(q\theta).$$

… and the quasi dual function is constructed:

$$\boldsymbol{K}_{n,m}^{(\alpha_j)}(\rho,\varphi,\theta) \stackrel{\text{def}}{=} \sum_{h=0,2,4\cdots}^{m} \partial_\theta^h B_j(\theta) \rho^{-\alpha_j} \sum_{f=0}^{n} \left(\frac{\rho}{R}\right)^{h+f} \boldsymbol{\psi}_{hjf}(\varphi)$$

Non axisymmetric case

To obtain a_{j_q} alone by $YCD_{\rho_0}[\boldsymbol{u}, \boldsymbol{K}_{n,m}^{(\alpha_j)}(B_{j_q})]$ we choose:

For ESIFs associated with mode I and mode II (i.e. $A_1, A_2, A_7, A_8, A_{13}, A_{14} \cdots$), b_{j_q} is:

$$b_{j_0} = \frac{\left(\alpha_j^2 - 1\right)(\lambda + \mu)}{32\pi^2 \alpha_j \mu(\lambda + 2\mu)R}, \quad q = 0$$

$$b_{j_q} = \frac{\left(\alpha_j^2 - 1\right)(\lambda + \mu)}{16\pi^2 \alpha_j \mu(\lambda + 2\mu)R}, \quad q \neq 0$$

For ESIFs associated with mode III (i.e. $A_3, A_9, A_{15} \cdots$), b_{j_q} is:

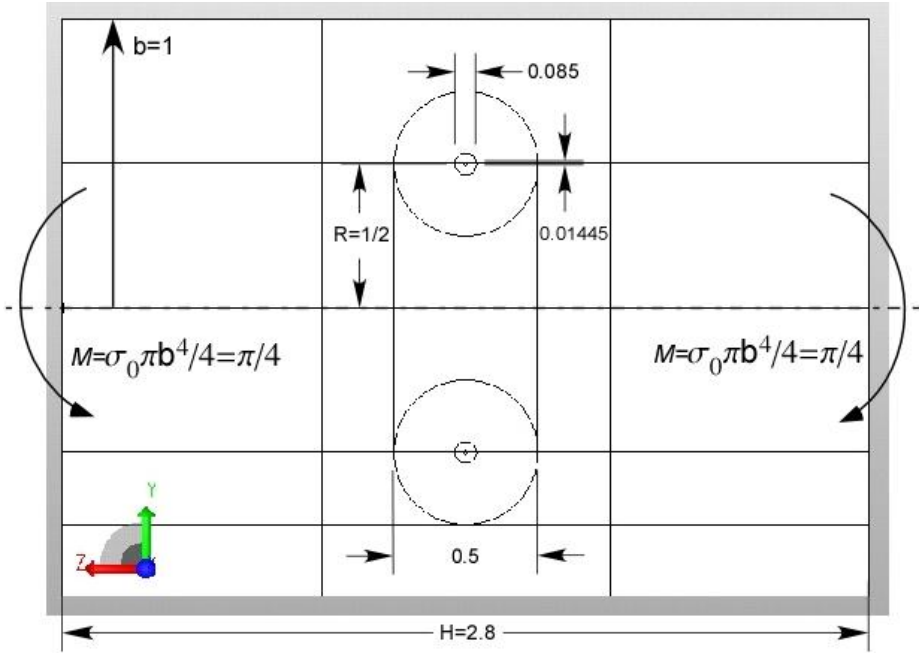
$$b_{j_0} = \frac{1}{4\pi^2 \alpha_j \mu R}, \quad q = 0$$

