Time Domain Electromagnetic Scattering from a Penetrable Medium

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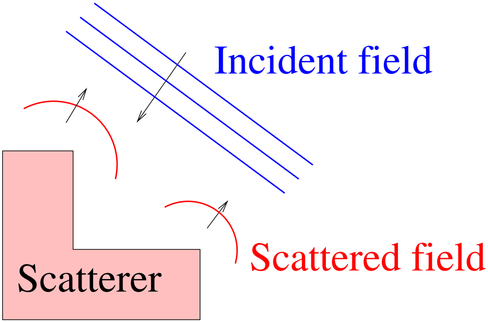
With best wishes to Martin Costabel on the occasion of his 65th birthday.

Outline

This talk is about time domain electromagnetic scattering from a penetrable homogeneous obstacle:

- Model problem.
- Time domain boundary integral equations (TDBIE).
- Numerical results of Wang and Weile.
- Mapping properties and error estimates.

General setup for scattering



Forward Problem: Given the physical properties of the scatterer, incident field, and background characteristics, predict the scattered field.

The field can penetrate the scatterer.

Maxwell's Equations: Penetrable Medium

Let Ω^- denote a bounded domain with connected complement denoted Ω^+, Lipschitz polyhedral boundary Γ and outward normal $\boldsymbol{\nu}$.

Suppose $\mathcal{E}^{inc}$ is a given incident wave, we seek the electric and magnetic fields $\mathcal{E}^{\pm} := \mathcal{E}^{\pm}(t, \mathbf{x})$ and $\mathcal{H}^{\pm} := \mathcal{H}^{\pm}(t, \mathbf{x})$ that satisfy

$$\left.\begin{aligned} \frac{1}{c^{\pm}}\frac{\partial \mathcal{E}^{\pm}}{\partial t} - \mathbf{curl}\,\mathcal{H}^{\pm} = 0 \\ \frac{1}{c^{\pm}}\frac{\partial \mathcal{H}^{\pm}}{\partial t} + \mathbf{curl}\,\mathcal{E}^{\pm} = 0 \end{aligned}\right\} \text{ in } (0, T) \times \Omega^{\pm},$$

$$\left.\begin{aligned} \mathcal{E}^- \times \mathbf{n} - (\mathcal{E}^+ + \mathcal{E}^{\text{inc}}) \times \mathbf{n} = 0 \\ \alpha\mathcal{H}^- \times \mathbf{n} - (\mathcal{H}^+ + \mathcal{H}^{\text{inc}}) \times \mathbf{n} = 0 \end{aligned}\right\} \text{ in } (0, T) \times \Gamma,$$

$$\mathcal{E}^{\pm}(t, \mathbf{x}) = \mathcal{H}^{\pm}(t, \mathbf{x}) = 0 \text{ for } t \le 0 \text{ and } \mathbf{x} \in \Omega^{\pm}.$$

where $\alpha = \sqrt{\frac{\epsilon^-\mu^+}{\epsilon^+\mu^-}}$

Methods in the Time Domain

- Useful review by Costabel[1]
- Space-Time Galerkin methods[2]
- We consider an alternative approach called Convolution Quadrature (CQ) due to Lubich[3]
- There are other approaches, for example the convolution approach of Davies and Duncan[4]

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[1] M. Costabel: Fundamentals, Encyclopedia of Computational Mechanics, vol. 1, chapter: Time-dependent Problems with the Boundary Integral Equation Method. John Wiley & Sons (2004)

[2] I. Terrasse, PhD Thesis, Ecole Polytechnique, France (1993), A. Bachelot, L. Bounhoure, A. Pujols: Numer. Math. 89, 257-306 (2001)

[3] C. Lubich, *Numer. Math.* **67**, 365-389 (1994).

