Numerical computation of non-scattering boundary deformations of a 2D acoustic waveguide.

A.S. Bonnet-Ben Dhia[1] E. Lunéville[1] Y. Mbeutcha[1]

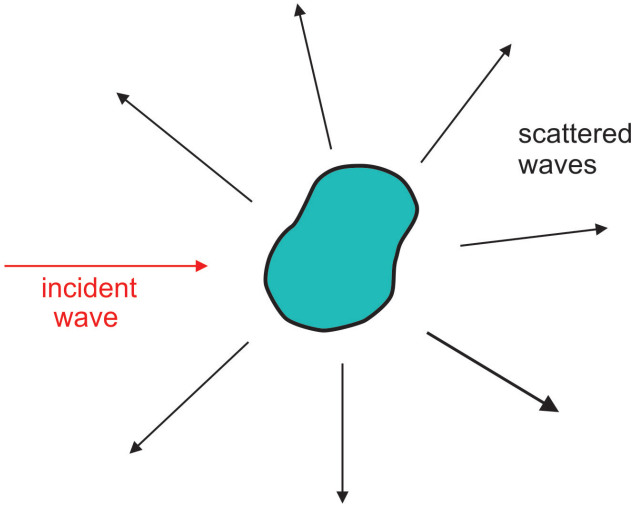
S. Nazarov[2]

[1] POEMS (UMR CNRS-ENSTA-INRIA), Palaiseau, France.
[2] Institute of Mechanical Engineering Problems, Saint-Petersburg, Russia.

Rennes, JSA 2013

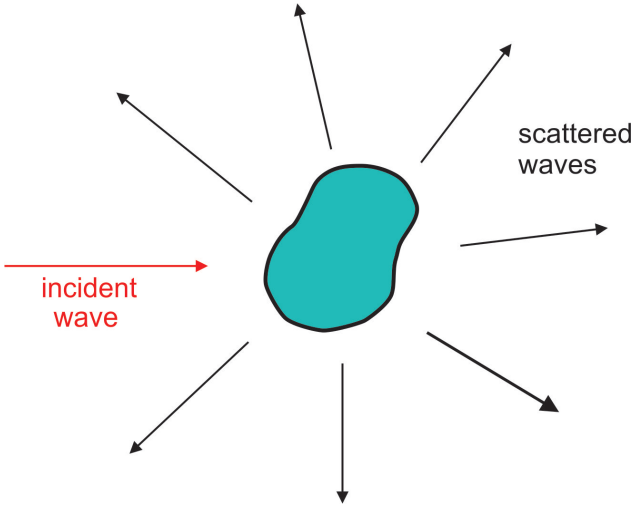
Non scattering obstacle and invisibility

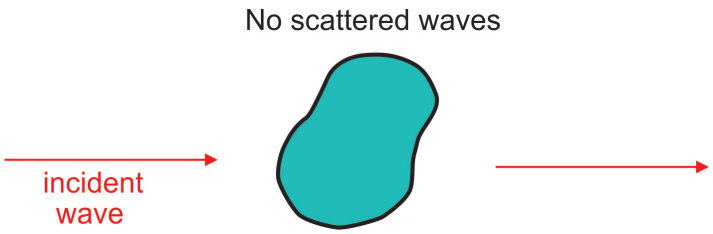
Generally, an obstacle can be detected by waves:



Non scattering obstacle and invisibility

Generally, an obstacle can be detected by waves:



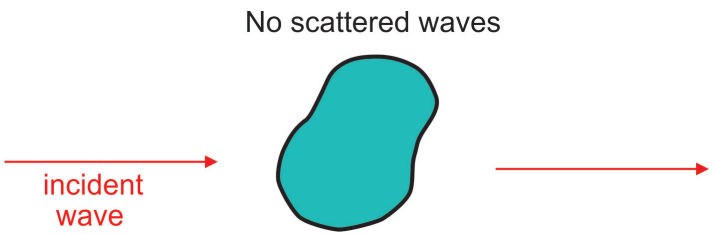


except if it is invisible, which means that it does not scatter.

Invisibility....what does-it mean ?

One should distinguish invisibility

at one (or some) frequency	at all frequencies
for one (or some) incident wave	for all incident waves
from far field measurements	from near field measurements

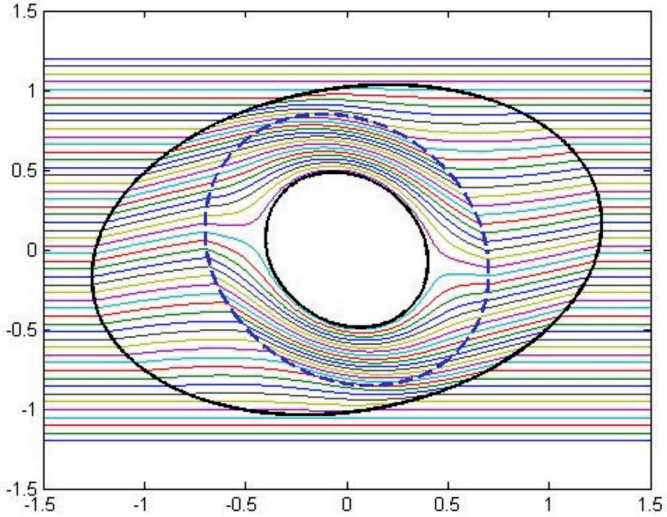


Invisibility....what does-it mean ?

Total invisibility (cloaking):

at one (or some) frequency	at all frequencies
for one (or some) incident wave	for all incident waves
from far field measurements	from near field measurements

requires a change of the medium around the obstacle (see e.g. A. Norris):

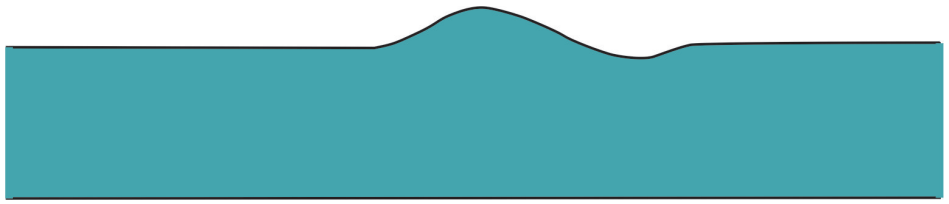


Invisibility....what does-it mean ?

Here, we consider invisibility

at one (or some) frequency	at all frequencies
for one (or some) incident wave	for all incident waves
from far field measurements	from near field measurements

in a homogeneous waveguide.



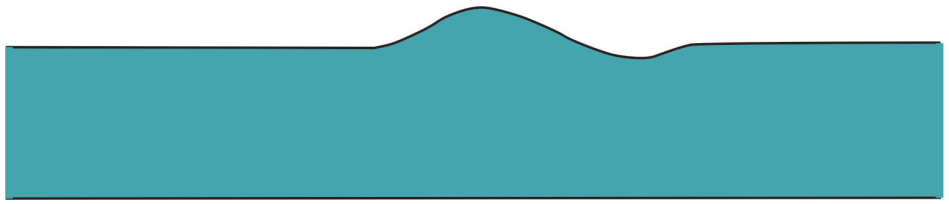
- There is no local change of the medium.
- The "obstacle" is a deformation of the surface of the waveguide.

Invisibility....what does-it mean ?

Here, we consider invisibility

at one (or some) frequency	at all frequencies
for one (or some) incident wave	for all incident waves
from far field measurements	from near field measurements

in a homogeneous waveguide.



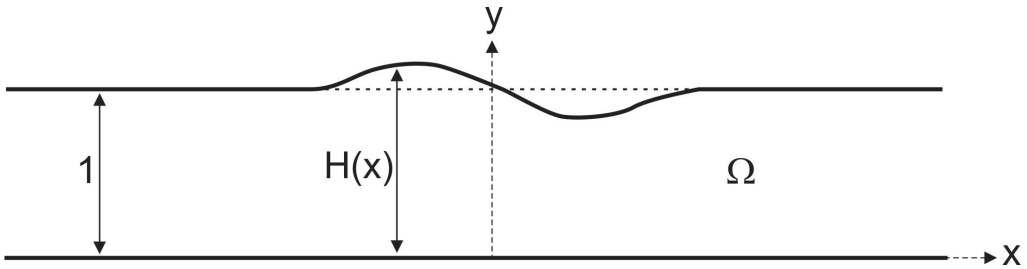
- There is no local change of the medium.
- The "obstacle" is a deformation of the surface of the waveguide.

Theoretical background: Obstacles in acoustic waveguides becoming "invisible" at given frequencies (A.-S. BBD and Sergei Nazarov) Acoustic Journal, to appear.

Outline

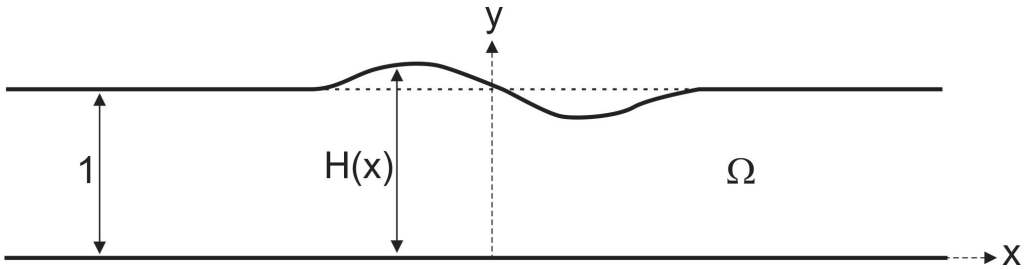
Notations and equations

We consider a 2D acoustic waveguide with rigid boundaries:



Notations and equations

We consider a 2D acoustic waveguide with rigid boundaries:



Time-harmonic equations ($k = \omega/c$):
(u is the pressure field)

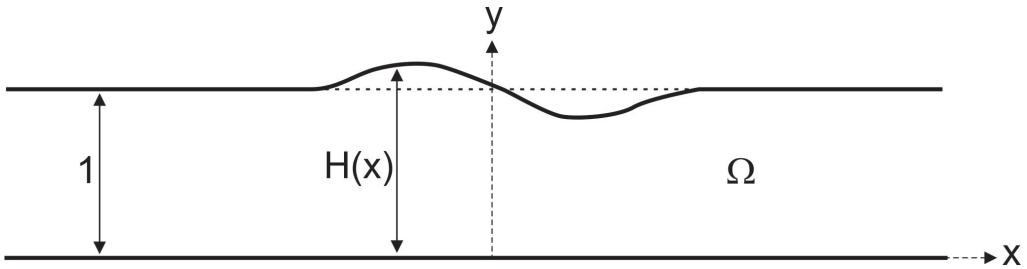
$$\begin{cases} \Delta u + k^2 u = 0 & (\Omega) \\ \dfrac{\partial u}{\partial n} = 0 & (\partial\Omega) \end{cases}$$

