On the Maxwell Constants in 3D

Dirk Pauly
Fakultät für Mathematik
Universität Duisburg-Essen, Campus Essen, Germany

JSA'13 & MC'65
Journées Singulières Augmentés 2013:
Conférence en l'honneur de Martin Costabel pour ses 65 ans

August 27 2013
Rennes, France

Sobolev Spaces

- $\Omega \subset \mathbb{R}^3$ bounded domain with Lipschitz (or weaker) boundary $\Gamma = \partial\Omega$
  crucial assumption MCP: embedding $\mathring{\mathrm{R}}(\Omega) \cap \mathrm{D}(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$ compact
- Sobolev spaces for rot = curl

$$\mathrm{R}(\Omega) := \{E \in \mathrm{L}^2(\Omega) \,:\, \operatorname{rot} E \in \mathrm{L}^2(\Omega)\} \qquad (= \mathrm{H}(\mathrm{curl};\Omega))$$

$$\mathrm{R}_0(\Omega) := \{E \in \mathrm{R}(\Omega) \,:\, \operatorname{rot} E = 0\}$$

$$\mathring{\mathrm{R}}(\Omega) := \overline{\mathring{\mathrm{C}}^\infty(\Omega)}^{\mathrm{R}(\Omega)} = \{E \in \mathrm{R}(\Omega) \,:\, \tau E = 0\}$$

$$\mathring{\mathrm{R}}_0(\Omega) := \mathring{\mathrm{R}}(\Omega) \cap \mathrm{R}_0(\Omega)$$

analogously for div

$$\mathrm{D}(\Omega), \quad \mathrm{D}_0(\Omega), \quad \mathring{\mathrm{D}}(\Omega), \quad \mathring{\mathrm{D}}_0(\Omega) \qquad (= \mathrm{H}(\mathrm{div};\Omega))$$

and Dirichlet resp. Neumann fields $\mathcal{H}_D(\Omega)$ resp. $\mathcal{H}_N(\Omega)$

$$\mathcal{H}_D(\Omega) := \mathring{\mathrm{R}}_0(\Omega) \cap \mathrm{D}_0(\Omega) \quad \text{(finite dimensional by compact embedding)}$$
$$= \{E \in \mathrm{L}^2(\Omega) \,:\, \operatorname{rot} E = 0,\ \operatorname{div} E = 0,\ \tau E = 0\}$$

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Estimates for the Maxwell Constants

open problem: estimates for the Maxwell constants $c_{\mathtt{m}}$ in 3D?

$$\forall\, E \in \overset{\circ}{\mathsf{R}}(\Omega) \cap \mathsf{D}(\Omega) \cap \mathcal{H}_D(\Omega)^\perp \qquad |E|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{m,t}} \big(|\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathsf{L}^2(\Omega)} \big)^{1/2}$$

$$\forall\, E \in \mathsf{R}(\Omega) \cap \overset{\circ}{\mathsf{D}}(\Omega) \cap \mathcal{H}_N(\Omega)^\perp \qquad |E|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{m,n}} \big(|\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathsf{L}^2(\Omega)} \big)^{1/2}$$

question:

$$\boxed{\, ? \leq c_{\mathtt{m,t}}, c_{\mathtt{m,n}} \leq \, ?}$$

in 2D well known

$$\overset{\circ}{c}_{\mathtt{p}} \leq c_{\mathtt{m,t}}, c_{\mathtt{m,n}} \leq c_{\mathtt{p}} \qquad \text{even} \qquad \overset{\circ}{c}_{\mathtt{p}} < c_{\mathtt{m,t}} = c_{\mathtt{m,n}} = c_{\mathtt{p}}$$

with Poincaré constants

$$\forall\, u \in \overset{\circ}{\mathsf{H}}{}^1(\Omega) \qquad |u|_{\mathsf{L}^2(\Omega)} \leq \overset{\circ}{c}_{\mathtt{p}} |\nabla u|_{\mathsf{L}^2(\Omega)} \qquad (\overset{\circ}{c}_{\mathtt{p}} \leq d, \text{ if } \Omega \text{ bd in one dir., trivial})$$