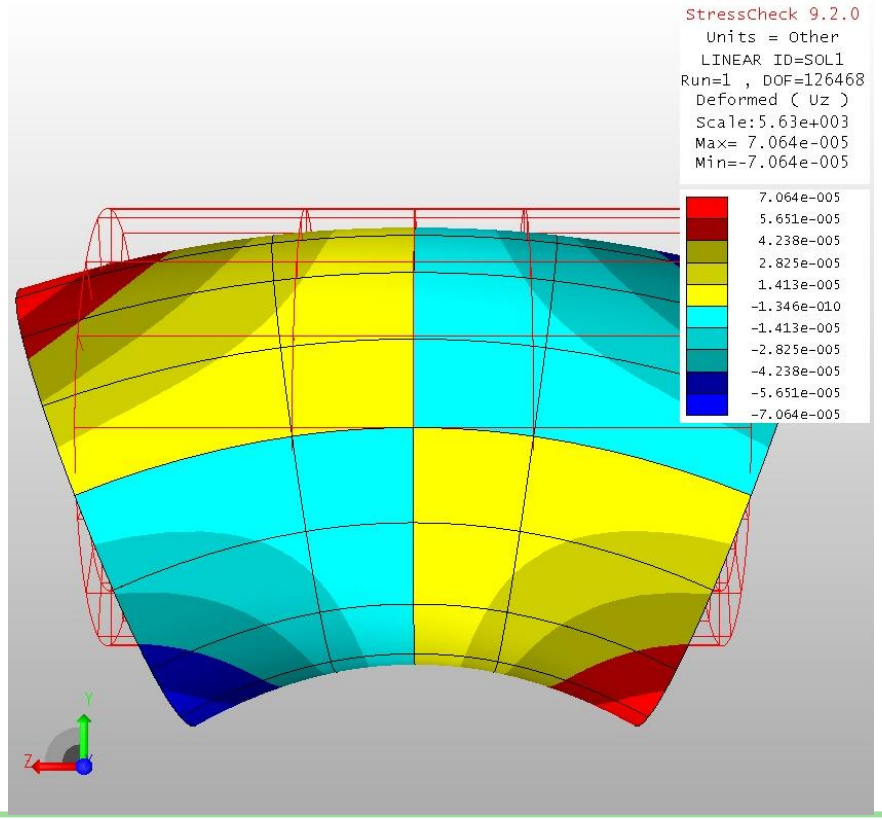
$$b_{j_q} = \frac{1}{2\pi^2 \alpha_j \mu R}, \quad q \neq 0$$

$$YCD_{\rho_0}[\boldsymbol{u}, \boldsymbol{K}_{n,m}^{(\alpha_j)}(B_{j_q})] = a_{j_q} + \mathcal{O}\left\{\rho_0^{\alpha_0 - \alpha_1}\left[\left(\frac{\rho_0}{R}\right)^{n+1} + \left(\frac{\rho_0}{R}\right)^{m+1}\right]\right\}$$