Modal solutions in the undeformed waveguide ($H = 1$):

$$u(x, y) = \cos(n\pi y) e^{\pm i\beta_n x} \text{ with } \beta_n = \sqrt{k^2 - n^2\pi^2} \text{ and } n \in \mathbb{N}$$

Notations and equations

We consider a 2D acoustic waveguide with rigid boundaries:



Time-harmonic equations ($k = \omega/c$):
(u is the pressure field)

$$\begin{cases} \Delta u + k^2 u = 0 & (\Omega) \\ \dfrac{\partial u}{\partial n} = 0 & (\partial\Omega) \end{cases}$$

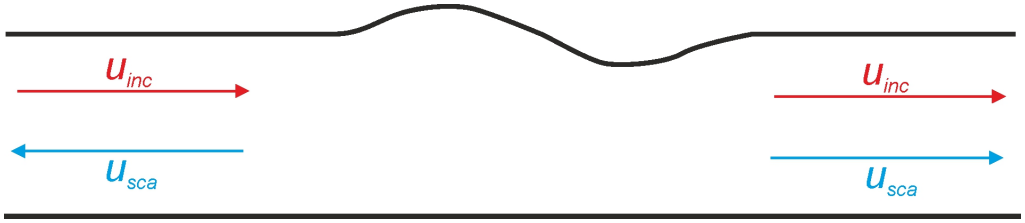
Modal solutions in the undeformed waveguide ($H = 1$):

$$u(x, y) = \cos(n\pi y) e^{\pm i\beta_n x} \text{ with } \beta_n = \sqrt{k^2 - n^2\pi^2} \text{ and } n \in \mathbb{N}$$

Key point

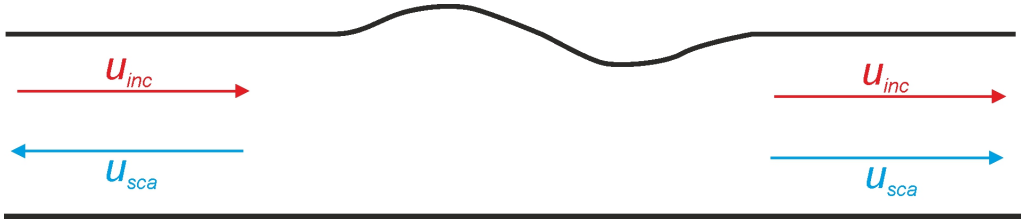
There are only a finite number of propagative modes ($\beta_n \in \mathbb{R}$).
Other modes are evanescent ($\beta_n \in i\mathbb{R}$).

The scattering problem



The total field u is such that $u = u_{inc} + u_{sca}$ where

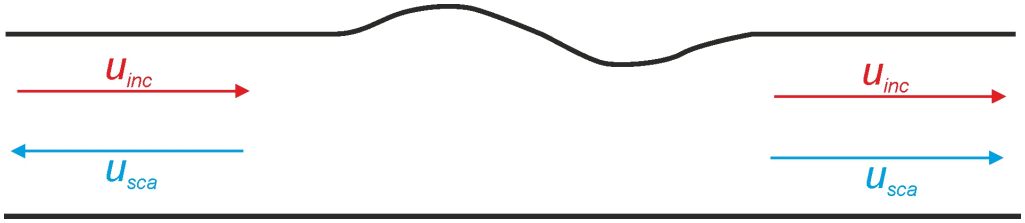
The scattering problem



The total field u is such that $u = u_{inc} + u_{sca}$ where

- the incident field u_{inc} is a propagative mode,

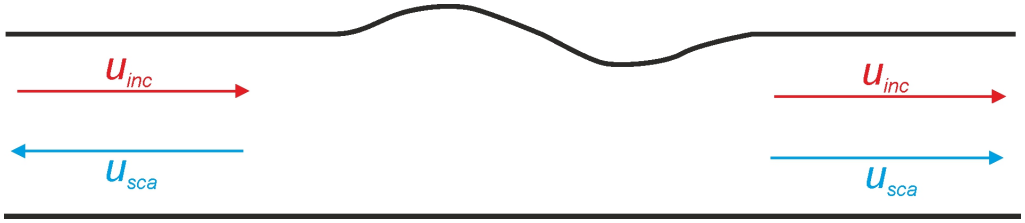
The scattering problem



The total field u is such that $u = u_{inc} + u_{sca}$ where

- the incident field u_{inc} is a propagative mode,
- the scattered field u_{sca} is outgoing.

The scattering problem



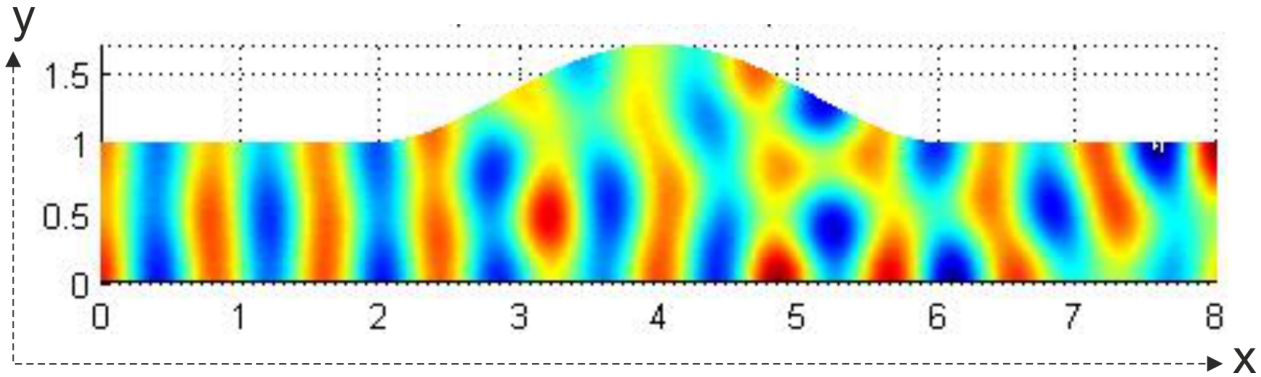
The total field u is such that $u = u_{inc} + u_{sca}$ where

- the incident field u_{inc} is a propagative mode,
- the scattered field u_{sca} is outgoing.

Invisibility

The deformation is invisible at the frequency k iff, for all incident waves, the scattered field u_{sca} vanishes exponentially at infinity.

The scattering problem: numerical solution



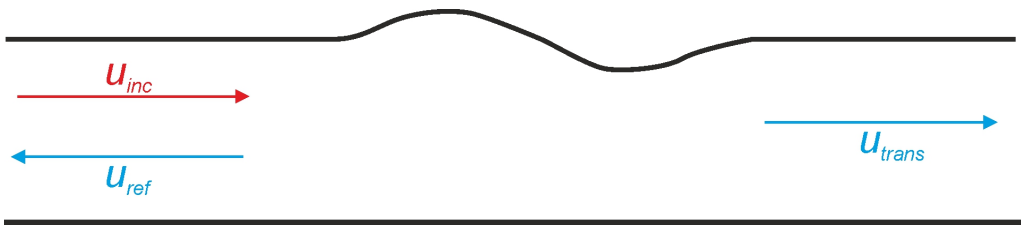
We use a multimodal approach (see Hazard and Lunéville, IMA, 2008):

Hybrid discretization:

- Modal expansion in y, at each x,
- Finite Element in x, for the modal amplitudes.

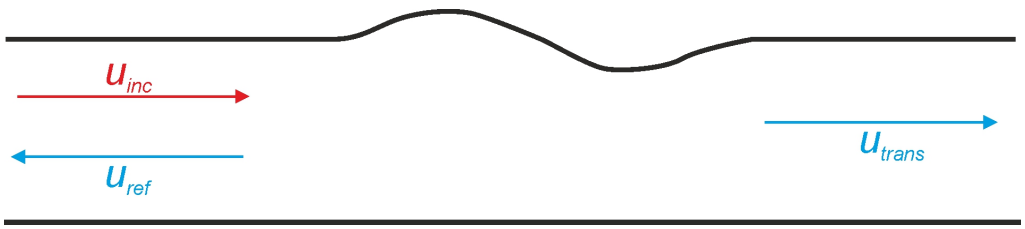
Main advantage: no mesh ! And therefore, no need of remeshing when changing the form.

The scattering problem - the single-mode case.



Suppose $0 < k < \pi$ so that there is only one propagative mode and consider first a rightgoing mode $u_{inc} = e^{ikx}$.

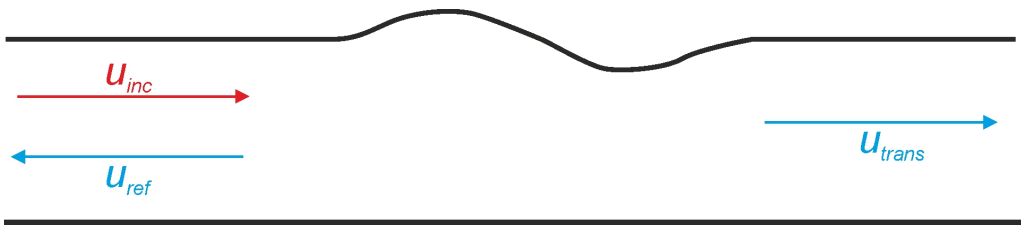
The scattering problem - the single-mode case.



Suppose $0 < k < \pi$ so that there is only one propagative mode and consider first a rightgoing mode $u_{inc} = e^{ikx}$. The total field u is such that

$$u(x,y) = \begin{cases} e^{ikx} + Re^{-ikx} + \sum \text{evanescent modes} & x \to -\infty, \\ Te^{ikx} + \sum \text{evanescent modes} & x \to +\infty. \end{cases}$$

The scattering problem - the single-mode case.



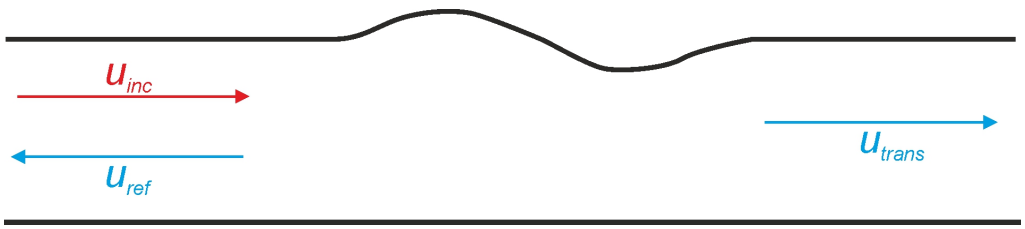
Suppose $0 < k < \pi$ so that there is only one propagative mode and consider first a rightgoing mode $u_{inc} = e^{ikx}$. The total field u is such that

$$u(x,y) = \begin{cases} e^{ikx} + Re^{-ikx} + \sum \text{evanescent modes} & x \to -\infty, \\ Te^{ikx} + \sum \text{evanescent modes} & x \to +\infty. \end{cases}$$

Invisibility

The deformation is invisible for a rightgoing incident wave iff $R = 0$ and $T = 1$.

