Numerical computation of non-scattering boundary deformations of a 2D acoustic waveguide.

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Generally, an obstacle can be detected by waves:
Non scattering obstacle and invisibility

Generally, an obstacle can be detected by waves:

except if it is invisible, which means that it does not scatter.
Invisibility...what does it mean?

One should distinguish invisibility

<table>
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<tr>
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<th>at all frequencies</th>
</tr>
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<tbody>
<tr>
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</tr>
<tr>
<td>from far field measurements</td>
<td>from near field measurements</td>
</tr>
</tbody>
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No scattered waves

incident wave
Invisibility...what does it mean?

Total invisibility (cloaking):

- at one (or some) frequency
- for one (or some) incident wave
- from far field measurements

requires a change of the medium around the obstacle (see e.g. A. Norris):
Invisibility....what does it mean?

Here, we consider invisibility

- at one (or some) frequency
- for one (or some) incident wave
- from far field measurements

versus

- at all frequencies
- for all incident waves
- from near field measurements

in a homogeneous waveguide.

- There is no local change of the medium.
- The “obstacle” is a deformation of the surface of the waveguide.

Theoretical background: Obstacles in acoustic waveguides becoming “invisible” at given frequencies (A.-S. BBD and Sergei Nazarov).
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We consider a 2D acoustic waveguide with rigid boundaries:

\[ \Delta u + k^2 u = 0 \quad (\Omega) \]
\[ \frac{\partial u}{\partial n} = 0 \quad (\partial \Omega) \]

Modal solutions in the undeformed waveguide \((H = 1)\):

\[ u(x, y) = \cos(n \pi y) e^{\pm i \beta_n x} \]

with

\[ \beta_n = \sqrt{k^2 - n^2 \pi^2} \]
\[ n \in \mathbb{N} \]

Key point:

There are only a finite number of propagative modes \((\beta_n \in \mathbb{R})\).

Other modes are evanescent \((\beta_n \in \mathbb{i} \mathbb{R})\).
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Notations and equations

We consider a 2D acoustic waveguide with rigid boundaries:

Time-harmonic equations ($k = \omega / c$):

\[
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\Delta u + k^2 u &= 0 \quad (\Omega) \\
\frac{\partial u}{\partial n} &= 0 \quad (\partial \Omega)
\end{align*}
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Modal solutions in the undeformed waveguide ($H = 1$):

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Key point

There are only a finite number of propagative modes ($\beta_n \in \mathbb{R}$). Other modes are evanescent ($\beta_n \in i\mathbb{R}$).
The total field $u$ is such that $u = u_{\text{inc}} + u_{\text{sca}}$ where
The scattering problem

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- the scattered field \( u_{sca} \) is outgoing.

**Invisibility**

The deformation is invisible at the frequency \( k \) iff, for all incident waves, the scattered field \( u_{sca} \) vanishes exponentially at infinity.
The scattering problem: numerical solution

We use a multimodal approach (see Hazard and Lunéville, IMA, 2008):

**Hybrid discretization:**

- Modal expansion in $y$, at each $x$,
- Finite Element in $x$, for the modal amplitudes.

**Main advantage:** no mesh! And therefore, no need of remeshing when changing the form.
The scattering problem - the single-mode case.

Suppose $0 < k < \pi$ so that there is only one propagative mode and consider first a rightgoing mode $u_{inc} = e^{ikx}$.
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$$u(x, y) = \begin{cases} 
  e^{ikx} + Re^{-ikx} + \sum \text{evanescent modes} & x \to -\infty, \\
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Invisibility

The deformation is invisible for a rightgoing incident wave iff $R = 0$ and $T = 1$. 
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**Invisibility**

The deformation is invisible for a rightgoing incident wave iff $R = 0$ and $T = 1$. And it works also for leftgoing waves! (consider $\bar{u}$)
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The deformation is invisible for a rightgoing and leftgoing incident waves iff \(R = 0\) and \(T = 1\).
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Energy conservation

$$|R|^2 + |T|^2 = 1$$

Invisibility

The deformation is invisible for a rightgoing and leftgoing incident waves iff $R = 0$ and $T = 1$.