[4] P. Davies and D. Duncan, SIAM J. Sci. Comput. **35** (2013), pp. B43-B61

Laplace domain equations

Classically CQ is analyzed in Laplace domain. Let $\mathbf{E}^{\pm}$ and $\mathbf{H}^{\pm}$ denote the temporal Laplace transforms of $\mathcal{E}^{\pm}$ and $\mathcal{H}^{\pm}$ respectively. They satisfy

$$\left.\begin{aligned} \frac{s}{c^{\pm}}\mathbf{E}^{\pm} - \mathbf{curl}\,\mathbf{H}^{\pm} &= 0 \\ \frac{s}{c^{\pm}}\mathbf{H}^{\pm} + \mathbf{curl}\,\mathbf{E}^{\pm} &= 0 \end{aligned}\right\} \quad \text{in} \quad \Omega^{\pm},$$

$$\left.\begin{aligned} \mathbf{E}^{-} \times \mathbf{n} - (\mathbf{E}^{+} + \mathbf{E}^{\text{inc}}) \times \mathbf{n} &= 0 \\ \alpha\mathbf{H}^{-} \times \mathbf{n} - (\mathbf{H}^{+} + \mathbf{H}^{\text{inc}}) \times \mathbf{n} &= 0 \end{aligned}\right\} \quad \text{on} \quad \Gamma,$$

where we assume $\Re s > \sigma > 0$.

Define the interior and exterior electric and magnetic currents by

$$\mathbf{j}^{\pm} = \mp \mathbf{H}^{\pm} \times \mathbf{n}, \qquad \mathbf{m}^{\pm} = \mp \mathbf{E}^{\pm} \times \mathbf{n}.$$

and the fundamental solution for the Helmholtz equation with wave speeds $c^{\pm}$:

$$G^{\pm}(\mathbf{x}, \mathbf{y}, s) = \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} e^{-s|\mathbf{x}-\mathbf{y}|/c^{\pm}}.$$

Define also

$$(S^{\pm}(s)\mathbf{p})(\mathbf{x}) := \int_{\Gamma} G^{\pm}(\mathbf{x}, \mathbf{y}, s)\mathbf{p}(\mathbf{y})\, d\Gamma(\mathbf{y}).$$

Let $k^{\pm} = s/c^{\pm}$ and set

$$\mathbb{S}^{\pm}_{\mathbf{E}}\mathbf{j} := -k^{\pm} S^{\pm}\mathbf{j} + \frac{1}{k^{\pm}} \nabla S^{\pm} \operatorname{div} \mathbf{j}, \text{ and } \mathbb{S}^{\pm}_{\mathbf{H}}\mathbf{m} := \mathbf{curl}\, S^{\pm}\mathbf{m},$$

Standard Stratton-Chu representation formulae then yield, away from Γ,

$$\begin{aligned} \mathbf{E}^{\pm} &= \mathbb{S}^{\pm}_{\mathbf{E}} \mathbf{j}^{\pm} + \mathbb{S}^{\pm}_{\mathbf{H}} \mathbf{m}^{\pm}, \\ \mathbf{H}^{\pm} &= -\mathbb{S}^{\pm}_{\mathbf{E}} \mathbf{m}^{\pm} + \mathbb{S}^{\pm}_{\mathbf{H}} \mathbf{j}^{\pm}. \end{aligned}$$

Define the trace operator

$$\gamma^{\pm}_{D} \mathbf{u} = \mathbf{u}|_{\Omega^{\pm}} \times \mathbf{n}$$

then the jump conditions for the electric and magnetic potentials allow us to define

$$\begin{aligned} M^{\pm}(\mathbf{m}) &:= \gamma^{\pm}_{D} \mathbb{S}^{\pm}_{\mathbf{H}} \mathbf{m} \pm \frac{1}{2}\mathbf{m} = \int_{\Gamma} \mathbf{curl}\big(G^{\pm}(\mathbf{x}, \mathbf{y}, s)\mathbf{m}(\mathbf{y})\big) \times \mathbf{n} \, d\Gamma(\mathbf{y}), \\ C^{\pm}(\mathbf{j}) &:= \gamma_{D} \mathbb{S}^{\pm}_{\mathbf{E}} \mathbf{j} = \Big(- k^{\pm} S^{\pm} \mathbf{j} + \frac{1}{k^{\pm}} \nabla S^{\pm} \operatorname{div} \mathbf{j} \Big) \times \mathbf{n}. \end{aligned}$$