The scattering problem - the single-mode case.



Suppose $0 < k < \pi$ so that there is only one propagative mode and consider first a rightgoing mode $u_{inc} = e^{ikx}$. The total field u is such that

$$u(x,y) = \begin{cases} e^{ikx} + Re^{-ikx} + \sum \text{evanescent modes} & x \to -\infty, \\ Te^{ikx} + \sum \text{evanescent modes} & x \to +\infty. \end{cases}$$

Invisibility

The deformation is invisible for a rightgoing incident wave iff $R = 0$ and $T = 1$. And it works also for leftgoing waves ! (consider $\overline{u}$)

The scattering problem - the single-mode case.

Suppose $0 < k < \pi$ and consider a rightgoing incident wave $u_{inc} = e^{ikx}$. The total field u is such that

$$u(x,y) = \begin{cases} e^{ikx} + Re^{-ikx} & x \to -\infty, \\ Te^{ikx} & x \to +\infty. \end{cases}$$

Invisibility

The deformation is invisible for a rightgoing and leftgoing incident waves iff $R = 0$ and $T = 1$.

The scattering problem - the single-mode case.

Suppose $0 < k < \pi$ and consider a rightgoing incident wave $u_{inc} = e^{ikx}$. The total field u is such that

$$u(x,y) = \begin{cases} e^{ikx} + Re^{-ikx} & x \to -\infty, \\ Te^{ikx} & x \to +\infty. \end{cases}$$

Energy conservation

$$|R|^2 + |T|^2 = 1$$

Invisibility

The deformation is invisible for a rightgoing and leftgoing incident waves iff $R = 0$ and $T = 1$.

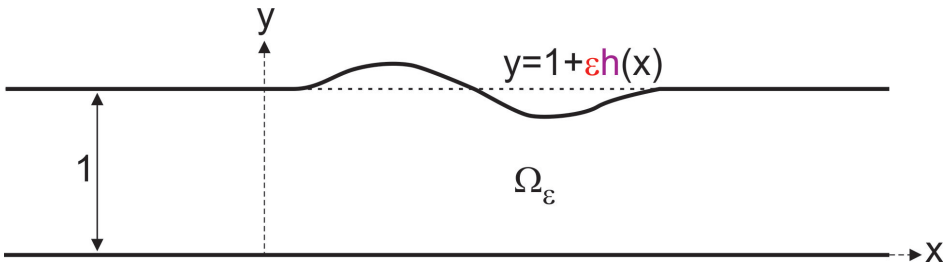
Near invisibility

The deformation is invisible up to a phase shift of the transmitted wave iff $R = 0$ (which implies $|T| = 1$).

Outline

The asymptotic expansion

We consider a small deformation of amplitude ε:



The solution u_ε of the scattering problem is such that:

$$u_\varepsilon = \begin{cases} e^{ikx} + R_\varepsilon e^{-ikx} & x \to -\infty \\ T_\varepsilon e^{ikx} & x \to +\infty \end{cases}$$

Ansatz:

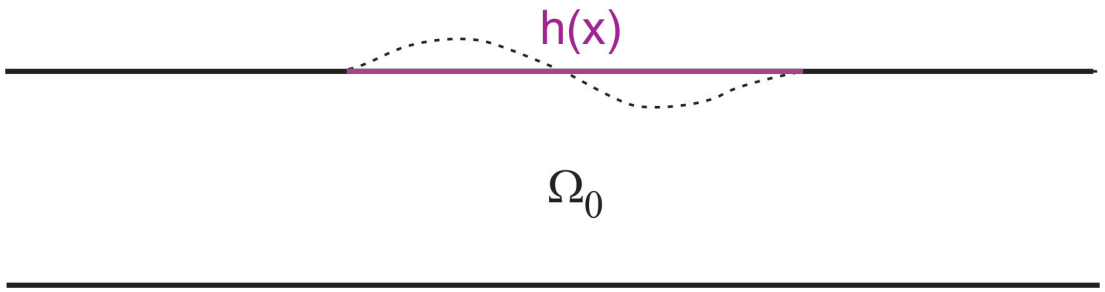
$$u_\varepsilon(x,y) = e^{ikx} + \varepsilon u'(x,y) + \text{high-order terms} \Rightarrow \begin{cases} R_\varepsilon = \varepsilon R' + \varepsilon^2 \tilde{R}(\varepsilon) \\ T_\varepsilon = 1 + \varepsilon T' + \varepsilon^2 \tilde{T}(\varepsilon) \end{cases}$$

with

$$|\tilde{R}(\varepsilon)| < C \text{ and } |\tilde{T}(\varepsilon)| < C$$

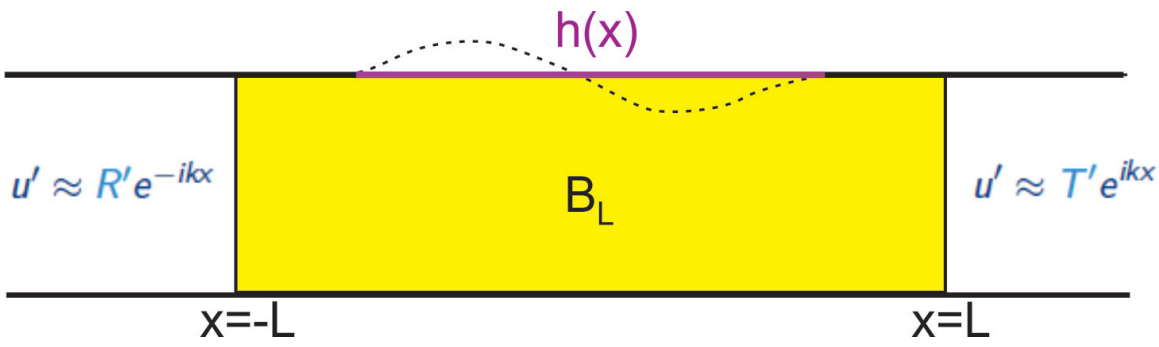
See Mazja-Nazarov-Plamenevski 91.

Computation of the first-order term u'



$$\begin{cases} \Delta u' + k^2 u' = 0 & (\Omega_0) \\ \partial_n u' = 0 & (y = 0) \\ \partial_n u' = g & (y = 1) \end{cases} \qquad \text{with } g(x) = ik\partial_x \left(h(x) e^{ikx} \right).$$

Computation of the first-order term u'



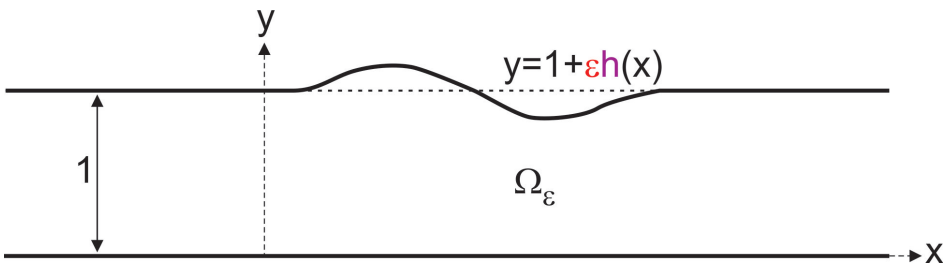
$$\begin{cases} \Delta u' + k^2 u' = 0 & (\Omega_0) \\ \partial_n u' = 0 & (y = 0) \\ \partial_n u' = g & (y = 1) \end{cases} \qquad \text{with } g(x) = ik\partial_x \left(h(x) e^{ikx} \right).$$

Then, from $\int_{\partial B_L} (\partial_n u' \, v - u' \, \partial_n v) = 0$ for $v = e^{\pm ikx}$ and $L \to +\infty$, we get

The first-order scattering coefficients

$$T' = -\frac{1}{2} \int \frac{dh}{dx}(x) dx = 0 \quad \text{and} \quad R' = ik \int h(x) e^{2ikx} dx$$

First-order expansion

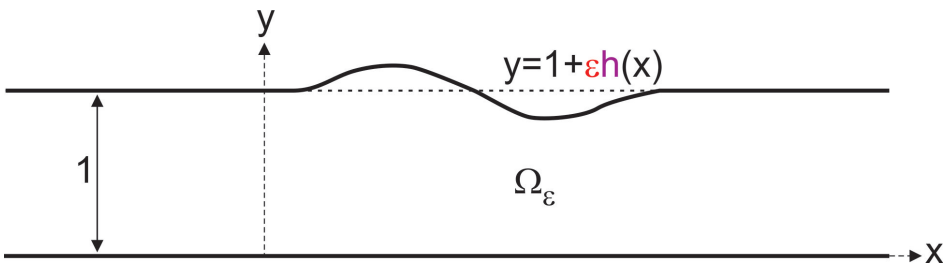


$$u_\varepsilon = \begin{cases} e^{ikx} + R_\varepsilon e^{-ikx} & x \to -\infty \\ T_\varepsilon e^{ikx} & x \to +\infty \end{cases}$$

Expansion of the scattering coefficients

$$\begin{aligned} R_\varepsilon &= ik\varepsilon \int h(x) e^{2ikx} dx + \varepsilon^2 \tilde{R}(\varepsilon) \\ T_\varepsilon &= 1 + \varepsilon^2 \tilde{T}(\varepsilon) \end{aligned}$$

First-order expansion



$$u_\varepsilon = \begin{cases} e^{ikx} + R_\varepsilon e^{-ikx} & x \to -\infty \\ T_\varepsilon e^{ikx} & x \to +\infty \end{cases}$$

Expansion of the scattering coefficients

$$\begin{aligned} R_\varepsilon &= \quad ik\varepsilon \int h(x) e^{2ikx} dx + \quad \varepsilon^2 \tilde{R}(\varepsilon) \\ T_\varepsilon &= 1 + \qquad\qquad\qquad\qquad\qquad \varepsilon^2 \tilde{T}(\varepsilon) \end{aligned}$$