$$\forall\, u \in \mathsf{H}^1(\Omega) \cap \mathbb{R}^\perp \qquad |u|_{\mathsf{L}^2(\Omega)} \leq c_{\mathtt{p}} |\nabla u|_{\mathsf{L}^2(\Omega)} \qquad (c_{\mathtt{p}} \leq \operatorname{diam}(\Omega)/\pi, \text{ if } \Omega \text{ bd \& convex,}$$

'60 Payne & Weinberger)

main result

Theorem ('13 DP)

$\Omega \subset \mathbb{R}^3$ *bounded & convex* $\quad \Rightarrow \quad \overset{\circ}{c}_{\mathtt{p}} \leq c_{\mathtt{m,t}} \leq c_{\mathtt{m,n}} = c_{\mathtt{p}} \leq \operatorname{diam}(\Omega)/\pi$

note always

$$\overset{\circ}{c}_{\mathtt{p}} = \frac{1}{\sqrt{\lambda_1}} < \frac{1}{\sqrt{\mu_2}} = c_{\mathtt{p}}$$

Step 1: Problem Reduction by Helmholtz Decomposition (as usual)

reminder

$$\forall\, E \in \overset{\circ}{\mathrm{R}}(\Omega) \cap \mathrm{D}(\Omega) \cap \mathcal{H}_D(\Omega)^{\perp} \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m,t}} \big(|\operatorname{rot} E|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)} \big)^{1/2}$$

$$\forall\, E \in \mathrm{R}(\Omega) \cap \overset{\circ}{\mathrm{D}}(\Omega) \cap \mathcal{H}_N(\Omega)^{\perp} \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m,n}} \big(|\operatorname{rot} E|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)} \big)^{1/2}$$

Helmholtz decomposition ⇒ splits 2 problems into 4 'nicer' problems

$$\forall\, E \in \mathrm{D}(\Omega) \cap \underbrace{\overset{\circ}{\mathrm{R}}_0(\Omega) \cap \mathcal{H}_D(\Omega)^{\perp}}_{=\nabla \overset{\circ}{\mathrm{H}}{}^1(\Omega)} \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m,t,div}} |\operatorname{div} E|_{\mathrm{L}^2(\Omega)}$$

$$\forall\, E \in \overset{\circ}{\mathrm{R}}(\Omega) \cap \underbrace{\mathrm{D}_0(\Omega) \cap \mathcal{H}_D(\Omega)^{\perp}}_{=\operatorname{rot} \mathrm{R}(\Omega)} \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m,t,rot}} |\operatorname{rot} E|_{\mathrm{L}^2(\Omega)}$$

$$\forall\, E \in \overset{\circ}{\mathrm{D}}(\Omega) \cap \underbrace{\mathrm{R}_0(\Omega) \cap \mathcal{H}_N(\Omega)^{\perp}}_{=\nabla \mathrm{H}^1(\Omega)} \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m,n,div}} |\operatorname{div} E|_{\mathrm{L}^2(\Omega)}$$

$$\forall\, E \in \mathrm{R}(\Omega) \cap \underbrace{\overset{\circ}{\mathrm{D}}_0(\Omega) \cap \mathcal{H}_N(\Omega)^{\perp}}_{=\operatorname{rot} \overset{\circ}{\mathrm{R}}(\Omega)} \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m,n,rot}} |\operatorname{rot} E|_{\mathrm{L}^2(\Omega)}$$