Using the jump conditions across Γ and the transmission conditions for the fields yields the problem of finding $(\mathbf{j}^-, \mathbf{m}^-)$ such that:

$$\begin{bmatrix} (C^- + \alpha C^+) & M^- + M^+ \\ M^- + M^+ & -(C^- + \frac{1}{\alpha} C^+) \end{bmatrix} \begin{bmatrix} \mathbf{j}^- \\ \mathbf{m}^- \end{bmatrix}$$

$$= \begin{bmatrix} C^+ & \frac{1}{2} I + M^+ \\ \frac{1}{\alpha}(\frac{1}{2} I + M^+) & -\frac{1}{\alpha} C^+ \end{bmatrix} \begin{bmatrix} \gamma_D \mathbf{H}^{inc} \\ \gamma_D \mathbf{E}^{inc} \end{bmatrix}$$

To derive the time dependent boundary integral equations we use the inverse Laplace transform. Let

$$\kappa^{\pm}(\mathbf{x}-\mathbf{y},t) := \mathcal{L}^{-1}\{\mathbf{G}^{\pm}(\mathbf{x},\mathbf{y},s)\}(t) = \frac{\delta(t-\|\mathbf{x}-\mathbf{y}\|/c^{\pm})}{4\pi\|\mathbf{x}-\mathbf{y}\|}$$

So that if

$$\begin{aligned}
\mathbb{M}^{\pm}(\mathbf{m})(t,\mathbf{x}) &:= \int_0^t \int_\Gamma \mathbf{curl}\big(\kappa^{\pm}(\mathbf{x}-\mathbf{y},t-\tau)\mathbf{m}(\mathbf{y},\tau)\big) \times \mathbf{n}\, d\Gamma_{\mathbf{y}}\, d\tau, \\
\mathbb{C}^{\pm}(\mathbf{j})(t,\mathbf{x}) &:= \Bigg(-\frac{1}{c^{\pm}} \int_0^t \int_\Gamma \kappa^{\pm}(t-\tau,\mathbf{x}-\mathbf{y})\, \mathbf{j}_t(\tau,\mathbf{y}) d\Gamma_{\mathbf{y}} d\tau \\
&\quad + c^{\pm}\nabla \int_0^t \int_\Gamma \kappa^{\pm}(t-\tau,\mathbf{x}-\mathbf{y})\, \mathbf{q}(\tau,\mathbf{y}) d\Gamma_{\mathbf{y}} d\tau\Bigg) \times \mathbf{n},
\end{aligned}$$

where the charge density $\mathbf{q}$ is given by

$$\mathbf{q}(t,x) = -\int_0^t \operatorname{div} \mathbf{j}(\tau,x) d\tau.$$

Time Domain System

So, we have a time domain boundary integral system, where, by an abuse of notation, we use $\mathbf{j}^-, \mathbf{m}^-$ also for time dependent quantities:

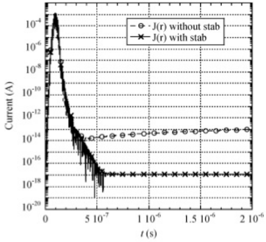
$$\begin{bmatrix} (\mathbb{C}^- + \alpha\mathbb{C}^+) & \mathbb{M}^- + \mathbb{M}^+ \\ \mathbb{M}^- + \mathbb{M}^+ & -(\mathbb{C}^- + \frac{1}{\alpha}\mathbb{C}^+) \end{bmatrix} \begin{bmatrix} \mathbf{j}^- \\ \mathbf{m}^- \end{bmatrix}$$

$$= \begin{bmatrix} \mathbb{C}^+ & \frac{1}{2}I + \mathbb{M}^+ \\ \frac{1}{\alpha}(\frac{1}{2}I + \mathbb{M}^+) & -\frac{1}{\alpha}\mathbb{C}^+ \end{bmatrix} \begin{bmatrix} \gamma_D \mathcal{H}^{inc} \\ \gamma_D \mathcal{E}^{inc} \end{bmatrix}.$$

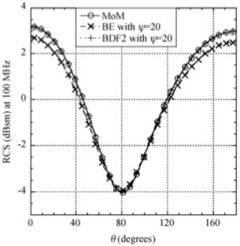
The following results are from

Wang, X., Weile, D.: Electromagnetic scattering from dispersive dielectric scatterers using the finite difference delay modeling method. IEEE Trans. Antennas Propagat. 58, 1720-1730 (2010)

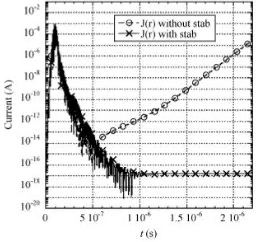
- Use Gaussian modulated plane incident wave.
- Compute the time domain BIE solution using CQ with RWG elements.
- Compare the far field computed using the same bounary mesh.