Consequence: Invisibility up to order 2 in ε is achieved if

$$\int h(x) \cos(2kx) dx = \int h(x) \sin(2kx) dx = 0$$

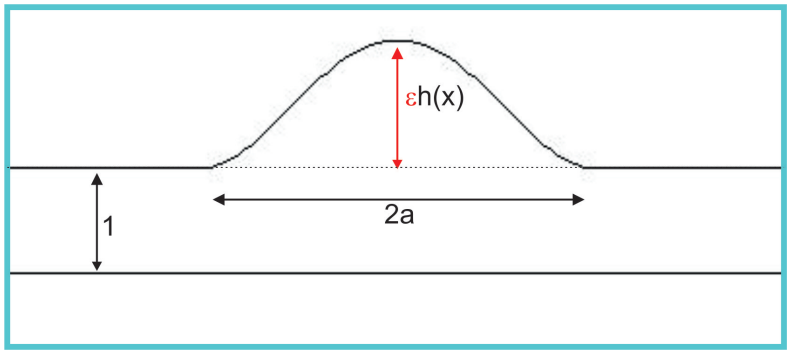
Numerical validation: the bump

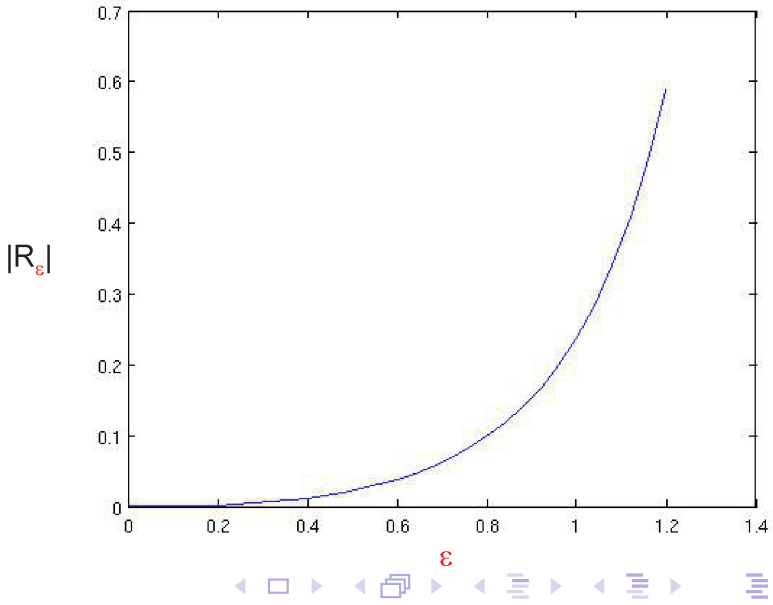
Consider

$$h(x) = \left(\left(\frac{x}{a} \right)^2 - 1 \right)^2 \chi_{(-a,a)}(x)$$

Then $\int h(x) \sin(2kx) dx = 0$ and $\exists a$ such that $\int h(x) \cos(2kx) dx = 0$.

$$\Rightarrow \exists a \text{ such that } \int h(x) e^{2ikx} dx = 0.$$





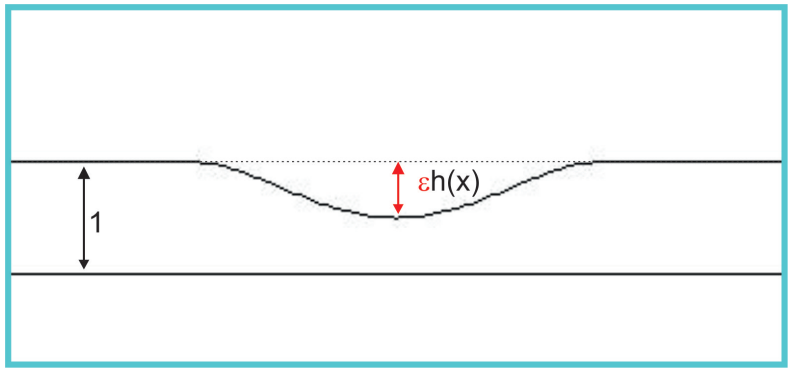
Numerical validation: the cavity

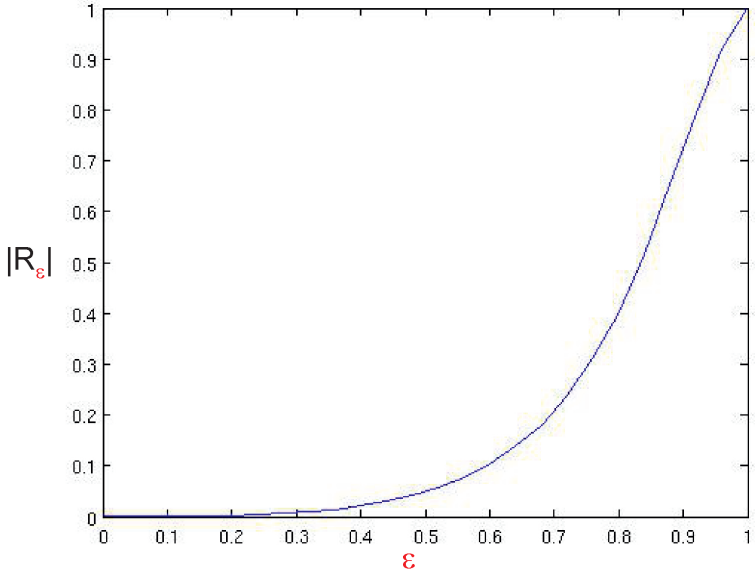
Consider

$$h(x) = -\left(\left(\frac{x}{a}\right)^2 - 1\right)^2 \chi_{(-a,a)}(x)$$

Then

$$\exists a \text{ such that } \int h(x) e^{2ikx} dx = 0.$$





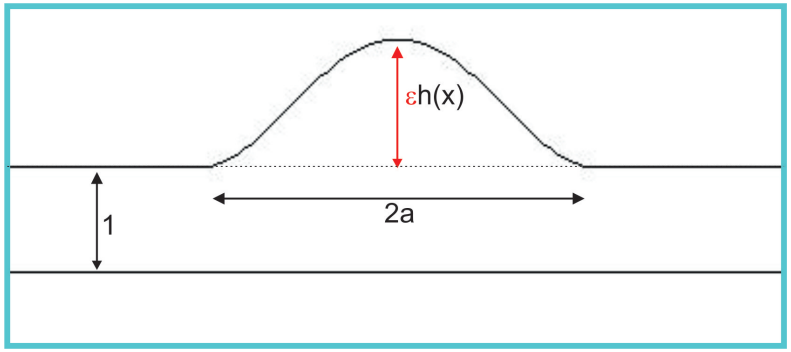
Numerical validation: the elementary bump

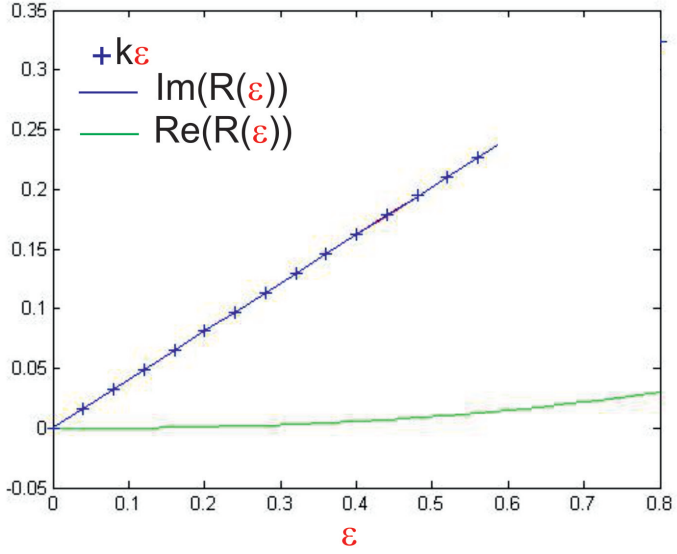
We can also chose $h(x) = \alpha \left(\left(\frac{x}{a}\right)^2 - 1 \right)^2 \chi_{(-a,a)}(x)$ with a and α such that

$$\int h(x) e^{2ikx} dx = 1$$

Then, the asymptotic theory says that:

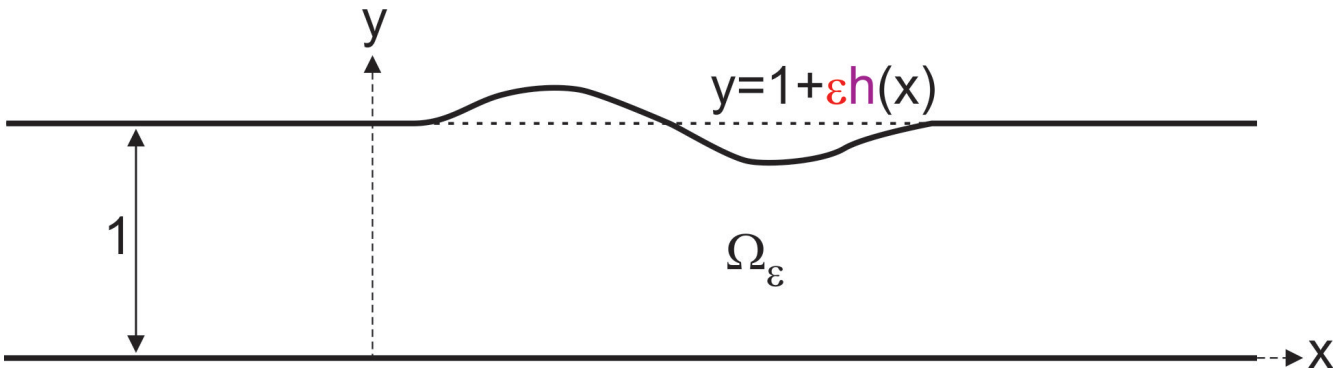
$$R_\varepsilon = ik\varepsilon + \varepsilon^2 \tilde{R}(\varepsilon)$$





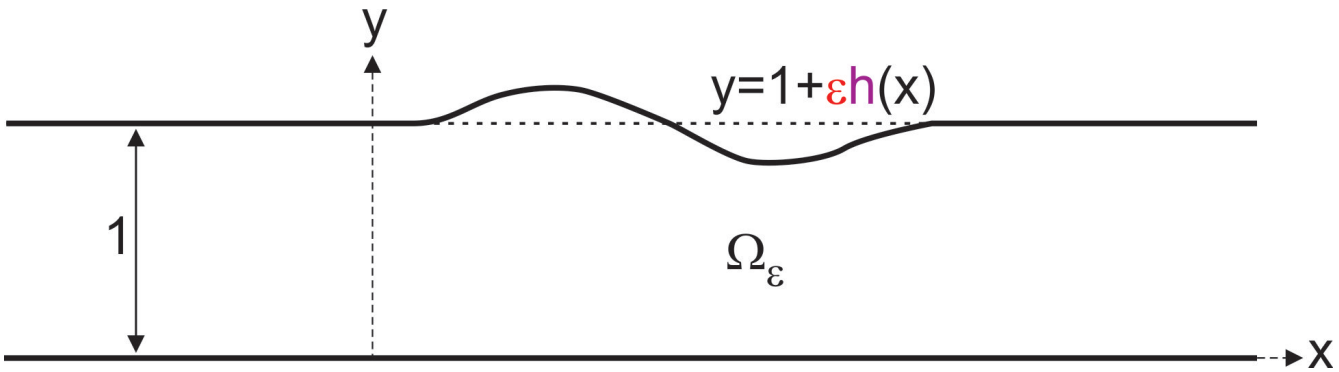
Outline

The objective



- For a given (small) ε and a given wavenumber k $(0 < k < \pi)$, find h such that the deformation is invisible, in the sense that $R_\varepsilon = 0$.