Near invisibility

The deformation is invisible up to a phase shift of the transmitted wave iff $R = 0$ (which implies $|T| = 1$).
The asymptotic expansion

We consider a small deformation of amplitude $\varepsilon$:

The solution $u_\varepsilon$ of the scattering problem is such that:

$$u_\varepsilon = \begin{cases} e^{ikx} + R_\varepsilon e^{-ikx} & x \to -\infty \\ T_\varepsilon e^{ikx} & x \to +\infty \end{cases}$$

Ansatz:

$$u_\varepsilon(x, y) = e^{ikx} + \varepsilon u'(x, y) + \text{high-order terms} \Rightarrow \begin{cases} R_\varepsilon = \varepsilon R' + \varepsilon^2 \tilde{R}(\varepsilon) \\ T_\varepsilon = 1 + \varepsilon T' + \varepsilon^2 \tilde{T}(\varepsilon) \end{cases}$$

with

$$|\tilde{R}(\varepsilon)| < C \text{ and } |\tilde{T}(\varepsilon)| < C$$

See Mazja-Nazarov-Plamenevski 91.
Computation of the first-order term $u'$

\[
\begin{aligned}
\Delta u' + k^2 u' &= 0 \quad (\Omega_0) \\
\partial_n u' &= 0 \quad (y = 0) \\
\partial_n u' &= g \quad (y = 1)
\end{aligned}
\]

with $g(x) = ik \partial_x \left( h(x) e^{ikx} \right)$. 

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\]

Then, from $\int_{\partial B_L} (\partial_n u' v - u' \partial_n v) = 0$ for $v = e^{\pm ikx}$ and $L \to +\infty$, we get

**The first-order scattering coefficients**

\[
T' = -\frac{1}{2} \int h(x) dx = 0 \quad \text{and} \quad R' = ik \int h(x)e^{2ix} dx
\]
First-order expansion

\[ u_\varepsilon = \begin{cases} 
  e^{ikx} + R_\varepsilon e^{-ikx} & x \to -\infty \\
  T_\varepsilon e^{ikx} & x \to +\infty 
\end{cases} \]

Expansion of the scattering coefficients

\[ R_\varepsilon = ik\varepsilon \int h(x)e^{2ikx} \, dx + \varepsilon^2 \tilde{R}(\varepsilon) \]

\[ T_\varepsilon = 1 + \varepsilon^2 \tilde{T}(\varepsilon) \]

Consequence: Invisibility up to order 2 in \( \varepsilon \) is achieved if

\[ \int h(x) \cos(2kx) \, dx = \int h(x) \sin(2kx) \, dx = 0 \]
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$$\int h(x) \cos(2kx) \, dx = \int h(x) \sin(2kx) \, dx = 0$$
Numerical validation: the bump

Consider

\[ h(x) = \left( \left( \frac{x}{a} \right)^2 - 1 \right)^2 \chi_{(-a,a)}(x) \]

Then \[ \int h(x) \sin(2kx) \, dx = 0 \] and \[ \exists a \text{ such that } \int h(x) \cos(2kx) \, dx = 0. \]

\[ \Rightarrow \exists a \text{ such that } \int h(x) e^{2ikx} \, dx = 0. \]
Numerical validation: the cavity

Consider

\[
h(x) = - \left( \left( \frac{x}{a} \right)^2 - 1 \right)^2 \chi_{(-a,a)}(x)
\]

Then

\[
\exists a \text{ such that } \int h(x) e^{2i k x} dx = 0.
\]
Numerical validation: the elementary bump

We can also choose \( h(x) = \alpha \left( \left( \frac{x}{a} \right)^2 - 1 \right)^2 \chi_{(-a,a)}(x) \) with \( a \) and \( \alpha \) such that

\[
\int h(x) e^{2ikx} \, dx = 1
\]

Then, the asymptotic theory says that:

\[
R_\varepsilon = ik\varepsilon + \varepsilon^2 \tilde{R}(\varepsilon)
\]
For a given (small) $\varepsilon$ and a given wavenumber $k$ ($0 < k < \pi$), find $h$ such that the deformation is invisible, in the sense that $R_\varepsilon = 0$. More generally, for a given (small) $\varepsilon$ and given wavenumbers $k_1, k_2, \ldots, k_N$, ($0 < k_j < \pi$, $j = 1, \ldots, N$), find $h$ such that the deformation is invisible at these frequencies, in the sense that $R_\varepsilon(k_1) = R_\varepsilon(k_2) = \cdots = R_\varepsilon(k_N) = 0$. 

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The objective

For a given (small) $\varepsilon$ and a given wavenumber $k$ ($0 < k < \pi$), find $h$ such that the deformation is invisible, in the sense that $R_\varepsilon = 0$.