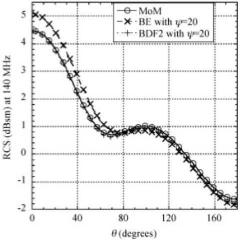
(a)



(a)



(b)



(b)

Unit sphere, Debye medium $\epsilon^{-}(s) = \epsilon_\infty + (\epsilon_s - \epsilon_\infty)/(1 + s\tau_e)$

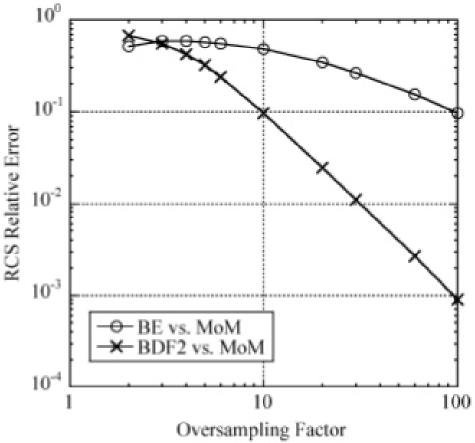


Fig. 4. Relative error in the bistatic RCS computed by the FDDM method compared to the MoM vs. oversampling factor ψ for the 1 m-diameter dispersive sphere.

Spatial function spaces

The framework for analyzing the Laplace domain integral equation system is provided by Buffa et al.[5].

Let $V_\pi^{1/2} = (\mathbf{n} \times H^{1/2}(\Gamma)^3) \times \mathbf{n}$ and

$$\mathbf{X} := \left\{ \lambda \in (V_\pi^{1/2})' \mid \mathrm{div}_\Gamma \lambda \in H^{-1/2}(\Gamma) \right\}$$

where div_Γ is the surface divergence. Note that for smooth domains $\mathbf{X} = H^{-1/2}(\mathrm{div}_\Gamma; \Gamma)$.

Define $b : \mathbf{X} \times \mathbf{X} \to \mathbb{C}$

$$b(\phi, \xi) = \int_\Gamma \phi \cdot \xi \times \mathbf{n} \, dA.$$

[5] Buffa, A., Hiptmair, R., von Petersdorff, T., Schwab, C.: Numer. Math. 95, 459-85 (2003)

Variational Formulation

We seek $(\mathbf{j}^-, \mathbf{m}^-) \in \mathbf{X}^2$ such that

$$\begin{cases} b(\phi, (C^- + \alpha C^+)\mathbf{j}^-) + b(\phi, (M^- + M^+)\mathbf{m}^-) \\ \qquad = b(\phi, C^+\gamma_D \mathbf{H}^{inc}) + b(\phi, (\frac{1}{2}I + M^+)\gamma_D \mathbf{E}^{inc}), \\ b(\psi, (M^- + M^+)\mathbf{j}^-) - b(\psi, (C^- + \frac{1}{\alpha}C^+)\mathbf{m}^-) \\ \qquad = \frac{1}{\alpha} b(\psi, (\frac{1}{2}I + M^+)\gamma_D \mathbf{H}^{inc}) - \frac{1}{\alpha} b(\psi, C^+\gamma_D \mathbf{E}^{inc}), \end{cases}$$

for all $(\phi, \psi) \in \mathbf{X}^2$.

The spatially discrete problem is obtained by replacing $\mathbf{X}$ by a Raviart-Thomas space $\mathbf{X}_h$ above (both test and trials spaces).

Continuous dependence

Buffa et al prove existence and estimates for the solution of the above system when $s = -i\omega$, $\omega \in \mathbb{R}$ for fixed ω using compactness. We are interested in s dependent bounds when $s = \sigma - i\omega$, $\sigma > \sigma_0 > 0$. We prove

Theorem

The above system has a unique solution and

$$\|\mathbf{j}^-\|_{\mathbf{X}} + \|\mathbf{m}^-\|_{\mathbf{X}} \leq C(\sigma_0)|s|^2 \left(\|\gamma_D \mathbf{H}^{inc}\|_{\mathbf{X}} + \|\gamma_D \mathbf{E}^{inc}\|_{\mathbf{X}}\right)$$

The same estimate holds for the discrete problem.