The objective



- For a given (small) ε and a given wavenumber k $(0 < k < \pi)$, find h such that the deformation is invisible, in the sense that $R_\varepsilon = 0$.
- More generally, for a given (small) ε and given wavenumbers k_1, k_2, $\cdots k_N$, $(0 < k_j < \pi,\ j = 1, \cdots, N)$, find h such that the deformation is invisible at these frequencies, in the sense that

$$R_\varepsilon(k_1) = R_\varepsilon(k_2) = \cdots R_\varepsilon(k_N) = 0$$

The deformation $y = 1+\varepsilon h(x)$

Search h of the form

$$h = h_0 + \tau_1 h_1 + \tau_2 h_2$$

where

- τ_1 and τ_2 are unknown real parameters
- h_0, h_1 and h_2 are given real valued functions such that:

$$\begin{cases} \displaystyle\int h_0(x)e^{2ikx}dx = 0 \\ \displaystyle\int h_1(x)e^{2ikx}dx = 1 \\ \displaystyle\int h_2(x)e^{2ikx}dx = i \end{cases}$$

The deformation $y = 1+\varepsilon h(x)$

Search h of the form

$$h = h_0 + \tau_1 h_1 + \tau_2 h_2$$

where

- τ_1 and τ_2 are unknown real parameters
- h_0, h_1 and h_2 are given real valued functions such that:

$$\begin{cases} \displaystyle\int h_0(x)e^{2ikx}dx = 0 \\ \displaystyle\int h_1(x)e^{2ikx}dx = 1 \\ \displaystyle\int h_2(x)e^{2ikx}dx = i \end{cases}$$

Remark: one can chose $h_2(x) = h_1(x - \pi/(4k))$.

The fixed-point equation

Equation of the boundary:

$$y = 1 + \varepsilon h(x) \text{ with}$$

$$h = h_0 + \tau_1 h_1 + \tau_2 h_2$$

$$\begin{cases} \displaystyle\int h_0(x) e^{2ikx} dx = 0 \\ \displaystyle\int h_1(x) e^{2ikx} dx = 1 \\ \displaystyle\int h_2(x) e^{2ikx} dx = i \end{cases}$$

The asymptotic theory says that:

$$\begin{aligned} R_\varepsilon &= ik\varepsilon \int h(x) e^{2ikx} dx + \varepsilon^2 \tilde{R}_\varepsilon \\ &= ik\varepsilon(\tau_1 + i\tau_2) + \varepsilon^2 \tilde{R}_\varepsilon \end{aligned}$$

The fixed-point equation

Equation of the boundary:

$$y = 1 + \varepsilon h(x) \text{ with}$$

$$h = h_0 + \tau_1 h_1 + \tau_2 h_2$$

$$\begin{cases} \displaystyle\int h_0(x) e^{2ikx} dx = 0 \\ \displaystyle\int h_1(x) e^{2ikx} dx = 1 \\ \displaystyle\int h_2(x) e^{2ikx} dx = i \end{cases}$$

The asymptotic theory says that:

$$\begin{aligned} R_\varepsilon &= ik\varepsilon \int h(x) e^{2ikx} dx + \varepsilon^2 \tilde{R}_\varepsilon \\ &= ik\varepsilon(\tau_1 + i\tau_2) + \varepsilon^2 \tilde{R}_\varepsilon \\ &= ik\varepsilon\tau + \varepsilon^2 \tilde{R}_\varepsilon(\varepsilon\tau) \end{aligned}$$

where we have set: $\tau = \tau_1 + i\tau_2 \in \mathbb{C}$.

The fixed-point equation

Equation of the boundary:

$$y = 1 + \varepsilon h(x) \text{ with}$$

$$h = h_0 + \tau_1 h_1 + \tau_2 h_2$$

$$\begin{cases} \displaystyle\int h_0(x) e^{2ikx} dx = 0 \\ \displaystyle\int h_1(x) e^{2ikx} dx = 1 \\ \displaystyle\int h_2(x) e^{2ikx} dx = i \end{cases}$$

Asymptotic theory: $R_\varepsilon = ik\varepsilon\tau + \varepsilon^2 \tilde{R}_\varepsilon(\varepsilon\tau)$ where $\tau = \tau_1 + i\tau_2$

The fixed-point equation

Equation of the boundary:

$$y = 1 + \varepsilon h(x) \text{ with}$$

$$h = h_0 + \tau_1 h_1 + \tau_2 h_2$$

$$\begin{cases} \displaystyle\int h_0(x) e^{2ikx} dx = 0 \\ \displaystyle\int h_1(x) e^{2ikx} dx = 1 \\ \displaystyle\int h_2(x) e^{2ikx} dx = i \end{cases}$$

Asymptotic theory: $R_\varepsilon = ik\varepsilon\tau + \varepsilon^2 \tilde{R}_\varepsilon(\varepsilon\tau)$ where $\tau = \tau_1 + i\tau_2$

Equation of (near-)invisibility

$$R_\varepsilon = 0 \Longleftrightarrow \tau = F_\varepsilon(\tau)$$

where $F_\varepsilon(\tau) = \dfrac{i\varepsilon}{k} \tilde{R}_\varepsilon(\varepsilon\tau) = \tau + \dfrac{i}{k\varepsilon} R_\varepsilon$

The fixed-point equation

Equation of the boundary:

$$y = 1 + \varepsilon h(x) \text{ with}$$

$$h = h_0 + \tau_1 h_1 + \tau_2 h_2$$

$$\begin{cases} \displaystyle\int h_0(x) e^{2ikx} dx = 0 \\ \displaystyle\int h_1(x) e^{2ikx} dx = 1 \\ \displaystyle\int h_2(x) e^{2ikx} dx = i \end{cases}$$

Asymptotic theory: $R_\varepsilon = ik\varepsilon\tau + \varepsilon^2 \tilde{R}_\varepsilon(\varepsilon\tau)$ where $\tau = \tau_1 + i\tau_2$

Equation of (near-)invisibility

$$R_\varepsilon = 0 \Longleftrightarrow \tau = F_\varepsilon(\tau)$$

where $F_\varepsilon(\tau) = \dfrac{i\varepsilon}{k} \tilde{R}_\varepsilon(\varepsilon\tau) = \tau + \dfrac{i}{k\varepsilon} R_\varepsilon$

Theorem

For ε small enough, F_ε is a contraction and $\exists! \tau \in \mathbb{C}$ such that $\tau = F_\varepsilon(\tau)$.

The fixed-point algorithm

- Initialization:
  - Choose h_0, h_1 and h_2 and fix ε
  - $\tau = 0$
  - $n = 0$
- Loop:
  1. Compute R_ε by solving the scattering problem for the deformation

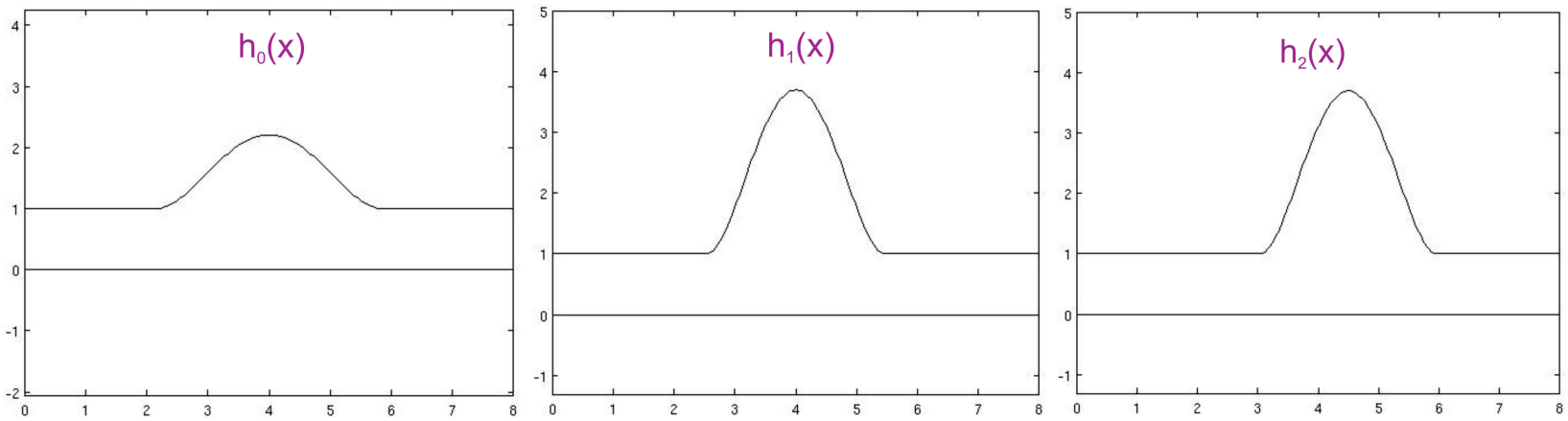
$$y = 1 + \varepsilon \left(h_0 + \tau_1 h_1 + \tau_2 h_2 \right)$$

  2. Update τ: $\quad \tau := \tau + \frac{i}{k\varepsilon} R_\varepsilon$
  3. Update n: $\quad n := n + 1$
  4. If $n < N_{max}$ goto 1.

Outline

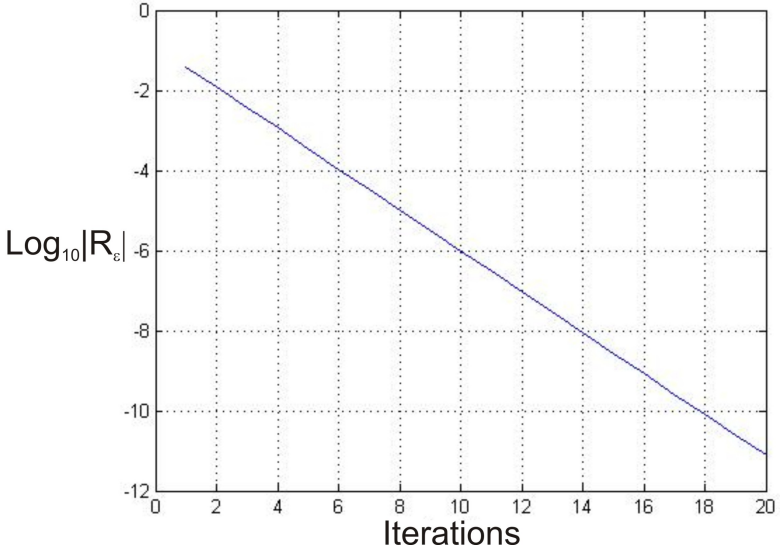
The form functions

We take the following functions:

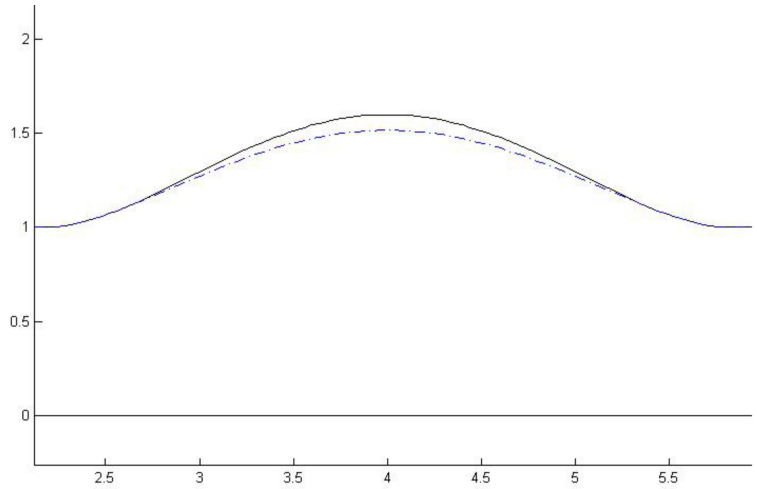


Results for the bump and for $\varepsilon = 0.6$

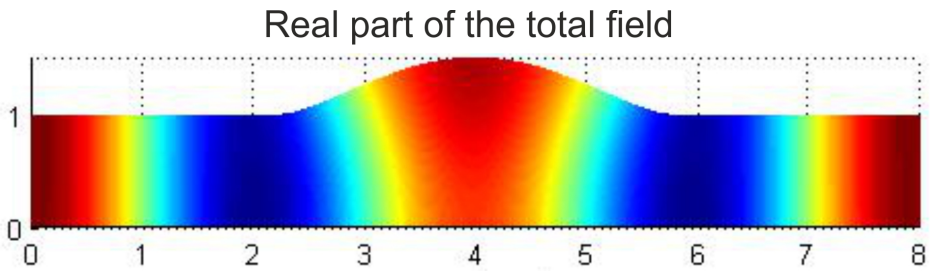
The convergence is monotonic as expected:

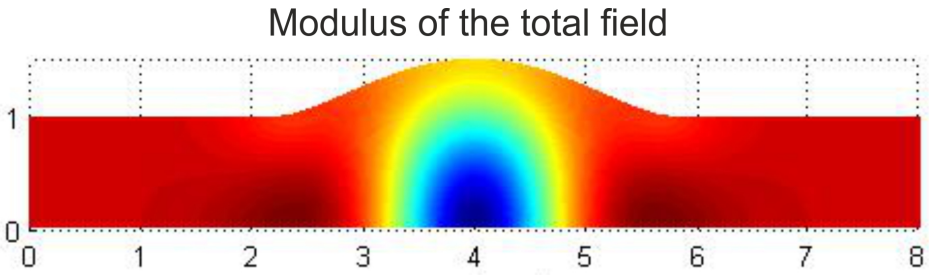


The bump is only slightly modified:

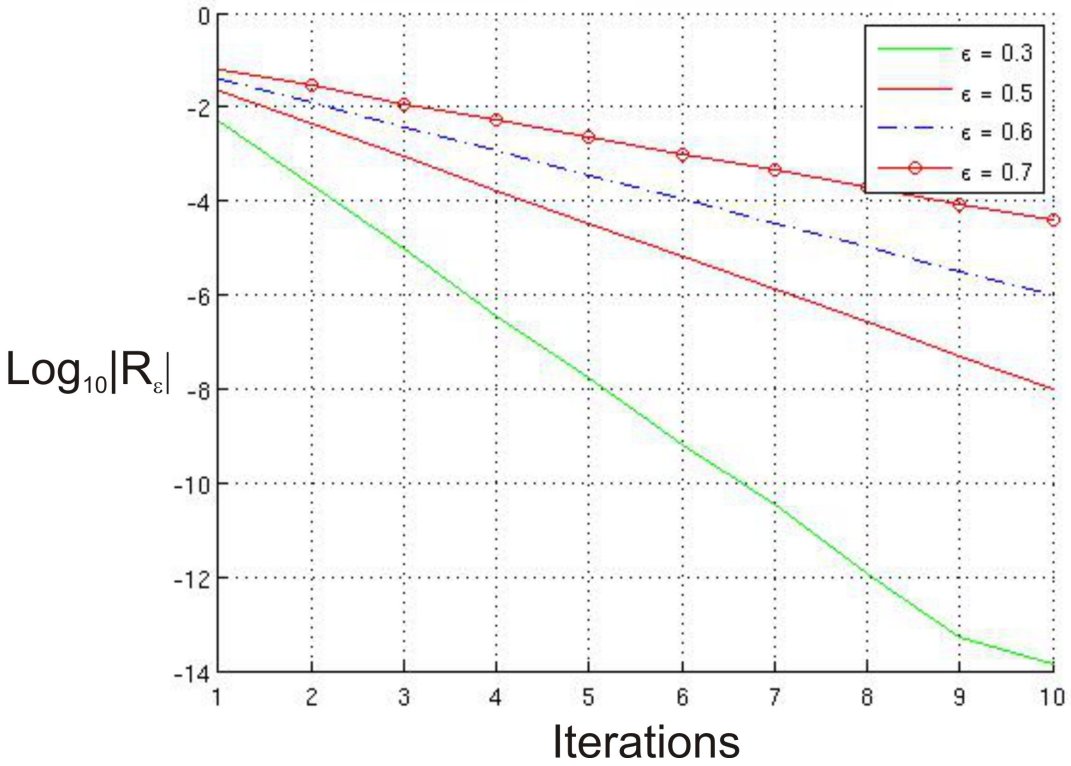


The bump is indeed invisible:





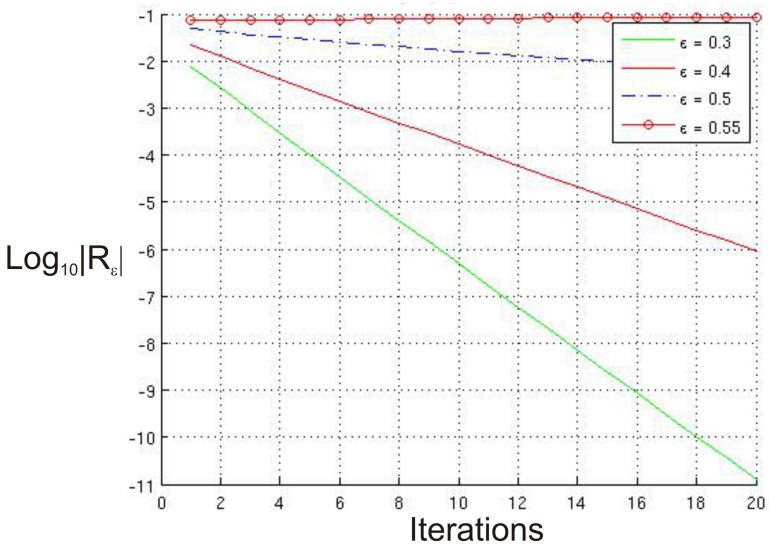
Influence of ε for the bump



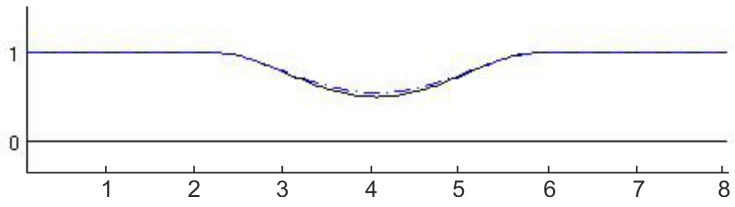
The results confirm the expected convergence in ε^{2n}.

Results for the cavity

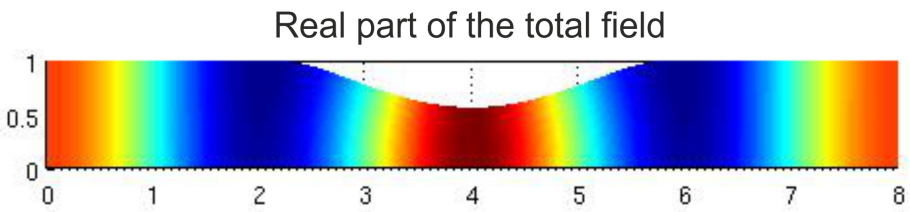
Divergence occurs at $\varepsilon \approx 0.5$ compared to $\varepsilon \approx 0.75$ for the bump.

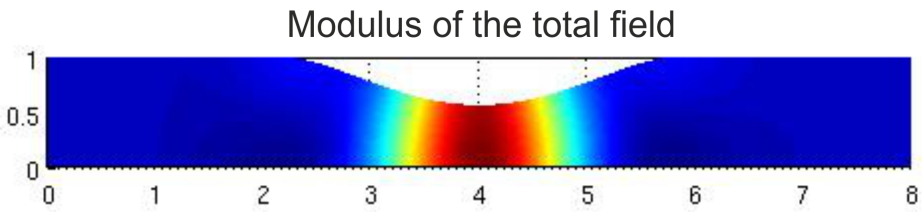


At $\varepsilon = 0.5$, the cavity is only slightly modified:



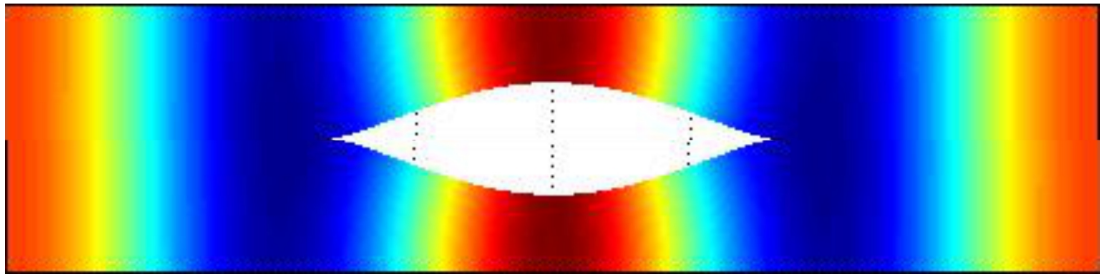
The cavity is indeed invisible:





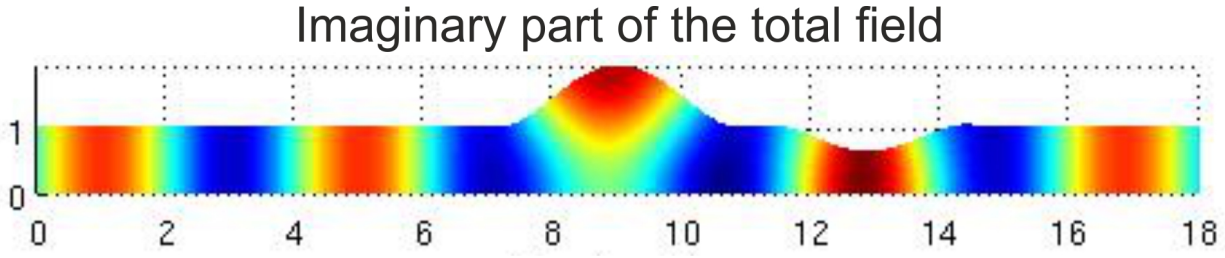
Invisible obstacle

Symmetrizing a cavity gives an obstacle !!