Non axisymmetric solution – cylinder with a penny-shaped crack in bending

e=0.1%





Non axisymmetric solution – cylinder with a penny-shaped crack in bending

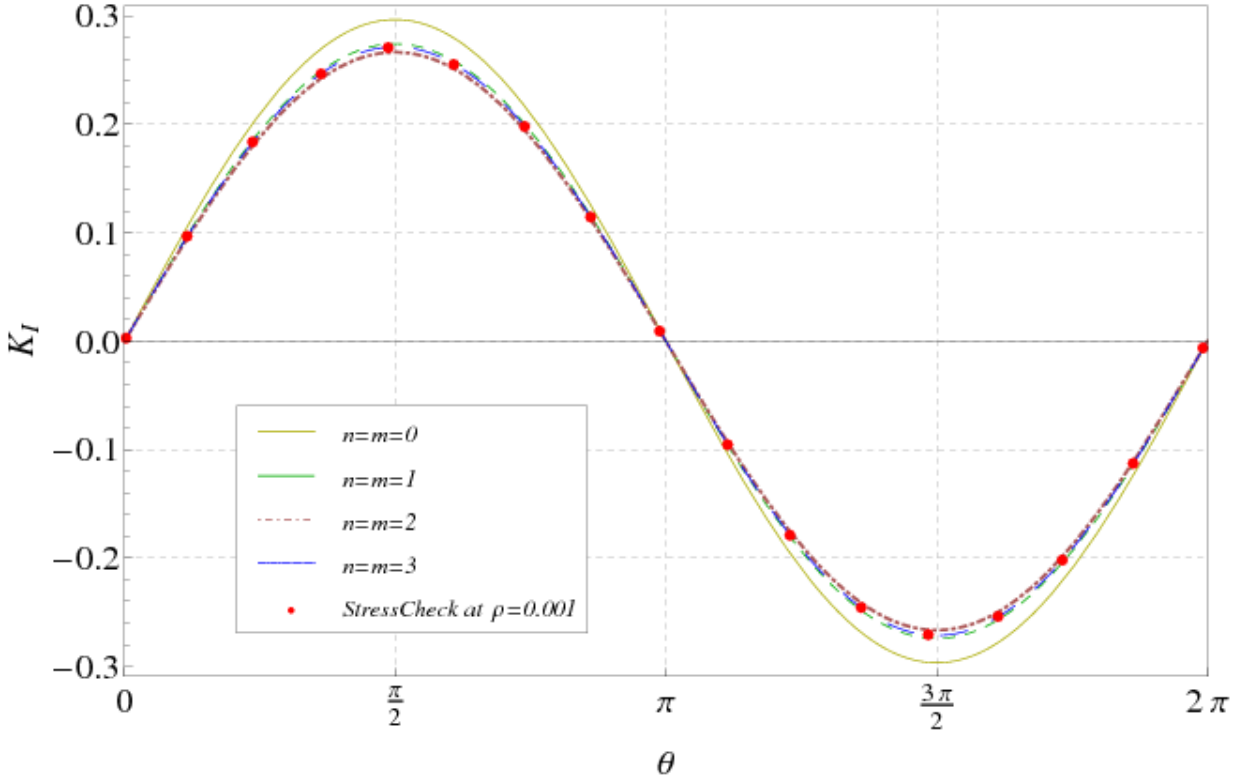


Figure 13: Extracted $\tilde{K}_I(\theta)$ by $YCD_{\rho_0=0.1}[\boldsymbol{u}, \boldsymbol{K}^{(1/2)}_{n,m=n}[B_{1_q}]]$ at $\rho_0 = 0.1$ and the pointwise values at $\rho_0 = 0.001$.

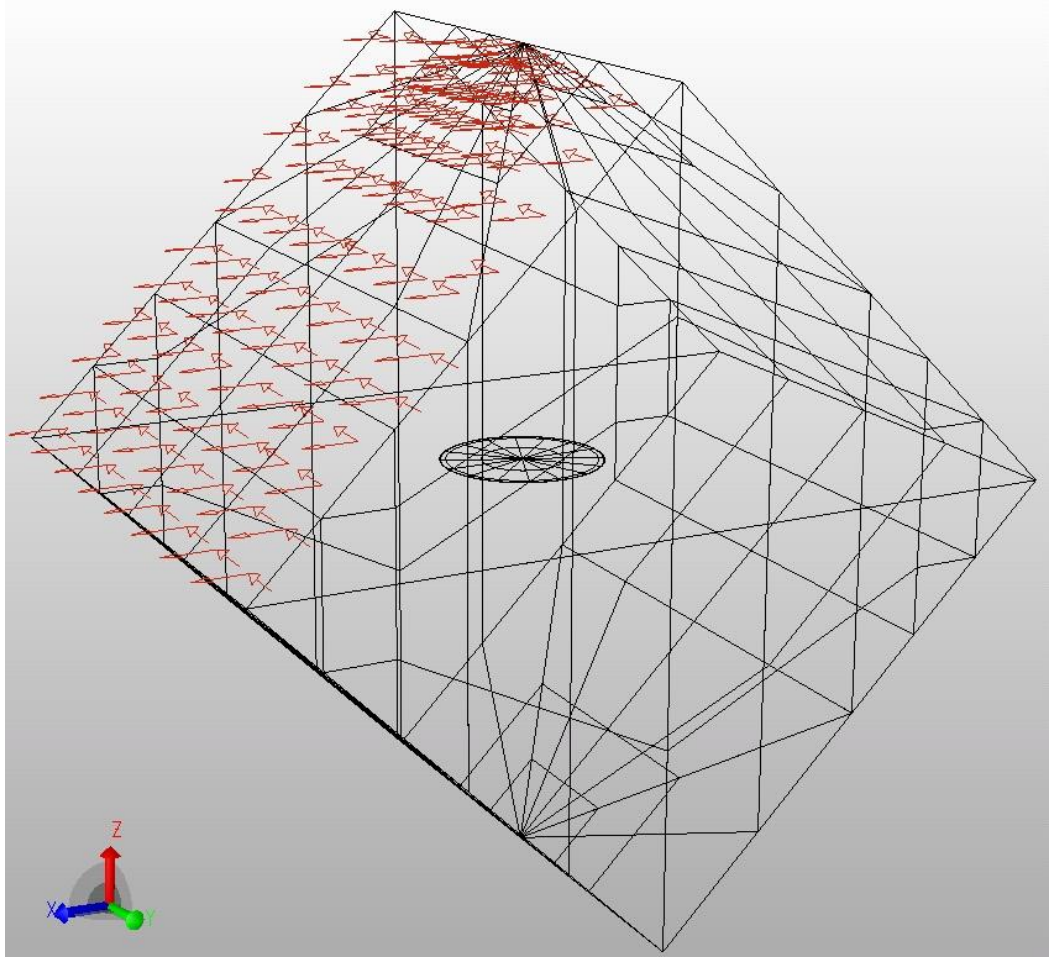
Table 12: Maximum value of K_I (the $\sin\theta$ coefficient) computed by $YCD_{0.1}[\boldsymbol{u}_{FE}, \boldsymbol{K}^{(1/2)}_{n,n}(B_{1_2})]$

