Maxwell's Equations & Relatives: On the Structure of Electromagnetic Wave Propagation.

Journées Singulières Augmentées en l'honneur de Martin Costabel.

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The Shape of Evolutionary Equations.

General Form of Evolutionary Problems:

$$\partial_0 V + AU = f \text{ on } \mathbb{R},\ V = \mathscr{M} U.$$

Evolutionary Equation:

$$(\partial_0 \mathscr{M} + A)\, U = f.$$

Solution Theory: Does the operator

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exist as a continuous linear mapping on a suitable Hilbert space? Which *"suitable"* Hilbert space?

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The Time Derivative as a Normal Operator

Exponential weight function $t \mapsto \exp(-\rho\, t)$, $\rho \in \mathbb{R}$, generates a weighted L^2-space $H_{\rho,0}(\mathbb{R},\mathbb{C})$ by completion of the space $\mathring{C}_\infty(\mathbb{R},\mathbb{C})$ of smooth complex-valued functions with compact support w.r.t. $\langle\cdot|\cdot\rangle_{\rho,0}$ (norm: $|\cdot|_{\rho,0}$)

$$(\varphi,\psi) \mapsto \int_{\mathbb{R}} \overline{\varphi(t)}\,\psi(t)\exp(-2\rho\, t)\,dt.$$

Time-differentiation ∂_0 as a closed operator in $H_{\rho,0}(\mathbb{R},\mathbb{C})$ induced by

$$\begin{aligned}\mathring{C}_\infty(\mathbb{R},\mathbb{C}) \subseteq H_{\rho,0}(\mathbb{R},\mathbb{C}) &\to H_{\rho,0}(\mathbb{R},\mathbb{C}),\\ \varphi &\mapsto \varphi'.\end{aligned}$$

The Time Derivative as a Normal Operator

Time-differentiation ∂_0 is a normal operator in $H_{\rho,0}(\mathbb{R},\mathbb{C})$

$$\partial_0 = \Re\mathfrak{e}\,\partial_0 + \mathrm{i}\,\Im\mathfrak{m}\,\partial_0 = \overline{\frac{1}{2}(\partial_0 + \partial_0^*)} + \mathrm{i}\,\overline{\frac{1}{2\mathrm{i}}(\partial_0 - \partial_0^*)}$$

with $\Re\mathfrak{e}\,\partial_0$, $\Im\mathfrak{m}\,\partial_0$ self-adjoint operators with commuting resolvents:

$$\Re\mathfrak{e}\,\partial_0 = \rho\,.$$

For $\rho \in \mathbb{R}\setminus\{0\}$: continuous invertibility of ∂_0, i.e. $0 \in \rho(\partial_0)$ (resolvent set):

$$\sigma(\partial_0) = \mathrm{i}\mathbb{R} + \rho \quad \text{(spectrum)}.$$

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The Time Derivative as a Normal Operator

Fourier-Laplace transform: unitary extension of

$$\mathring{C}_\infty(\mathbb{R},\mathbb{C}) \subseteq H_{\rho,0}(\mathbb{R},\mathbb{C}) \to H_{0,0}(\mathbb{R},\mathbb{C}) = L^2(\mathbb{R},\mathbb{C})$$

$$\varphi \mapsto \mathscr{L}_\rho \varphi$$

with $\mathscr{L}_\rho \varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp(-\mathrm{i}\, x\, t) \exp(-\rho\, t)\, \varphi(t)\, dt,\ x \in \mathbb{R}.$

is spectral representation for $\mathfrak{Im}\,\partial_0$:

$$\mathfrak{Im}\,\partial_0 = \mathscr{L}_\rho^{-1} \mathbf{m}_0 \mathscr{L}_\rho, \quad \partial_0 = \mathscr{L}_\rho^{-1} (\mathrm{i}\, \mathbf{m}_0 + \rho) \mathscr{L}_\rho.$$

Here $\mathbf{m}_0$ is the selfadjoint multiplication-by-argument operator in $L^2(\mathbb{R},\mathbb{C})$:

$$(\mathbf{m}_0 \varphi)(x) = x\varphi(x)$$

for $x \in \mathbb{R}$ and $\varphi \in \mathring{C}_\infty(\mathbb{R},\mathbb{C})$.