In the upcoming discussion we will need the analogue of the normal derivative for Maxwell's equations:

$$\gamma_{N,k}\mathbf{u} := k^{-1}(\mathbf{curl}\,\mathbf{u} \times \mathbf{n})|_\Gamma,$$

Comments on the Proof

We follow the argument of Laliena and Sayas[6] developed for the Helmholtz equation. I will only talk about the continuous case here. The key step is to see that the integral equation system is equivalent to the generalized transmission problem of finding $(\mathbf{E}, \mathbf{E}^*) \in \mathbf{H}(\mathbf{curl}, \mathbb{R}^3 \setminus \Gamma)^2$:

$$\begin{cases} \mathbf{curl}\,\mathbf{curl}\,\mathbf{E} + (k^-)^2\mathbf{E} = 0, \text{ in } \mathbb{R}^3\backslash\Gamma, \\ \mathbf{curl}\,\mathbf{curl}\,\mathbf{E}^* + (k^+)^2\mathbf{E}^* = 0, \text{ in } \mathbb{R}^3\backslash\Gamma, \\ [\![\gamma_D\mathbf{E}]\!] - [\![\gamma_D\mathbf{E}^*]\!] = \gamma_D\mathbf{E}^{inc}, \\ \alpha\left[\left[\gamma_{N,k^-}\mathbf{E}\right]\right] - \left[\left[\gamma_{N,k^+}\mathbf{E}^*\right]\right] = \gamma_D\mathbf{H}^{inc}, \\ \gamma_D^+\mathbf{E} + \gamma_D^-\mathbf{E}^* = 0, \\ \alpha\gamma_{N,k^-}^-\mathbf{E} + \gamma_{N,k^+}^+\mathbf{E}^* = \gamma_D\mathbf{H}^{inc}. \end{cases}$$

[6] Laliena, A., Sayas, F.: Numer. Math. 112, 637-78 (2009)

Equivalence is proved by using, on the one hand,

$$\mathbf{E} = \mathbb{S}_{\mathbf{E}}^{-}\mathbf{j}^{-} + \mathbb{S}_{H}^{-}\mathbf{m}^{-},\ \mathbf{E}^{*} = \mathbb{S}_{\mathbf{E}}^{+}(\alpha\mathbf{j}^{-} - \gamma_D \mathbf{H}^{inc}) + \mathbb{S}_{H}^{+}(\mathbf{m}^{-} - \gamma_D \mathbf{E}^{inc})$$

and on the other hand $(\left[\left[\gamma_{N,k^-}\mathbf{E}\right]\right], [\![\gamma_D \mathbf{E}]\!])$ gives a solution to the integral equation system.

Analysis of the generalized transmission problem

The generalized transmission problem involves essential and natural boundary conditions.
The relevant space for the fields $(\mathbf{E}, \mathbf{E}^*)$ is

$$H(\mathbf{curl}, \mathbb{R}^3 \backslash \Gamma) \times H(\mathbf{curl}, \mathbb{R}^3 \backslash \Gamma).$$

with the generalized trace operator $\hat{\gamma} : \hat{H} \to \mathbf{X} \times \mathbf{X}$ as follows

$$\hat{\gamma}(\mathbf{E}, \mathbf{E}^*) := \left(\gamma_D^+ \mathbf{E} + \gamma_D^- \mathbf{E}^*, [\![\gamma_D \mathbf{E}]\!] - [\![\gamma_D \mathbf{E}^*]\!]\right) \in \mathbf{X} \times \mathbf{X}$$

This has a continuous right inverse.