How to do Step 1 (Helmholtz Decomposition)

e.g. tangential case

Helmholtz decomposition

$$\mathsf{L}^2(\Omega) = \underbrace{\nabla\overset{\circ}{\mathsf{H}}{}^1(\Omega)}_{\cap} \oplus \underbrace{\mathsf{D}_0(\Omega)}_{\cup}$$

$$= \overbrace{\overset{\circ}{\mathsf{R}}_0(\Omega)} \oplus \overbrace{\operatorname{rot}\mathsf{R}(\Omega)}$$

$$\Rightarrow \qquad \mathsf{L}^2(\Omega) = \nabla\overset{\circ}{\mathsf{H}}{}^1(\Omega) \oplus \mathcal{H}_D(\Omega) \oplus \operatorname{rot}\mathsf{R}(\Omega), \qquad \mathcal{H}_D(\Omega) = \overset{\circ}{\mathsf{R}}_0(\Omega) \cap \mathsf{D}_0(\Omega)$$

pick some $E \in \overset{\circ}{\mathsf{R}}(\Omega) \cap \mathsf{D}(\Omega) \cap \mathcal{H}_D(\Omega)^\perp$

$$\Rightarrow \qquad E = E_\nabla \oplus E_{\mathrm{rot}} \in \big(\underline{\nabla\overset{\circ}{\mathsf{H}}{}^1(\Omega)} \cap \mathsf{D}(\Omega)\big) \oplus \big(\underline{\operatorname{rot}\mathsf{R}(\Omega)} \cap \overset{\circ}{\mathsf{R}}(\Omega)\big)$$

as well as $\quad \operatorname{rot} E_{\mathrm{rot}} = \operatorname{rot} E \quad$ and $\quad \operatorname{div} E_\nabla = \operatorname{div} E$

$$\Rightarrow \qquad |E|^2_{\mathsf{L}^2(\Omega)} = |E_\nabla|^2_{\mathsf{L}^2(\Omega)} + |E_{\mathrm{rot}}|^2_{\mathsf{L}^2(\Omega)}$$

$$\leq c^2_{\mathtt{m},\mathtt{t},\mathrm{div}} |\operatorname{div} E_\nabla|^2_{\mathsf{L}^2(\Omega)} + c^2_{\mathtt{m},\mathtt{t},\mathrm{rot}} |\operatorname{rot} E_{\mathrm{rot}}|^2_{\mathsf{L}^2(\Omega)}$$

$$\leq \max\{c_{\mathtt{m},\mathtt{t},\mathrm{div}}, c_{\mathtt{m},\mathtt{t},\mathrm{rot}}\}^2 \big(|\operatorname{div} E|^2_{\mathsf{L}^2(\Omega)} + |\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)}\big)$$

Step 2: First Results

reminder

$$\forall E \in \overset{\circ}{\mathrm{R}}(\Omega) \cap \mathrm{D}(\Omega) \cap \mathcal{H}_D(\Omega)^\perp \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{m,t}} \big(|\operatorname{rot} E|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)} \big)^{1/2}$$

$$\forall E \in \mathrm{R}(\Omega) \cap \overset{\circ}{\mathrm{D}}(\Omega) \cap \mathcal{H}_N(\Omega)^\perp \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{m,n}} \big(|\operatorname{rot} E|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)} \big)^{1/2}$$

$$\forall E \in \mathrm{D}(\Omega) \cap \nabla \overset{\circ}{\mathrm{H}}{}^1(\Omega) \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{m,t,div}} |\operatorname{div} E|_{\mathrm{L}^2(\Omega)}$$

$$\forall E \in \overset{\circ}{\mathrm{R}}(\Omega) \cap \operatorname{rot} \mathrm{R}(\Omega) \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{m,t,rot}} |\operatorname{rot} E|_{\mathrm{L}^2(\Omega)}$$

$$\forall E \in \overset{\circ}{\mathrm{D}}(\Omega) \cap \nabla \mathrm{H}^1(\Omega) \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{m,n,div}} |\operatorname{div} E|_{\mathrm{L}^2(\Omega)}$$

$$\forall E \in \mathrm{R}(\Omega) \cap \operatorname{rot} \overset{\circ}{\mathrm{R}}(\Omega) \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathrm{m,n,rot}} |\operatorname{rot} E|_{\mathrm{L}^2(\Omega)}$$