$n=0$	$n=1$	$n=2$	$n=3$	Benthem&Koiter [2]	Kuo [10]
0.296589	0.274168	0.266477	0.271342	0.271317	0.266013

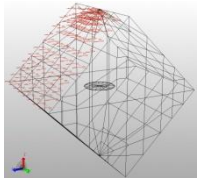
Benthem, J.P., Koiter, W.T. (Edited by G.C. Sih): Asymptotic approximations to crack problems in mechanics of fracture. *Noordhoff, Leyden*, 131-178 (1973).

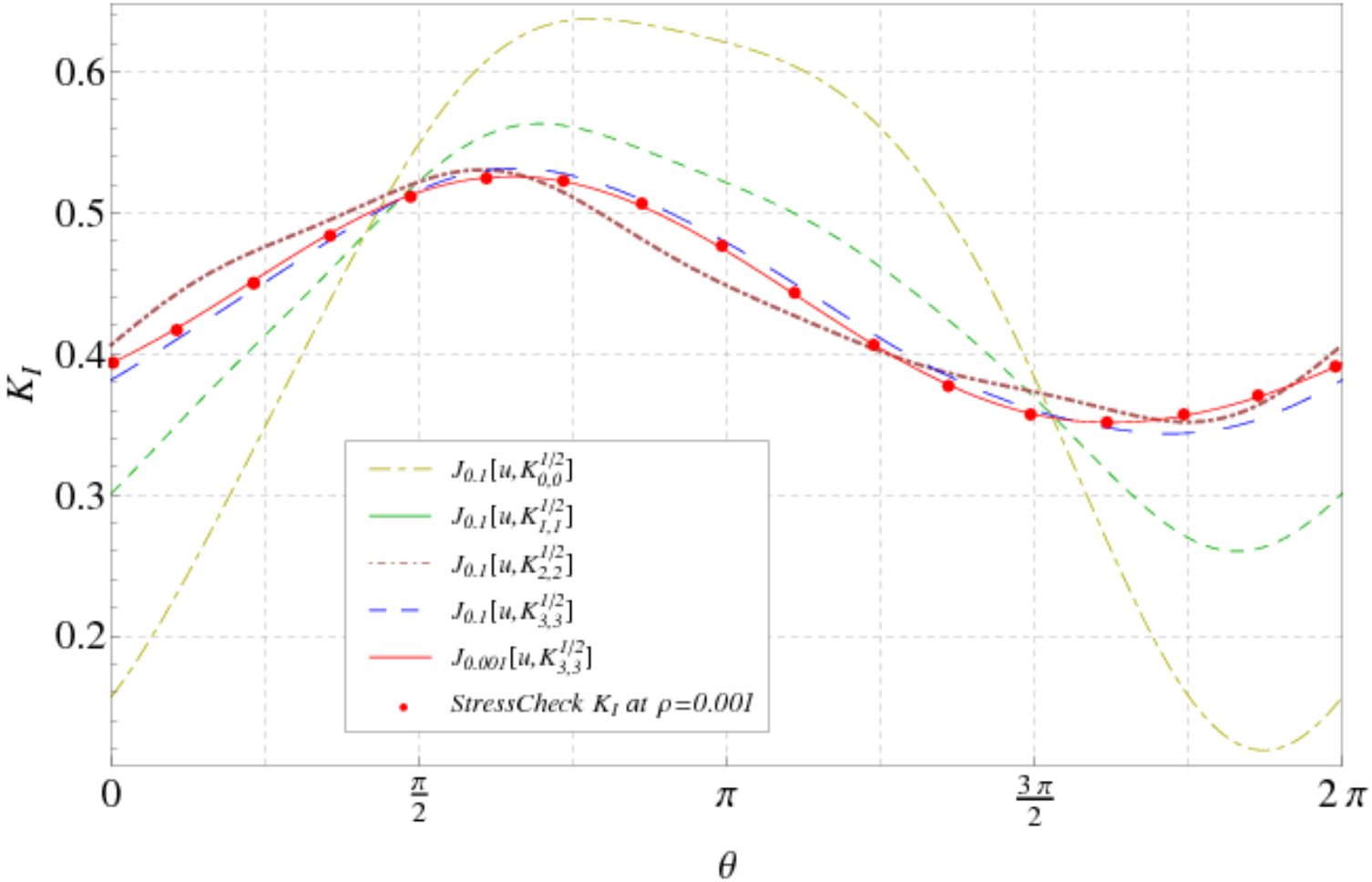
Kuo A.Y.: On the use of a Path-Independent line integral for Axisymmetric cracks with Nonaxisymmetric loading. *Journal of Applied Mech*anics **54**, 833-837 (1987).