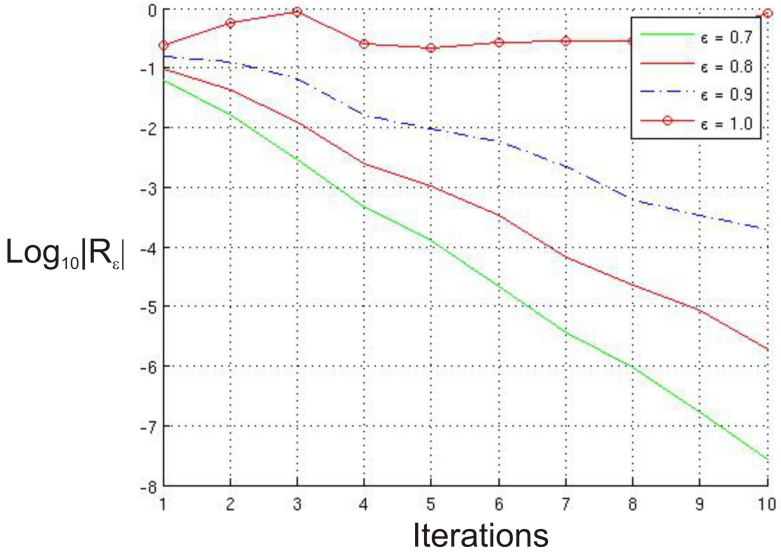


Making invisible by external deformations

The main form h_0 and the corrections h_1 and h_2 may have disjoint supports:



but the convergence is more chaotic...



The multifrequency case

We want a deformation invisible at the two frequencies $k = k_1$ and $k = k_2$ with $0 < k_1 < k_2 < \pi$.

Equation of the boundary:

$$y = 1 + \varepsilon h(x) \quad \text{with} \quad h = h_0 + \sum_{j=1}^{2} \left(\tau_1^{(j)} h_1^{(j)} + \tau_2^{(j)} h_2^{(j)} \right)$$

$$\text{with} \quad \begin{cases} \displaystyle\int h_0(x) e^{2ik_j x} dx = 0 \\ \displaystyle\int h_1^{(j)}(x) e^{2ik_\ell x} dx = \delta_{j\ell} \\ \displaystyle\int h_2^{(j)}(x) e^{2ik_\ell x} dx = i\delta_{j\ell} \end{cases} \qquad \text{for } j, \ell = 1, 2.$$

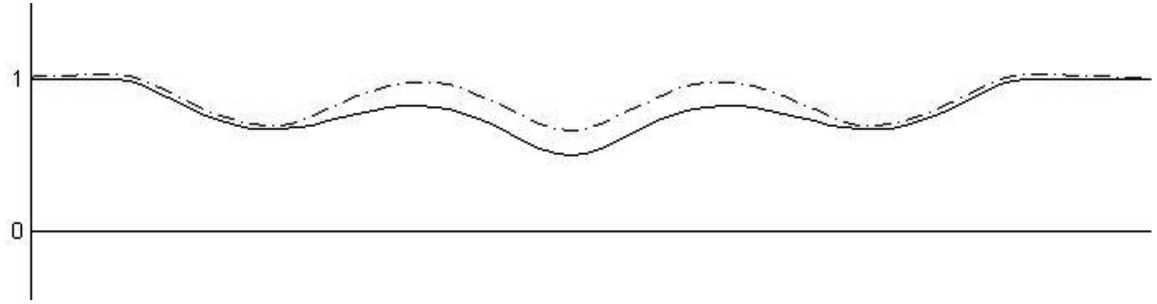
The multifrequency case

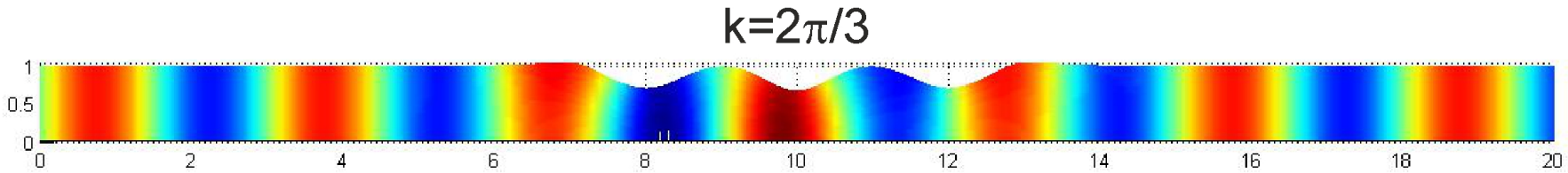
The form h_0 and the corrections $h_i^{(j)}$ are more oscillatory.

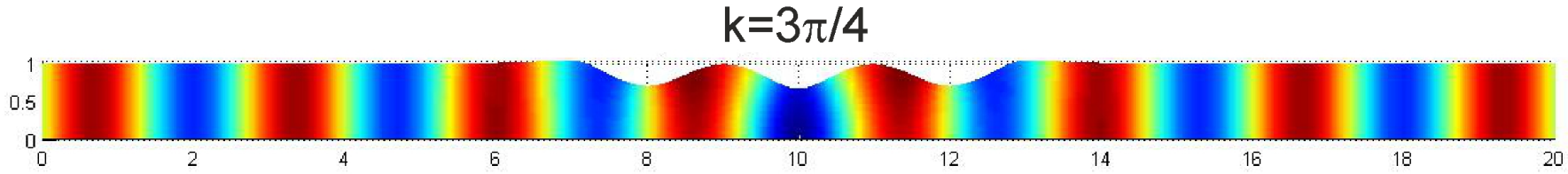
The multifrequency case

The form h_0 and the corrections $h_i^{(j)}$ are more oscillatory.

This is a result for $\varepsilon = 0.5$, $k_1 = 2\pi/3$ and $k_2 = 3\pi/4$:





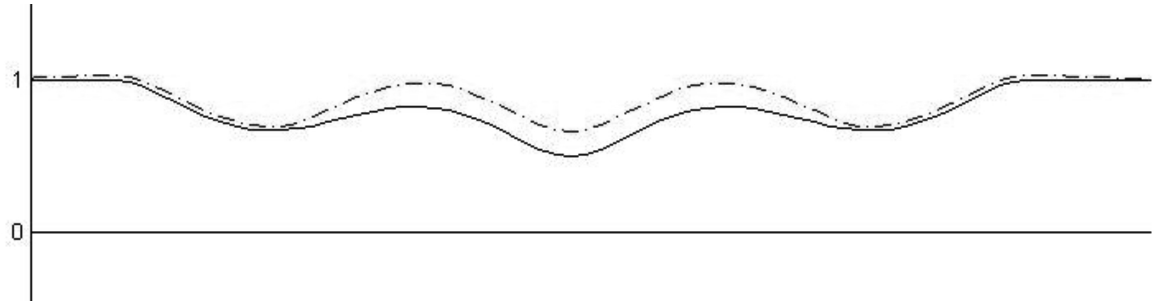


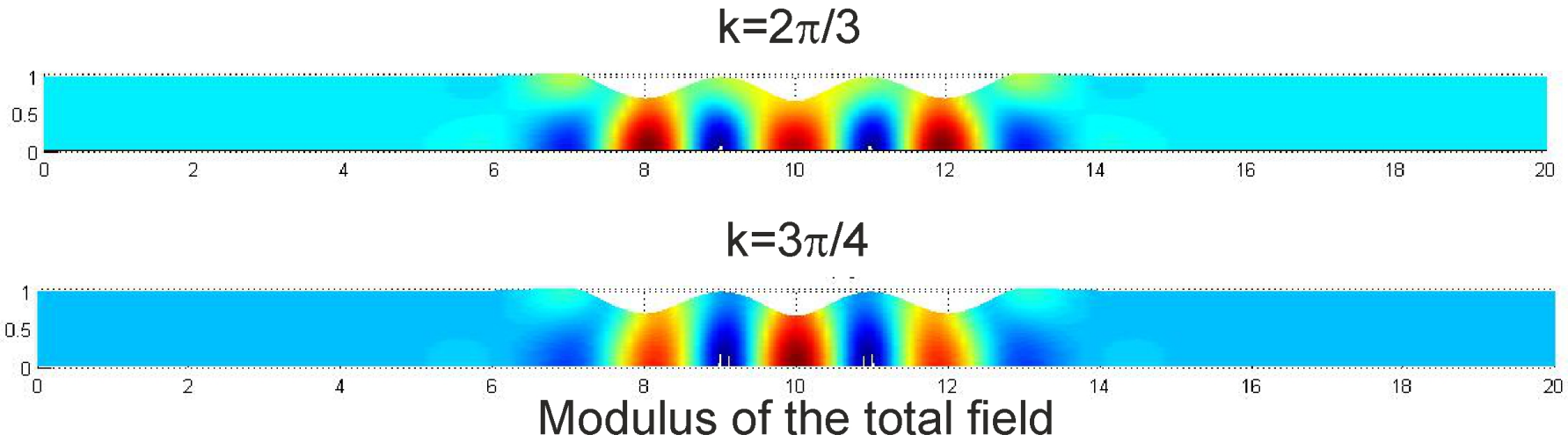
Imaginary part of the total field

The multifrequency case

The form h_0 and the corrections $h_i^{(j)}$ are more oscillatory.