The Time Derivative as a Normal Operator

The canonical extension of ∂_0 to the X-valued case, X a Hilbert space, inherits the normality:

∂_0 is still a normal operator in $H_{\rho,0}(\mathbb{R},X)$

$$\rho = \mathfrak{Re}\,\partial_0.$$

With the extended Fourier-Laplace transform

$$\mathscr{L}_\rho : H_{\rho,0}(\mathbb{R},X) \to L^2(\mathbb{R},X)$$

we still get

$$\partial_0 = \mathscr{L}_\rho^{-1}\left(\mathrm{i}\,\mathbf{m}_0 + \rho\right)\mathscr{L}_\rho.$$

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Material Law Operators as Functions of the Time Derivative

- We also have that

$$\partial_0^{-1} = \mathcal{L}_\rho^{-1} \frac{1}{\mathrm{i}\,\mathbf{m}_0 + \rho} \mathcal{L}_\rho,$$

and so

$$\sum_{k=0}^{N} M_k \partial_0^{-k} = \mathcal{L}_\rho^{-1} \sum_{k=0}^{N} M_k \frac{1}{(\mathrm{i}\,\mathbf{m}_0 + \rho)^k} \mathcal{L}_\rho$$

with continuous linear operators M_k on X as coefficients, $k = 0, \ldots, N$.

- Note that for $\rho \in]0, \infty[$

$$\left\| \partial_0^{-1} \right\| = \frac{1}{\rho} \text{ and } \left(\partial_0^{-1} \varphi\right)(x) = \int_{-\infty}^{x} \varphi(t)\, dt$$

for all $\varphi \in \mathring{C}_\infty(\mathbb{R})$ and $x \in \mathbb{R}$.

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$$\mathscr{M} = M\left(\partial_0^{-1}\right).$$

It is
$$M\left(\partial_0^{-1}\right) := \mathscr{L}_\rho^{-1} M\left(\frac{1}{\mathrm{i}\,\mathbf{m}_0+\rho}\right)\mathscr{L}_\rho,$$

where
$$M\left(\frac{1}{\mathrm{i}\,\mathbf{m}_0+\rho}\right)\Phi := \left(\omega \mapsto M\left(\frac{1}{\mathrm{i}\,\omega+\rho}\right)\Phi(\omega)\right)$$

for $\Phi \in \mathring{C}_\infty(\mathbb{R}, X)$.

Here $\left(M(z)\right)_{z\in B_{\mathbb{C}}(r,r)}$ is a uniformly bounded, holomorphic family of linear operators in H with $r \geq \frac{1}{2\rho} > 0$. The operator $M\left(\partial_0^{-1}\right)$ will be referred to as the **material law operator**. The operator-valued function M will be referred to as the **material law function**.

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Basic Solution Theory $H_{\rho,0}(\mathbb{R},H)$

Evolutionary Problem:

$$\overline{\left(\partial_0 M\left(\partial_0^{-1}\right)+A\right)}U=F$$

When is $\left(\partial_0 M\left(\partial_0^{-1}\right)+A\right)$ (and its adjoint) strictly positive definite in $H_{\rho,0}(\mathbb{R},H)$ (for all sufficiently large $\rho\in\,]0,\infty[$)?

Assumptions (E):

- A **skew-selfadjoint** in H (lifted to $H_{\rho,0}(\mathbb{R},H)$),
- $M(z)=M_0+z\left(M_1+M^{(2)}(z)\right)$, $M^{(2)}$ a **causal material law function** (values in $L(H,H)$), e.g. analytic at 0,
- $\limsup_{\rho\to\infty}\left\|M^{(2)}(\mathrm{i}\cdot+\rho)\right\|=0$,
- $M_0\geq 0$ **selfadjoint**, **strictly positive definite on its range**,
- $\mathfrak{Re}\, M_1$ **strictly positive definite on the null space of** M_0.

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The Basic Solution Theorem

Theorem

Let M and A satisfy ***Assumptions (E)****. Then we have for all sufficiently large $\rho \in]0,\infty[$ that for every $f \in H_{\rho,0}(\mathbb{R},H)$ there is a unique solution $U \in H_{\rho,0}(\mathbb{R},H)$ of the problem*

$$\overline{\left(\partial_0 M\left(\partial_0^{-1}\right) + A\right)} U = f.$$

The solution operator $\left(\overline{\partial_0 M\left(\partial_0^{-1}\right) + A}\right)^{-1}$ is continuous and causal on $H_{\rho,0}(\mathbb{R},H)$.