Non axisymmetric solution – a slanted penny-shaped crack in a cube

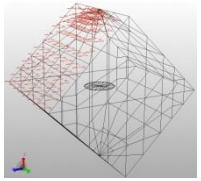


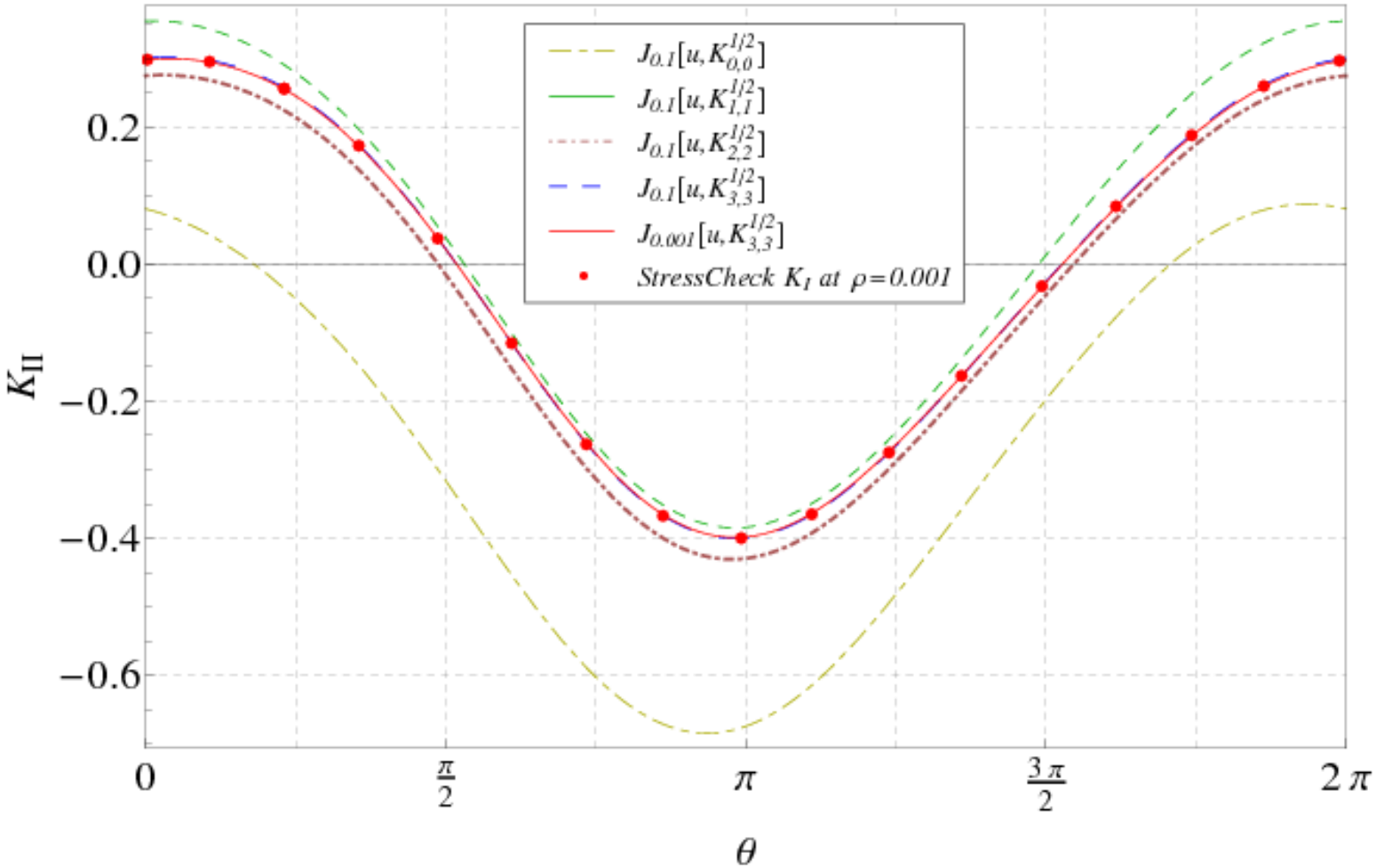
Non axisymmetric solution – a slanted penny-shaped crack in a cube