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trivially: $c_{\mathrm{m,t,rot}}, c_{\mathrm{m,t,div}} \leq c_{\mathrm{m,t}}$ and $c_{\mathrm{m,n,rot}}, c_{\mathrm{m,n,div}} \leq c_{\mathrm{m,n}}$
trivially: $c_{\mathrm{m,t}} \leq \max\{c_{\mathrm{m,t,rot}}, c_{\mathrm{m,t,div}}\}$ and $c_{\mathrm{m,n}} \leq \max\{c_{\mathrm{m,n,rot}}, c_{\mathrm{m,n,div}}\}$ (Helmholtz)
trivially: $c_{\mathrm{m,t}} = \max\{c_{\mathrm{m,t,rot}}, c_{\mathrm{m,t,div}}\}$ and $c_{\mathrm{m,n}} = \max\{c_{\mathrm{m,n,rot}}, c_{\mathrm{m,n,div}}\}$

Lemma

$$c_{\mathrm{m,t,div}} = \overset{\circ}{c}_{\mathrm{p}} \qquad c_{\mathrm{m,n,div}} = c_{\mathrm{p}} \qquad c_{\mathrm{m,t,rot}} = c_{\mathrm{m,n,rot}}$$

remains to estimate $\boxed{c_{\mathrm{m,rot}} := c_{\mathrm{m,t,rot}} = c_{\mathrm{m,n,rot}}}$

Step 3: Main Results

reminder

$$\forall E \in \mathring{\mathrm{R}}(\Omega) \cap \mathrm{D}(\Omega) \cap \mathcal{H}_D(\Omega)^{\perp} \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m,t}} \big(|\operatorname{rot} E|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)} \big)^{1/2}$$

$$\forall E \in \mathrm{R}(\Omega) \cap \mathring{\mathrm{D}}(\Omega) \cap \mathcal{H}_N(\Omega)^{\perp} \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m,n}} \big(|\operatorname{rot} E|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)} \big)^{1/2}$$

$$\forall E \in \mathrm{D}(\Omega) \cap \nabla \mathring{\mathrm{H}}^1(\Omega) \qquad |E|_{\mathrm{L}^2(\Omega)} \leq \mathring{c}_{\mathtt{p}} |\operatorname{div} E|_{\mathrm{L}^2(\Omega)}$$

$$\forall E \in \mathring{\mathrm{R}}(\Omega) \cap \operatorname{rot} \mathrm{R}(\Omega) \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m,rot}} |\operatorname{rot} E|_{\mathrm{L}^2(\Omega)}$$

$$\forall E \in \mathring{\mathrm{D}}(\Omega) \cap \nabla \mathrm{H}^1(\Omega) \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{p}} |\operatorname{div} E|_{\mathrm{L}^2(\Omega)}$$

$$\forall E \in \mathrm{R}(\Omega) \cap \operatorname{rot} \mathring{\mathrm{R}}(\Omega) \qquad |E|_{\mathrm{L}^2(\Omega)} \leq c_{\mathtt{m,rot}} |\operatorname{rot} E|_{\mathrm{L}^2(\Omega)}$$

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trivially: $c_{\mathtt{m,t}} = \max\{c_{\mathtt{m,rot}}, \mathring{c}_{\mathtt{p}}\}$ and $c_{\mathtt{m,n}} = \max\{c_{\mathtt{m,rot}}, c_{\mathtt{p}}\}$
remains to estimate $\boxed{c_{\mathtt{m,rot}}}$

Theorem ('13 DP)

Let Ω be bounded and convex. Then $c_{\mathtt{m,rot}} \leq c_{\mathtt{p}}$. Moreover, $\mathring{c}_{\mathtt{p}} \leq c_{\mathtt{m,t}} \leq c_{\mathtt{m,n}} = c_{\mathtt{p}}$.

equivalent formulation for eigenvalues

Proof of First Theorem

Proof ... by some functional analysis ...