Causal? For every $a \in \mathbb{R}$ we have:

If $F \in H_{\rho,0}(\mathbb{R},H)$ vanishes on the time interval $]-\infty,a]$, then so does $\left(\overline{\partial_0 M\left(\partial_0^{-1}\right) + A}\right)^{-1} F$.

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Some Applications to a Particular Class of Problems

The structure of A as a block operator matrix is frequently of the form

$$A = \begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix} \tag{1}$$

with $G : D(G) \subseteq H_0 \to H_1$ a closed, densely defined linear operator between Hilbert spaces H_0 and H_1, and the material laws are often given simply as

$$M\left(\partial_0^{-1}\right) = M_0 + \partial_0^{-1} M_1 ,$$

where M_0 is self-adjoint and strictly positive definite in $H := H_0 \oplus H_1$. The term $M^{(2)}$ can be treated as a perturbation.

Some Applications to a Particular Class of Problems

Maxwell's equations, acoustics equations, elasticity equations etc. are of this specific form if memory effects are not considered:

$$\partial_0 M_0 + M_1 + \begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix}.$$

M_0, M_1 block diagonal in simple cases.

Metamaterials and Other Complex Media

Complex materials: general material law operators

$$M\left(\partial_0^{-1}\right)$$

- not block-diagonal
- (linear) delay
- memory terms (such as temporal convolution operators or fractional derivatives).

New materials!

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A Side Note: The "Mother" of "All" Evolutionary PDE

The Maxwell system is a "descendant" of the "Mother":

$$A = \begin{pmatrix} 0 & -\nabla^* \\ \nabla & 0 \end{pmatrix} \tag{2}$$

with a suitable domain making A skew-selfadjoint in the Hilbert space

$$H = \left(\bigoplus_{k \in \mathbb{N}} L_k^2(\Omega) \right) \oplus \left(\bigoplus_{k \in \mathbb{N}} L_k^2(\Omega) \right).$$

$L_k^2(\Omega)$ tensors of order k with $L^2(\Omega)$-coefficients.

∇ co-variant derivative and $-\nabla^*$ its skew-adjoint (tensorial divergence).

The "Mother" of "All" Evolutionary PDE

Dirichlet boundary condition $G = \mathring{\nabla}$:

$$A := \begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix}$$

Initial boundary value problems of classical mathematical physics can be produced from this particular "mother" operator A by choosing suitable projections for constructing "descendants".

The "Mother" of "All" Evolutionary PDE

Theorem

Let $C : D(C) \subseteq H_0 \to H_1$ be a closed densely defined linear operator, H_k, $k = 0, 1$, Hilbert spaces. If $B_k : H_k \to X_k$ are continuous linear mappings, X_k Hilbert space, $k = 0, 1$, such that

- *$C^* B_1^*$ densely defined and B_0 is a bijection*
  or
- *$C B_0^*$ densely defined and B_1 is a bijection.*

Then $\overline{\begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix} \begin{pmatrix} 0 & -C^* \\ C & 0 \end{pmatrix}} \begin{pmatrix} B_0^* & 0 \\ 0 & B_1^* \end{pmatrix}$ *is skew-selfadjoint.*

"Mother" and "descendant".

The "Mother" of "All" Evolutionary PDE

Examples:

- tensor order (or degree; "Stufe")
- symmetric/alternating

3-dimensional		
order 0, 1	————	acoustics
order 1, 2	symmetric	elastics
order 1, 2	alternating	electrodynamics

- descend in space dimension
- vanishing trace condition (divergence-free; incompressible Stokes equation)

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Coupling of Different Physical Phenomena

Without coupling, block-diagonal operator matrix:

$$\partial_0 \begin{pmatrix} V_0 \\ \vdots \\ \vdots \\ V_n \end{pmatrix} + A \begin{pmatrix} U_0 \\ \vdots \\ \vdots \\ U_n \end{pmatrix} = \begin{pmatrix} f_0 \\ \vdots \\ \vdots \\ f_n \end{pmatrix},$$

where

$$A = \begin{pmatrix} A_0 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & A_n \end{pmatrix}$$

skew-selfadjoint in $H = \bigoplus_{k=0,\dots,n} H_k$, since diagonal block entries $A_k : D(A_k) \subseteq H_k \to H_k$, $k = 0, \dots, n$, are skew-self-adjoint.