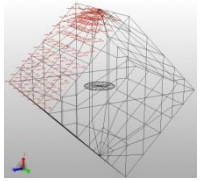


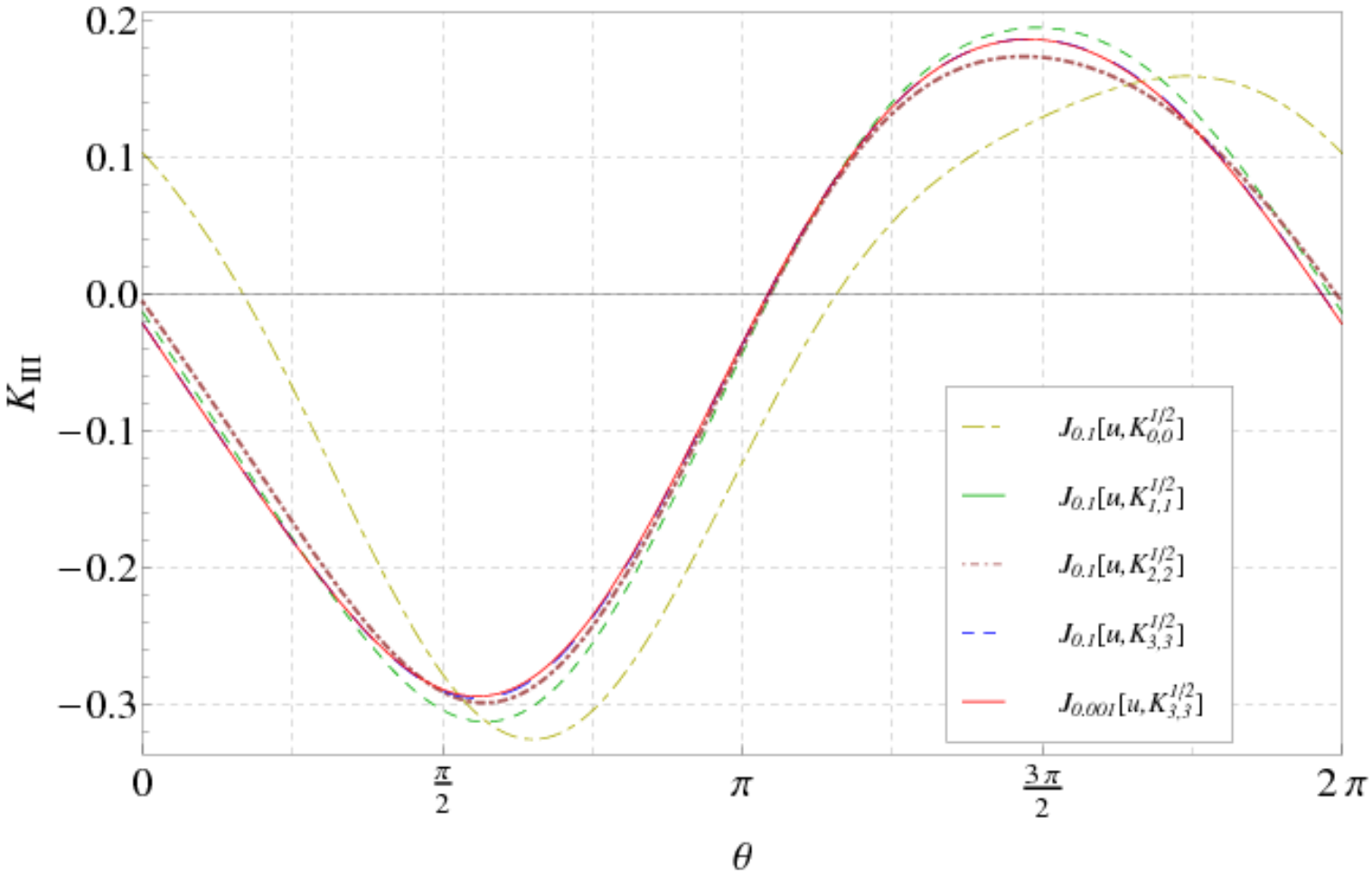
Non axisymmetric solution – a slanted penny-shaped crack in a cube





Non axisymmetric solution – a slanted penny-shaped crack in a cube

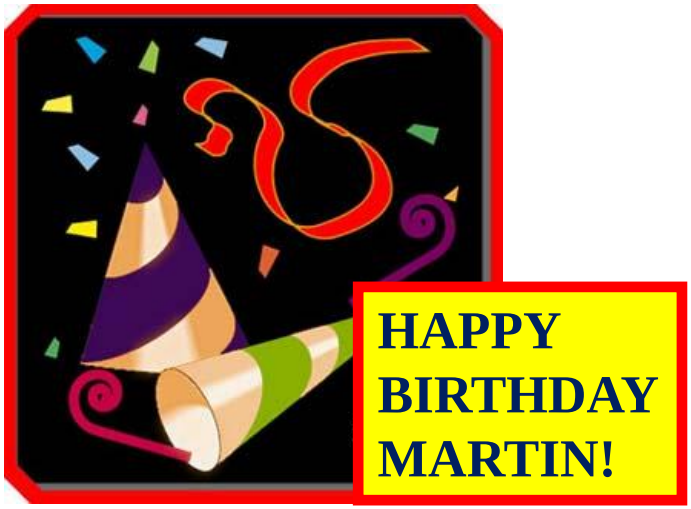




Summary

• The QDFM has been extended to circular singular edges. It has three important advantages:
a) It allows to extract the ESIFs away from the singular edge, thus avoiding the need for a refined FE mesh,
b) The ESIFs are obtained as a function along the edge and not as pointwise values,
c) The method is general in the sense that it is applicable to any circular edge (be it a penny shaped crack, a cylindrical crack or a circular external crack).

• Efforts are underway to develop an explicit singular solution along an elliptical 3D crack – see Samuel Shannon's poster….



Thank you for your attention