- $A : D(A) \subset \mathrm{H}_1 \to \mathrm{H}_2$ lin., dens. def., closed with adjoint $A^* : D(A^*) \subset \mathrm{H}_2 \to \mathrm{H}_1$
- assume $D(A) \cap \overline{R(A^*)} \hookrightarrow \mathrm{H}_1$ compact! note $\mathrm{H}_1 = \overline{R(A^*)} \oplus N(A)$
- define $M := \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$ and note $M^2 := \begin{bmatrix} A^*A & 0 \\ & AA^* \end{bmatrix}$

$\Rightarrow M, M^2, A^*A, AA^*$ self-adjoint with compact resolvent $\Rightarrow$ pure point spectra

$$\sigma_{\mathrm{p}}(M) = \pm\sqrt{\sigma_{\mathrm{p}}(A^*A)} = \pm\sqrt{\sigma_{\mathrm{p}}(AA^*)} = \pm\{\kappa_1, \kappa_2, \ldots\}, \quad 0 \le \kappa_n \nearrow \infty$$

looking at first resp. second eigenvalues $\Rightarrow$

Lemma (school of Rolf Leis: R. Leis & R. Picard, N. Weck, K.-J. Witsch, . . .)

$$\inf_{0 \neq u \in D(A) \cap R(A^*)} \frac{|Au|^2_{\mathrm{H}_2}}{|u|^2_{\mathrm{H}_1}} = \inf_{0 \neq v \in D(A^*) \cap R(A)} \frac{|A^*v|^2_{\mathrm{H}_1}}{|v|^2_{\mathrm{H}_2}}$$

Proof of First Theorem...

reminder

$A : D(A) \subset \mathrm{H}_1 \to \mathrm{H}_2$ lin., dens. def., closed, adjoint $A^* : D(A^*) \subset \mathrm{H}_2 \to \mathrm{H}_1$

$D(A) \cap R(A^*) \hookrightarrow \mathrm{H}_1$ compact

$(R(A^*) = N(A)^\perp$ and $R(A), R(A^*)$ closed and $\forall\, u \in D(A) \cap R(A^*) : |u|_{\mathrm{H}_1} \leq c|Au|_{\mathrm{H}_2})$

$$\inf_{0 \neq u \in D(A) \cap R(A^*)} \frac{|Au|^2_{\mathrm{H}_2}}{|u|^2_{\mathrm{H}_1}} = \inf_{0 \neq v \in D(A^*) \cap R(A)} \frac{|A^* v|^2_{\mathrm{H}_1}}{|v|^2_{\mathrm{H}_2}}$$

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especially

$A := \mathring{\nabla} : \mathring{\mathrm{H}}^1(\Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$, $A^* := -\operatorname{div} : \mathrm{D}(\Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$

$R(A^*) = N(A)^\perp = \{0\}^\perp = \mathrm{L}^2(\Omega)$

$\mathring{\mathrm{H}}^1(\Omega) \cap \mathrm{L}^2(\Omega) = \mathring{\mathrm{H}}^1(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$ compact by Rellich's selection theorem

$$\frac{1}{\mathring{c}_{\mathtt{p}}^2} = \lambda_1 = \inf_{0 \neq u \in \mathring{\mathrm{H}}^1(\Omega)} \frac{|\nabla u|^2_{\mathrm{L}^2(\Omega)}}{|u|^2_{\mathrm{L}^2(\Omega)}} = \inf_{0 \neq E \in \mathrm{D}(\Omega) \cap \nabla \mathring{\mathrm{H}}^1(\Omega)} \frac{|\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)}}{|E|^2_{\mathrm{L}^2(\Omega)}} = \frac{1}{c^2_{\mathtt{m,t,div}}}$$

$\Rightarrow c_{\mathtt{m,t,div}} = \mathring{c}_{\mathtt{p}}$ known!