More generally, for a given (small) $\varepsilon$ and given wavenumbers $k_1, k_2, \cdots k_N$, ($0 < k_j < \pi$, $j = 1, \cdots, N$), find $h$ such that the deformation is invisible at these frequencies, in the sense that

$$R_\varepsilon(k_1) = R_\varepsilon(k_2) = \cdots R_\varepsilon(k_N) = 0$$
The deformation $y = 1 + \varepsilon h(x)$

Search $h$ of the form

$$h = h_0 + \tau_1 h_1 + \tau_2 h_2$$

where

- $\tau_1$ and $\tau_2$ are unknown real parameters
- $h_0$, $h_1$ and $h_2$ are given real valued functions such that:

$$\begin{cases}
\int h_0(x) e^{2i k x} \, dx = 0 \\
\int h_1(x) e^{2i k x} \, dx = 1 \\
\int h_2(x) e^{2i k x} \, dx = i
\end{cases}$$

Remark: one can choose $h_2(x) = h_1(x - \pi/(4k))$. 
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\end{align*}$$

Remark: one can chose $h_2(x) = h_1(x - \pi/(4k))$. 
The fixed-point equation

Equation of the boundary:

\[ y = 1 + \varepsilon h(x) \text{ with } h = h_0 + \tau_1 h_1 + \tau_2 h_2 \]

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The asymptotic theory says that:

\[ R_{\varepsilon} = ik\varepsilon \int h(x)e^{2ikx} \, dx + \varepsilon^2 \tilde{R}_{\varepsilon} = ik\varepsilon(\tau_1 + i\tau_2) + \varepsilon^2 \tilde{R}_{\varepsilon} \]
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\]
\[
= ik\varepsilon(\tau_1 + i\tau_2) + \varepsilon^2 \tilde{R}_\varepsilon
\]
\[
= ik\varepsilon\tau + \varepsilon^2 \tilde{R}_\varepsilon(\varepsilon\tau)
\]

where we have set: \( \tau = \tau_1 + i\tau_2 \in \mathbb{C} \).
The fixed-point equation

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Asymptotic theory: \( R_\varepsilon = ik\varepsilon\tau + \varepsilon^2 \tilde{R}_\varepsilon(\varepsilon \tau) \)

where \( \tau = \tau_1 + i\tau_2 \)
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Equation of (near-)invisibility

\[ R_\varepsilon = 0 \iff \tau = F_\varepsilon (\tau) \]

where \( F_\varepsilon (\tau) = \frac{i\varepsilon}{k} \tilde{R}_\varepsilon (\varepsilon \tau) = \tau + \frac{i}{k\varepsilon} R_\varepsilon \)
The fixed-point equation

**Equation of the boundary:**

\[ y = 1 + \varepsilon h(x) \text{ with } \]

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where \( F_\varepsilon(\tau) = \frac{i\varepsilon}{k} \tilde{R}_\varepsilon(\varepsilon\tau) = \tau + \frac{i}{k\varepsilon} R_\varepsilon \)

**Theorem**

For \( \varepsilon \) small enough, \( F_\varepsilon \) is a contraction and \( \exists! \tau \in \mathbb{C} \) such that \( \tau = F_\varepsilon(\tau) \).
The fixed-point algorithm

- **Initialization:**
  - Choose $h_0$, $h_1$ and $h_2$ and fix $\varepsilon$
  - $\tau = 0$
  - $n = 0$

- **Loop:**
  1. Compute $R_\varepsilon$ by solving the scattering problem for the deformation
     
     $$y = 1 + \varepsilon (h_0 + \tau_1 h_1 + \tau_2 h_2)$$

  2. Update $\tau$: $\tau := \tau + \frac{i}{k\varepsilon} R_\varepsilon$

  3. Update $n$: $n := n + 1$

  4. If $n < N_{\text{max}}$ goto 1.
The form functions

We take the following functions:
Results for the bump and for $\varepsilon = 0.6$

The convergence is monotonic as expected:

The bump is only slightly modified:

The bump is indeed invisible:
Influence of $\varepsilon$ for the bump

The results confirm the expected convergence in $\varepsilon^{2n}$. 
Results for the cavity

Divergence occurs at $\varepsilon \approx 0.5$ compared to $\varepsilon \approx 0.75$ for the bump.

At $\varepsilon = 0.5$, the cavity is only slightly modified:

The cavity is indeed invisible:
Symmetrizing a cavity gives an obstacle!!
Making invisible by external deformations

The main form $h_0$ and the corrections $h_1$ and $h_2$ may have disjoint supports:

but the convergence is more chaotic...
The multifrequency case

We want a deformation invisible at the two frequencies $k = k_1$ and $k = k_2$ with $0 < k_1 < k_2 < \pi$.