A variational formulation

Let

$$\langle A_{\Omega^\pm}(k)\mathbf{u},\mathbf{v}\rangle := \int_{\Omega^\pm} \mathbf{curl}\,\mathbf{u}\cdot\mathbf{curl}\,\mathbf{v} + k^2\mathbf{u}\cdot\mathbf{v}.$$

The generalized transmission problem is equivalent to finding $(\mathbf{E},\mathbf{E}^*)\in H(\mathbf{curl},\mathbb{R}^3\backslash\Gamma)\times H(\mathbf{curl},\mathbb{R}^3\backslash\Gamma)$ satisfying the essential boundary conditions $\hat{\gamma}(\mathbf{E},\mathbf{E}^*) = (0,\gamma_D\mathbf{E}^{inc})$, such that

$$\alpha\,\frac{1}{k^-}\left\langle A_{\mathbb{R}^3\backslash\Gamma}(k_2)\mathbf{E},\mathbf{v}\right\rangle + \frac{1}{k^+}\left\langle A_{\mathbb{R}^3\backslash\Gamma}(k_1)\mathbf{E}^*,\mathbf{v}^*\right\rangle$$
$$= b(\gamma_D\mathbf{H}^{inc},[\gamma_D\mathbf{v}]) - b(\gamma_D\mathbf{H}^{inc},\gamma_D^-\mathbf{v}^*)$$

for all $(\mathbf{v},\mathbf{v}^*)\in\left\{H(\mathbf{curl},\mathbb{R}^3\backslash\Gamma)\times H(\mathbf{curl},\mathbb{R}^3\backslash\Gamma)\mid\hat{\gamma}(\mathbf{v},\mathbf{v}^*)=0\right\}$.

Final steps

Since $k = \sigma - i\omega$, $\sigma > 0$,

$$\langle A_{\Omega^\pm}(k)\mathbf{u}, \mathbf{v}\rangle := \int_{\Omega^\pm} \mathbf{curl}\,\mathbf{u} \cdot \mathbf{curl}\,\mathbf{v} + k^2 \int_{\Omega^\pm} \mathbf{u} \cdot \mathbf{v},$$

the technique of Bamberger and Ha-Duong[7] can be used to derive estimates on $\mathbf{E}$ and $\mathbf{E}^*$ and hence the desired estimate for $\mathbf{j}^-$ and $\mathbf{m}^-$.

[7] Bamberger, A., Duong, T.H.: Math. Meth. Appl. Sci. 8, 405-435 (1986)

Error estimates

Theorem

Suppose that convolution quadrature based on a pth order multi-step discretization that is A-stable (with other technical assumptions) is applied in time and Raviart-Thomas elements on Γ are used to compute the discrete fluxes $[\mathbf{j}^-_{h,\Delta t}, \mathbf{m}^-_{h,\Delta t}]$. Then provided the vector data $\mathbf{g}$ is smooth enough the following holds where $t_k = k\Delta t$, $\Delta t = T/N$, $0 \le k \le N$:

$$\left\| \begin{bmatrix} \mathbf{j}^- \\ \mathbf{m}^- \end{bmatrix} (t_k) - \begin{bmatrix} \mathbf{j}^-_{h,\Delta t} \\ \mathbf{m}^-_{h,\Delta t} \end{bmatrix} (t_k) \right\|_{\mathbf{X}} \le C\Delta t^2 \int_0^t \left\| \partial_t^{r+1} \mathbf{g}(\tau, \cdot) \right\|_{\mathbf{X}} d\tau + C \left\| \begin{bmatrix} \mathbf{j}^- \\ \mathbf{m}^- \end{bmatrix} - \begin{bmatrix} \xi_h \\ \eta_h \end{bmatrix} \right\|_{W_0^{r,1}((0,T);\mathbf{X})},$$

for any $(\xi_h, \eta_h) \in W_0^{r,1}((0, T); \mathbf{X}_h)$ for $r > \max\{p + 4, 5\}$.

Conclusions

- We have shown that CQ can be used in the usual way to compute solutions of Maxwell's equations for homogeneous penetrable media.
- Wang and Weile suggest that stabilization is needed in practice.
- Other formulations (e.g. single field) could be considered perhaps.

Best wishes to Martin Costabel

Thanks - amongst many things - for

- M. Costabel: Boundary integral operators on Lipschitz domains: elementary results. SIAM J. Math. Anal. 19, 613-626 (1988)
- M. Costabel: A coercive bilinear form for Maxwell's equations. J.Math. Anal. Appl. 157, 527Ð541 (1991).
- M. Costabel and M. Dauge: Singularities of electromagnetic fields in polyhedral domains, Archive for Rational Mechanics and Analysis, 151, 221-275 (2000).
- M. Costabel: Fundamentals, Encyclopedia of Computational Mechanics, vol. 1, chap. Time-dependent Problems with the Boundary Integral Equation Method. John Wiley & Sons (2004)