Proof of First Theorem......

reminder

$A : D(A) \subset \mathrm{H}_1 \to \mathrm{H}_2$ lin., dens. def., closed, adjoint $A^* : D(A^*) \subset \mathrm{H}_2 \to \mathrm{H}_1$

$D(A) \cap R(A^*) \hookrightarrow \mathrm{H}_1$ compact

$(R(A^*) = N(A)^\perp$ and $R(A), R(A^*)$ closed and $\forall\, u \in D(A) \cap R(A^*) : |u|_{\mathrm{H}_1} \leq c|Au|_{\mathrm{H}_2})$

$$\inf_{0 \neq u \in D(A) \cap R(A^*)} \frac{|Au|^2_{\mathrm{H}_2}}{|u|^2_{\mathrm{H}_1}} = \inf_{0 \neq v \in D(A^*) \cap R(A)} \frac{|A^* v|^2_{\mathrm{H}_1}}{|v|^2_{\mathrm{H}_2}}$$

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especially

$A := \nabla : \mathrm{H}^1(\Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega),\ A^* := -\mathring{\operatorname{div}} : \mathring{\mathrm{D}}(\Omega) \subset \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$

$R(A^*) = N(A)^\perp = \mathbb{R}^\perp$

$\mathrm{H}^1(\Omega) \cap \mathbb{R}^\perp \subset \mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$ compact by Rellich's selection theorem

$$\frac{1}{c_{\mathrm{p}}^2} = \mu_2 = \inf_{0 \neq u \in \mathrm{H}^1(\Omega) \cap \mathbb{R}^\perp} \frac{|\nabla u|^2_{\mathrm{L}^2(\Omega)}}{|u|^2_{\mathrm{L}^2(\Omega)}} = \inf_{0 \neq E \in \mathring{\mathrm{D}}(\Omega) \cap \nabla \mathrm{H}^1(\Omega)} \frac{|\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)}}{|E|^2_{\mathrm{L}^2(\Omega)}} = \frac{1}{c^2_{\mathtt{m},\mathtt{n},\mathrm{div}}}$$

$\Rightarrow c_{\mathtt{m},\mathtt{n},\mathrm{div}} = c_{\mathrm{p}}$ known!

Proof of First Theorem.........

reminder

$A : D(A) \subset \mathsf{H}_1 \to \mathsf{H}_2$ lin., dens. def., closed, adjoint $A^* : D(A^*) \subset \mathsf{H}_2 \to \mathsf{H}_1$

$D(A) \cap R(A^*) \hookrightarrow \mathsf{H}_1$ compact

$(R(A^*) = N(A)^\perp$ and $R(A), R(A^*)$ closed and $\forall\, u \in D(A) \cap R(A^*) : |u|_{\mathsf{H}_1} \leq c|Au|_{\mathsf{H}_2})$

$$\inf_{0 \neq u \in D(A) \cap R(A^*)} \frac{|Au|^2_{\mathsf{H}_2}}{|u|^2_{\mathsf{H}_1}} = \inf_{0 \neq v \in D(A^*) \cap R(A)} \frac{|A^* v|^2_{\mathsf{H}_1}}{|v|^2_{\mathsf{H}_2}}$$

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especially

$A := \mathring{\operatorname{rot}} : \mathring{\mathsf{R}}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega)$, $A^* := \operatorname{rot} : \mathsf{R}(\Omega) \subset \mathsf{L}^2(\Omega) \to \mathsf{L}^2(\Omega)$