Equation of the boundary:

$$y = 1 + \varepsilon h(x) \quad \text{with} \quad h = h_0 + \sum_{j=1}^{2} \left( \tau_1^{(j)} h_1^{(j)} + \tau_2^{(j)} h_2^{(j)} \right)$$

with

$$\left\{ \begin{array}{ll}
\int h_0(x) e^{2ik_j x} \, dx = 0 \\
\int h_1^{(j)}(x) e^{2ik_\ell x} \, dx = \delta_{j\ell} & \text{for } j, \ell = 1, 2. \\
\int h_2^{(j)}(x) e^{2ik_\ell x} \, dx = i\delta_{j\ell}
\end{array} \right.$$
The multifrequency case

The form $h_0$ and the corrections $h^{(j)}_i$ are more oscillatory.
The multifrequency case

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This is a result for $\varepsilon = 0.5$, $k_1 = 2\pi/3$ and $k_2 = 3\pi/4$:
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This is a result for $\varepsilon = 0.5$, $k_1 = 2\pi/3$ and $k_2 = 3\pi/4$:
The multimode case

For \((N-1)\pi < k < N\pi\), the scattering properties of the deformation are summarized in the unitary symmetric scattering matrix \(S\):

\[
S = \begin{pmatrix}
R_{00}^- & R_{01}^- & R_{02}^- & \cdots & T_{00}^- & T_{01}^- & T_{02}^- & \cdots \\
R_{10}^- & R_{11}^- & R_{12}^- & \cdots & T_{10}^- & T_{11}^- & T_{12}^- & \cdots \\
R_{20}^- & R_{21}^- & R_{22}^- & \cdots & T_{20}^- & T_{21}^- & T_{22}^- & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
T_{00}^+ & T_{01}^+ & T_{02}^+ & \cdots & R_{00}^+ & R_{01}^+ & R_{02}^+ & \cdots \\
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\end{pmatrix}
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The multimode case

Near invisibility is achieved if all coefficients in the blue part vanish:

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\vdots & \vdots & \vdots & \ddots \\
T_{-00}^- & T_{-01}^- & T_{-02}^- & \cdots \\
T_{-10}^- & T_{-11}^- & T_{-12}^- & \cdots \\
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\end{pmatrix}
\]

As $S$ is symmetric and unitary:

This implies $|T_{\pm m}| = 1$ for all $m = 0, 1, \cdots, n-1$.

It is sufficient to cancel $N_2^2$ complex coefficients.
Near invisibility is achieved if all coefficients in the blue part vanish:

$$S = \begin{pmatrix}
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\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\
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- This implies
  $$|T_{mm}^\pm| = 1 \quad \forall m = 0, 1 \cdots, n - 1$$

- It is sufficient to cancel $N^2$ complex coefficients.
The multimode case: asymptotic formulae

Expansion of the scattering coefficients

\[ R_{m\ell,\varepsilon}^+ = \varepsilon \, r_{m\ell} \int h(x) e^{i(\beta_\ell + \beta_m) x} \, dx + \varepsilon^2 \tilde{R}_{m\ell}^+(\varepsilon) \]

\[ T_{m\ell,\varepsilon}^+ = 1 + \varepsilon \, t_{m\ell} \int h(x) e^{i(\beta_\ell - \beta_m) x} \, dx + \varepsilon^2 \tilde{T}_{m\ell}^+(\varepsilon) \]

with \( \beta_m = \sqrt{k^2 - m^2 \pi^2} \)
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\[ R^+_{m\ell, \varepsilon} = \varepsilon r_{m\ell} \int h(x)e^{i(\beta_\ell + \beta_m)x} \, dx + \varepsilon^2 \tilde{R}^+_{m\ell}(\varepsilon) \]

\[ T^+_{m\ell, \varepsilon} = 1 + \varepsilon t_{m\ell} \int h(x)e^{i(\beta_\ell - \beta_m)x} \, dx + \varepsilon^2 \tilde{T}^+_{m\ell}(\varepsilon) \]

with \( \beta_m = \sqrt{k^2 - m^2\pi^2} \)

Consequence: Invisibility up to order 2 in \( \varepsilon \) is achieved if

\[ \int h(x)e^{i\kappa x} \, dx = 0 \text{ for } \kappa = \beta_\ell \pm \beta_m, \ 0 \leq m, \ell < N \]
The two-mode case: a result

When $\pi < k < 2\pi$: we have to build 9 form functions satisfying 8 constraints and to solve a fixed-point equation for 8 coefficients $\tau_j$!
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Compared to the monomode case, the behavior of the auxiliary form functions deteriorates: in particular, some of them present a large amplitude, which forces \( \varepsilon \) to be small...
1. How to increase $\varepsilon$ in the previous results...? What is the best choice for the elementary form functions?
Perspectives

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2. Extension to the case of water waves (with J. Taskinen): only 1 propagative mode and perfect invisibility can be achieved.
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Extension to the case of water waves (with J. Taskinen): only 1 propagative mode and perfect invisibility can be achieved.

A theoretical question is: for a given deformation, is the spectrum of invisible frequencies discrete?