This is a result for $\varepsilon = 0.5$, $k_1 = 2\pi/3$ and $k_2 = 3\pi/4$:





The multimode case

For $(N-1)\pi < k < N\pi$, the scattering properties of the deformation are summarized in the unitary symmetric scattering matrix S:

$$S = \left(\begin{array}{cccc|cccc} R^-_{00} & R^-_{01} & R^-_{02} & \cdots & T^-_{00} & T^-_{01} & T^-_{02} & \cdots \\ R^-_{10} & R^-_{11} & R^-_{12} & \cdots & T^-_{10} & T^-_{11} & T^-_{12} & \cdots \\ R^-_{20} & R^-_{21} & R^-_{22} & \cdots & T^-_{20} & T^-_{21} & T^-_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline T^+_{00} & T^+_{01} & T^+_{02} & \cdots & R^+_{00} & R^+_{01} & R^+_{02} & \cdots \\ T^+_{10} & T^+_{11} & T^+_{12} & \cdots & R^+_{10} & R^+_{11} & R^+_{12} & \cdots \\ T^+_{20} & T^+_{21} & T^+_{22} & \cdots & R^+_{20} & R^+_{21} & R^+_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right)$$

The multimode case

Near invisibility is achieved if all coefficients in the blue part vanish:

$$S = \left(\begin{array}{cccc|cccc} R^-_{00} & R^-_{01} & R^-_{02} & \cdots & T^-_{00} & T^-_{01} & T^-_{02} & \cdots \\ R^-_{10} & R^-_{11} & R^-_{12} & \cdots & T^-_{10} & T^-_{11} & T^-_{12} & \cdots \\ R^-_{20} & R^-_{21} & R^-_{22} & \cdots & T^-_{20} & T^-_{21} & T^-_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline T^+_{00} & T^+_{01} & T^+_{02} & \cdots & R^+_{00} & R^+_{01} & R^+_{02} & \cdots \\ T^+_{10} & T^+_{11} & T^+_{12} & \cdots & R^+_{10} & R^+_{11} & R^+_{12} & \cdots \\ T^+_{20} & T^+_{21} & T^+_{22} & \cdots & R^+_{20} & R^+_{21} & R^+_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right)$$

The multimode case

Near invisibility is achieved if all coefficients in the blue part vanish:

$$S = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & \cdots & T_{00}^{-} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & T_{11}^{-} & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & T_{22}^{-} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline T_{00}^{+} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & T_{11}^{+} & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & T_{22}^{+} & \cdots & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right)$$

The multimode case

Near invisibility is achieved if all coefficients in the blue part vanish:

$$S = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & \cdots & T^-_{00} & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & T^-_{11} & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & T^-_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \hline T^+_{00} & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & T^+_{11} & 0 & \cdots & 0 & 0 & 0 & \cdots \\ 0 & 0 & T^+_{22} & \cdots & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}\right)$$

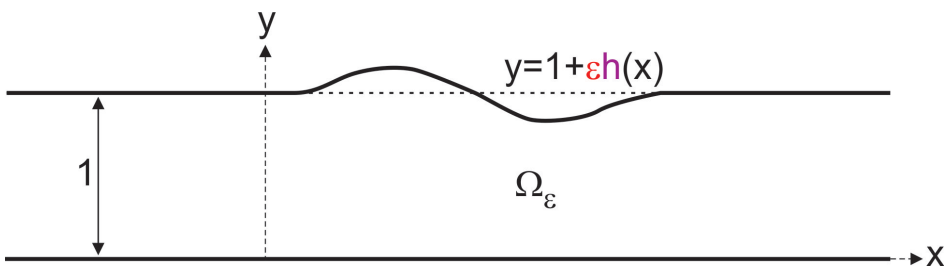
As S is symmetric and unitary:

- This implies

$$|T^{\pm}_{mm}| = 1 \quad \forall m = 0, 1 \cdots, n-1$$

- It is sufficient to cancel N^2 complex coefficients.

The multimode case: asymptotic formulae



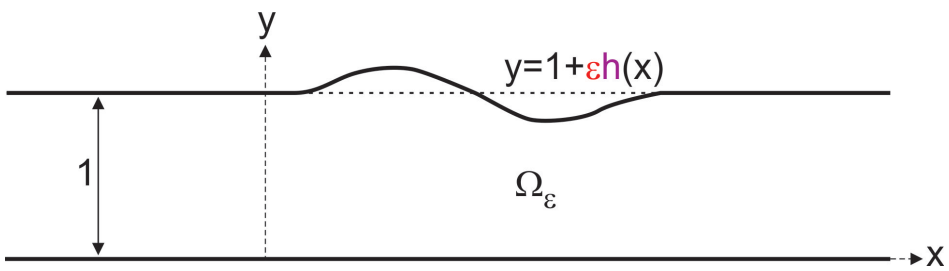
Expansion of the scattering coefficients

$$R^+_{m\ell,\varepsilon} = \varepsilon\, r_{m\ell} \int h(x) e^{i(\beta_\ell+\beta_m)x} dx + \varepsilon^2 \tilde{R}^+_{m\ell}(\varepsilon)$$

$$T^+_{m\ell,\varepsilon} = 1 + \varepsilon\, t_{m\ell} \int h(x) e^{i(\beta_\ell-\beta_m)x} dx + \varepsilon^2 \tilde{T}^+_{m\ell}(\varepsilon)$$

with $\beta_m = \sqrt{k^2 - m^2\pi^2}$

The multimode case: asymptotic formulae



Expansion of the scattering coefficients

$$R^+_{m\ell,\varepsilon} = \varepsilon\, r_{m\ell} \int h(x) e^{i(\beta_\ell+\beta_m)x} dx + \varepsilon^2 \tilde{R}^+_{m\ell}(\varepsilon)$$

$$T^+_{m\ell,\varepsilon} = 1 + \varepsilon\, t_{m\ell} \int h(x) e^{i(\beta_\ell-\beta_m)x} dx + \varepsilon^2 \tilde{T}^+_{m\ell}(\varepsilon)$$

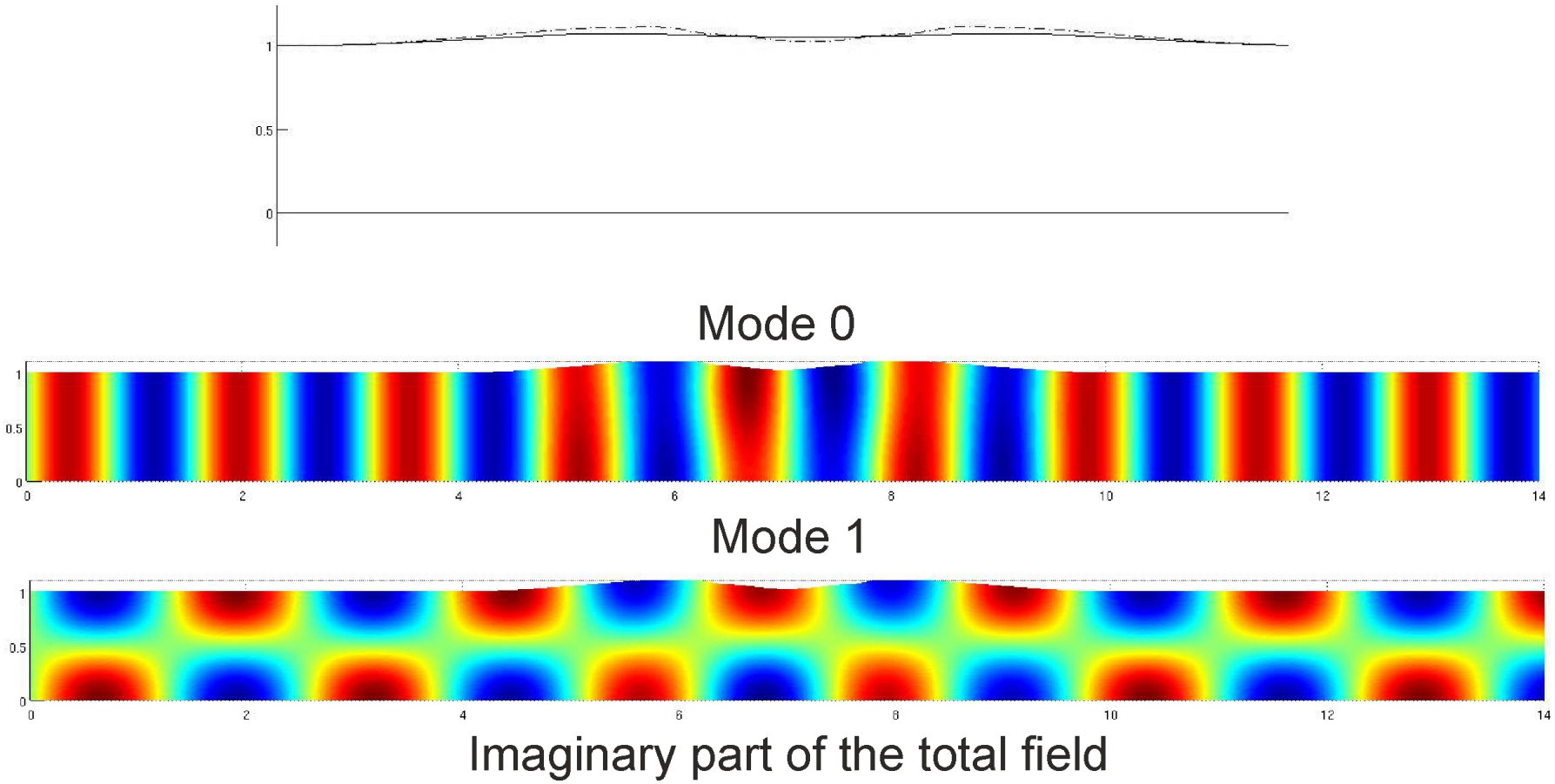
with $\beta_m = \sqrt{k^2 - m^2\pi^2}$

Consequence: Invisibility up to order 2 in ε is achieved if

$$\int h(x) e^{i\kappa x} dx = 0 \text{ for } \kappa = \beta_\ell \pm \beta_m,\ 0 \le m, \ell < N$$

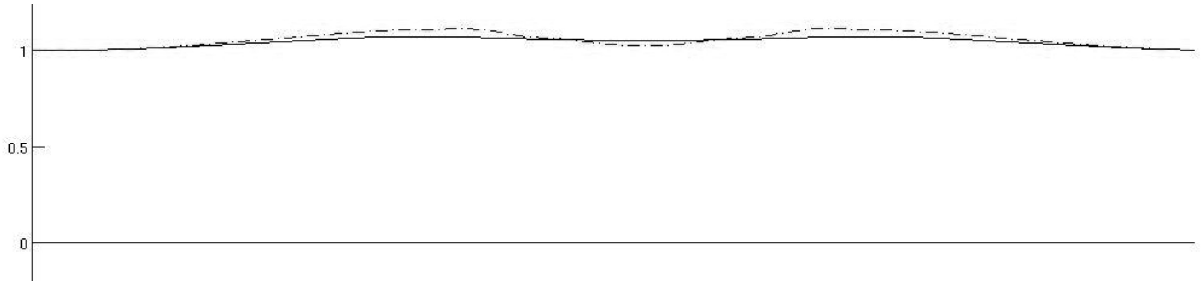
The two-mode case: a result

When $\pi < k < 2\pi$: we have to build 9 form functions satisfying 8 constraints and to solve a fixed-point equation for 8 coefficients τ_j !



The two-mode case: a result

When $\pi < k < 2\pi$: we have to build 9 form functions satisfying 8 constraints and to solve a fixed-point equation for 8 coefficients τ_j !



Compared to the monomode case, the behavior of the auxiliary form functions deteriorates: in particular, some of them present a large amplitude, which forces ε to be small...

Perspectives

1. How to increase ε in the previous results...? What is the best choice for the elementary form functions ?

Perspectives

1. How to increase ε in the previous results...? What is the best choice for the elementary form functions ?
2. Extension to the case of water waves (with J. Taskinen) : only 1 propagative mode and perfect invisibility can be achieved.

Perspectives

1. How to increase ε in the previous results...? What is the best choice for the elementary form functions ?
2. Extension to the case of water waves (with J. Taskinen) : only 1 propagative mode and perfect invisibility can be achieved.
3. A theoretical question is: for a given deformation, is the spectrum of invisible frequencies discrete ?