$R(A^*) = \operatorname{rot} \mathsf{R}(\Omega)$

$\mathring{\mathsf{R}}(\Omega) \cap \operatorname{rot} \mathsf{R}(\Omega) \subset \mathring{\mathsf{R}}(\Omega) \cap \mathsf{D}(\Omega) \hookrightarrow \mathsf{L}^2(\Omega)$ compact by MCP

$$\frac{1}{c^2_{\mathtt{m},\mathtt{t},\mathrm{rot}}} = \kappa_2 = \inf_{0 \neq E \in \mathring{\mathsf{R}}(\Omega) \cap \operatorname{rot} \mathsf{R}(\Omega)} \frac{|\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)}}{|E|^2_{\mathsf{L}^2(\Omega)}} = \inf_{0 \neq E \in \mathsf{R}(\Omega) \cap \operatorname{rot} \mathring{\mathsf{R}}(\Omega)} \frac{|\operatorname{rot} E|^2_{\mathsf{L}^2(\Omega)}}{|E|^2_{\mathsf{L}^2(\Omega)}} = \frac{1}{c^2_{\mathtt{m},\mathtt{n},\mathrm{rot}}}$$

$\Rightarrow c_{\mathtt{m},\mathtt{t},\mathrm{rot}} = c_{\mathtt{m},\mathtt{n},\mathrm{rot}} := c_{\mathtt{m},\mathrm{rot}}$ $\square$

$\Rightarrow$ remains to estimate only one constant $c_{\mathtt{m},\mathrm{rot}}$!

Proof of Second (Main) Theorem

Proof

crucial estimate for convex domains

Lemma ('98 C. Amrouche, C. Bernardi, M. Dauge, V. Girault)

$\Omega \subset \mathbb{R}$ *bd and convex. Then* $E \in \overset{\circ}{\mathrm{R}}(\Omega) \cap \mathrm{D}(\Omega), \mathrm{R}(\Omega) \cap \overset{\circ}{\mathrm{D}}(\Omega) \subset \mathrm{H}^1(\Omega)$ *continuous and*

$$|\nabla E|^2_{\mathrm{L}^2(\Omega)} \leq 1 \cdot \big(|\operatorname{rot} E|^2_{\mathrm{L}^2(\Omega)} + |\operatorname{div} E|^2_{\mathrm{L}^2(\Omega)}\big).$$

related, earlier, partial results by
J. Kadlec ('64), R. Leis ('68), P. Grisvard ('72, '85), J. Saranen ('82), J.-C. Nedelec ('82), V. Girault & P.-A. Raviart ('86), M. Costabel ('91)

$\Rightarrow$ pick $E \in \mathrm{R}(\Omega) \cap \operatorname{rot} \overset{\circ}{\mathrm{R}}(\Omega) = \mathrm{R}(\Omega) \cap \overset{\circ}{\mathrm{D}}_0(\Omega) \cap \mathcal{H}_N(\Omega)^{\perp} = \mathrm{R}(\Omega) \cap \overset{\circ}{\mathrm{D}}_0(\Omega)$ (Ω convex)

$\Rightarrow \langle E, a\rangle_{\mathrm{L}^2(\Omega)} = \langle \operatorname{rot} H, a\rangle_{\mathrm{L}^2(\Omega)} = 0$ for all $a \in \mathbb{R}^3$ since $H \in \overset{\circ}{\mathrm{R}}(\Omega)$

$\Rightarrow E \in \mathrm{H}^1(\Omega) \cap (\mathbb{R}^3)^{\perp}$

$$\Rightarrow |E|_{\mathrm{L}^2(\Omega)} \overset{\text{Poincaré}}{\leq} c_{\mathrm{p}} |\nabla E|_{\mathrm{L}^2(\Omega)} \overset{\text{Lemma}}{\leq} 1 \cdot c_{\mathrm{p}} |\operatorname{rot} E|_{\mathrm{L}^2(\Omega)}$$

$\Rightarrow c_{\mathrm{m,rot}} \leq c_{\mathrm{p}}$

very simple!!! □

Last Slide!

Merci / Thank You

more results:

- also inhomogeneous media, i.e., $\varepsilon \neq \mathrm{id}$, $\mu \neq \mathrm{id}$ non-smooth
- also ND-case, i.e., $\Omega \subset \mathbb{R}^N$ with differential forms, same result
- also non-convex polygons (not too pointy) or combinations

applications:

- functional a posteriori error estimates for problems with rot
- preconditioning in numerical algorithms with rot
- ...