Coupling of Different Physical Phenomena

The combined material laws now take the simple diagonal form

$$V=\begin{pmatrix} V_0 \\ \vdots \\ \vdots \\ V_n \end{pmatrix}=\begin{pmatrix} M_{00}\left(\partial_0^{-1}\right) & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & M_{nn}\left(\partial_0^{-1}\right) \end{pmatrix}\begin{pmatrix} U_0 \\ \vdots \\ \vdots \\ U_n \end{pmatrix}.$$

Proper coupling: M contains off-diagonal block entries

$$M\left(\partial_0^{-1}\right):=\begin{pmatrix} M_{00}\left(\partial_0^{-1}\right) & \cdots & \cdots & M_{0n}\left(\partial_0^{-1}\right) \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ M_{n0}\left(\partial_0^{-1}\right) & \cdots & \cdots & M_{nn}\left(\partial_0^{-1}\right) \end{pmatrix}.$$

Coupling of Different Physical Phenomena

Canonical Form:

If
$$A_k = \begin{pmatrix} 0 & -G_k^* \\ G_k & 0 \end{pmatrix},$$

then, with the unitary permutation matrix

$$P = \left(e_0\, e_2 \cdots e_{2n} e_1\, e_3 \cdots e_{2n+1}\right),$$

based on $\begin{array}{rcl} \{0,\dots,2n+1\} & \to & \{0,\dots,2n+1\} \\ k & \mapsto & \frac{1-(-1)^k}{2}(n+1) + \left\lfloor \frac{k}{2} \right\rfloor \end{array}$, we obtain

$PAP^* = \begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix}$ with

$$G = \begin{pmatrix} G_0 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & G_n \end{pmatrix}.$$

Example: Plasma Field Equations

Plasma field equations, [Felsen-Marcuvitz-1973]: Maxwell equation and acoustic equation coupled (average electron velocity v, electron pressure p).

$$\left(\partial_0 M_0 + M_1 + A\right)\begin{pmatrix}\begin{pmatrix} p \\ E \end{pmatrix} \\ \begin{pmatrix} v \\ H \end{pmatrix}\end{pmatrix} = F$$

$$M_0 = \begin{pmatrix} \begin{pmatrix} \frac{1}{\gamma p_0} & 0 \\ 0 & \varepsilon_0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} n_0 m & 0 \\ 0 & \mu_0 \end{pmatrix} \end{pmatrix}, \; M_1 = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ -n_0 q & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & n_0 q \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -n_0 m \omega_c b_0 \times & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix},$$

$$A = \begin{pmatrix} 0 & -G^* \\ G & 0 \end{pmatrix}, \quad G = \begin{pmatrix} \mathring{\operatorname{grad}} & 0 \\ 0 & \mathring{\operatorname{curl}} \end{pmatrix}$$

M_0 strictly positive definite, M_1 skew-selfadjoint.

Example: Extended Maxwell System

A different coupling of acoustic equations with Maxwell's equations, [Pi-1984] (precursor Ohmura 1956 (quaternions!)):

$$A_{\mathrm{Max}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\operatorname{curl} & 0 \\ 0 & \mathring{\operatorname{curl}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$A_{\mathrm{AC}} := \begin{pmatrix} 0 & \operatorname{div} & 0 & 0 \\ \mathring{\operatorname{grad}} & 0 & 0 & 0 \\ 0 & 0 & 0 & \operatorname{grad} \\ 0 & 0 & \mathring{\operatorname{div}} & 0 \end{pmatrix} =: A_{\mathrm{ACD}} + A_{\mathrm{ACN}}$$

$$\left(\partial_0 + \sqrt{\mathscr{E}^{-1}} A_{\mathrm{Max}} \sqrt{\mathscr{E}^{-1}} + \sqrt{\mathscr{E}} A_{\mathrm{ACD}} \sqrt{\mathscr{E}} + \sqrt{\mathscr{E}} A_{\mathrm{ACN}} \sqrt{\mathscr{E}}\right) \left(\sqrt{\mathscr{E}}\, U\right) = \widetilde{F},$$

here $\widetilde{F} = \sqrt{\mathscr{E}^{-1}} F + \partial_0^{-1} \sqrt{\mathscr{E}} A_{\mathrm{AC}} F$.

Example: Extended Maxwell System

Extended Maxwell system:

$$(\partial_0 + A)\,U = \widetilde{F},$$

where

$$A = \sqrt{\mathcal{E}^{-1}}A_{\mathrm{Max}}\sqrt{\mathcal{E}^{-1}} + \sqrt{\mathcal{E}}A_{\mathrm{AC}}\sqrt{\mathcal{E}}$$

is skew-selfajoint in $L^2(\Omega)^8$.

Actually: *no* coupling between $\sqrt{\mathcal{E}^{-1}}A_{\mathrm{Max}}\sqrt{\mathcal{E}^{-1}}$ and $\sqrt{\mathcal{E}}A_{\mathrm{AC}}\sqrt{\mathcal{E}}$ occurring!

$$\sqrt{\mathcal{E}^{-1}}A_{\mathrm{Max}}\sqrt{\mathcal{E}^{-1}}\sqrt{\mathcal{E}}A_{\mathrm{AC}}\sqrt{\mathcal{E}} = \sqrt{\mathcal{E}^{-1}}A_{\mathrm{Max}}A_{\mathrm{AC}}\sqrt{\mathcal{E}} = 0,$$

$$\sqrt{\mathcal{E}}A_{\mathrm{AC}}\sqrt{\mathcal{E}}\sqrt{\mathcal{E}^{-1}}A_{\mathrm{Max}}\sqrt{\mathcal{E}^{-1}} = \sqrt{\mathcal{E}}A_{\mathrm{AC}}A_{\mathrm{Max}}\sqrt{\mathcal{E}^{-1}} = 0,$$

on $D\left(A_{\mathrm{AC}}\sqrt{\mathcal{E}}\right)$ and on $D\left(A_{\mathrm{Max}}\sqrt{\mathcal{E}^{-1}}\right)$, respectively.

Example: Extended Maxwell System

It may still be useful to consider this extended Maxwell system, e.g. for numerical purposes, see [Taskinen-Vänskä-2007, Weggler-2012], or low frequency limits [Pauly-2006, Pauly-2008, Pi-1984], since the spatial operator is now a differential operator of elliptic type, yielding a "small" null space.

For $\mathscr{E} = 1$ the formal determinant is Δ^4, leading to a weakly singular Green's tensor. We see that

$$(\partial_0 + (A_{\mathrm{Max}} + A_{\mathrm{ACD}} + A_{\mathrm{ACN}}))(\partial_0 - (A_{\mathrm{Max}} + A_{\mathrm{ACD}} + A_{\mathrm{ACN}})) = \partial_0^2 - \Delta.$$

Solved: Dirac's idea (1928) of finding a first order "root" of the differential operator $\Box$ of the wave equation (Klein-Gordon). Ivanenko-Landau equation (1928), Dirac-Kähler equation (1962), EM-Dirac: Kravchenko & Shapiro (1995, quaternions!), Simulik (1997, quaternions!)

Example: Dirac Equations

- Dirac equation (4×4, separating real and imaginary part 8×8) unitarily equivalent by permutation to extended Maxwell!

  Dirac: $\mathscr{E} = 1, \quad M_0 = 1, \; M_1 = \begin{pmatrix} 0 & -S^* \\ S & 0 \end{pmatrix}$,

$$S = \begin{pmatrix} 0 & \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{pmatrix}.$$

Example: GEM

- The linear equations of gravitation, A. Singh, [Singh-1982], received renewed interest in the context of modern **gravito-electromagnetism** (**GEM**) [Ulrych-2005] (quaternions!). $\mathscr{E} = 1$, $M_0 \gg 0$ block-diagonal

$$\partial_0 M_0 + (A_{\mathrm{Max}} + A_{\mathrm{ACD}} + A_{\mathrm{ACN}})$$

Recall: choice of data filters out the acoustics part or the electrodynamics part.

Summary

- The key to well-posedness of evolutionary problems is strict positive definiteness.
- Causality is a characterizing property for evolutionary equations.
- The framework provides for an abundance of applications with a single highly unified approach (here only shown for some Maxwell type equations).

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