Mapping properties of retarded potentials and applications to a posteriori estimates

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Outline

Work in progress…

Boundary integral operators

- Two differential operators: $-\Delta_x$, $\partial_t^2 - \Delta_x \quad (x \in \mathbb{R}^3)$
- Fundamental solutions $\mathcal{G}(x)$ resp. $\mathcal{G}(t,x)$:
  $-\Delta\left(\frac{1}{4\pi|x|}\right) = \delta(x)$, $(\partial_t^2 - \Delta)\left(\frac{\delta(t-x)}{4\pi|x|}\right) = \delta(t,x)$
- Ω bounded Lipschitz, $\Gamma = \partial\Omega$,
  trace operator $\gamma_0 = \cdot\,|_\Gamma : H^s(\mathbb{R}^3) \to H^{s-\frac{1}{2}}(\Gamma)$, $s \in (\frac{1}{2}, \frac{3}{2})$
- **Single layer potential** $\mathcal{S} = \mathcal{G} \circ \gamma_0^*$
  $\mathcal{S}\phi(x) = \int_\Gamma \frac{\phi(x')}{4\pi|x-x'|} d\Gamma_{x'}$ continuous, solves $-\Delta u = 0 \ (\mathbb{R}^3 \setminus \Gamma)$
  $\mathcal{S}\phi(t,x) = \int_\Gamma \frac{\phi(x', t-|x-x'|)}{4\pi|x-x'|} d\Gamma_{x'}$ cont. , solves $\partial_t^2 u - \Delta u = 0 \ (\mathbb{R}^3 \setminus \Gamma)$
- **Boundary integral operator** $\mathcal{V} = \mathcal{S}|_\Gamma$
- $\mathcal{K}\phi(x) = \gamma_0 \int_\Gamma \partial_{\nu_{x'}} \frac{1}{4\pi|x-x'|} \phi(x')\, d\Gamma_{x'}$, $\qquad \mathcal{K}' = (\mathcal{K})^*$
  $\mathcal{W}\phi(x) = \partial_{\nu_x}|_\Gamma \int_\Gamma \phi(x')\, \partial_{\nu_{x'}} \frac{1}{4\pi|x-x'|}\, d\Gamma_{x'}$

Costabel for Δ: Mapping properties / coercivity

M. Costabel, SIAM J. Math. Anal. (1988), *Boundary integral operators on Lipschitz domains: Elementary results.*

Theorem

- $\begin{pmatrix} -\mathcal{K} & \mathcal{V} \\ \mathcal{W} & \mathcal{K}' \end{pmatrix}$ bounded on $H^{s+\frac{1}{2}}(\Gamma) \times H^{s-\frac{1}{2}}(\Gamma)$ for $s \in (-\frac{1}{2}, \frac{1}{2})$.
- $\mathcal{V}$ coercive on $H^{-\frac{1}{2}}(\Gamma)$, $\mathcal{W}$ coercive on $H^{-\frac{1}{2}}(\Gamma)$ mod $\mathbb{K}$.

- Mapping properties: $\mathcal{V} = \gamma_0 \circ \mathcal{G} \circ \gamma_0^*$,
  Costabel's trace theorem(s) for γ_0, and $\mathcal{G}$ ΨDO$(\mathbb{R}^3)$ order -2.
- deduction of properties for $\mathcal{K}, \mathcal{K}', \mathcal{W}$:
  Solution operator to Dir. problem $T : H^{\frac{1}{2}}(\Gamma) \to H^1_\Delta(\Omega)$ continuous.
  Nečas: Dir. -Neumann op. $\gamma_1 T : H^{s+\frac{1}{2}}(\Gamma) \to H^{s-\frac{1}{2}}(\Gamma), s \in [-\frac{1}{2}, \frac{1}{2}]$.
  Identities: $\mathcal{K} = \gamma_0(-1 + \mathcal{S}\gamma_1)T$ etc.
- $s = \pm\frac{1}{2}$ more subtle: Verchota, Jerison–Kenig.

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- $\mathcal{V}$ coercive on $H^{-\frac{1}{2}}(\Gamma)$, $\mathcal{W}$ coercive on $H^{-\frac{1}{2}}(\Gamma)$ mod $\mathbb{K}$.

- Coercivity: Green's formula for Ω, $\mathbb{R}^3 \setminus \Omega$ translates Gårding's inequality for $-\Delta$ to Gårding's inequality for $\mathcal{V}$.

Think PDE, NOT integral operators!

BEM for Δ: a posteriori error analysis

Dirichlet problem

$$\Delta u = 0 \ \ (\Omega), \qquad u|_\Gamma = f$$

Ansatz $u = \mathcal{S}\phi \implies \mathcal{V}\phi = f$, Galerkin solution ϕ_h.

Theorem (Carstensen–Stephan '95, Carstensen '96, 2d)

- Ω polygonal domain, f continuous and smooth on each side of Γ
- quasi-uniform triangulation of Γ, pw. constant ansatz functions
- $\mathcal{R}_h = f - \mathcal{V}\phi_h$

$$\implies \forall s \in [0,1] \ \forall 0 < h < h_0 : \ \|\phi - \phi_h\|_{H^{-s}(\Gamma)} \simeq h^s \|\partial_\Gamma \mathcal{R}_h\|_{L^2(\Gamma)}$$

Motivation: $\eta(E) = h_E^s \|\partial_\Gamma \mathcal{R}_h\|_{L^2(E)}$ indicates local error on edge E
Adaptively refine mesh where η_E is large!

BEM for Δ: a posteriori error analysis

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Upper bound:

- coercivity $\|\phi - \phi_h\|^2_{-\frac{1}{2}} \lesssim \langle \mathcal{V}(\phi - \phi_h), \phi - \phi_h \rangle = \langle f - \mathcal{V}\phi_h, \phi - \phi_h \rangle$
  $\Longrightarrow \ \|\phi - \phi_h\|_{H^{-\frac{1}{2}}(\Gamma)} \lesssim \|\mathcal{R}_h\|_{H^{\frac{1}{2}}(\Gamma)}$
- residual orthogonal: $\mathcal{R} \perp \phi_h$
- interpolation $\rightsquigarrow \ h$

BEM for Δ: a posteriori error analysis

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$$\Longrightarrow \ \forall s \in [0,1] \ \forall 0 < h < h_0 : \ \|\phi - \phi_h\|_{H^{-s}(\Gamma)} \simeq h^s \|\partial_\Gamma \mathcal{R}_h\|_{L^2(\Gamma)}$$

Lower bound:

- Mapping properties $\mathcal{V} : H^{s-\frac{1}{2}}(\Gamma) \to H^{s+\frac{1}{2}}(\Gamma)$
- approximation: $\frac{\|\phi-\phi_h\|_{H^{-1}(\Gamma)}}{\|\phi-\phi_h\|_{L^2(\Gamma)}}$ ($\hookleftarrow$ singular expansion @ vertices)
- inverse estimates
- stable interpolation operators, etc.

To sum up: Proof almost functional analytic!
(except for singular expansion @ vertices)

What about the wave equation? – Motivation

$u = u(t,x)$ sound pressure

$\partial_t^2 u - \Delta u = 0$ in $\mathbb{R}_t \times \Omega_x$, $\Omega = \mathbb{R}^3_+ \setminus \{$ tire $\}$

$u = 0$ for $t \leq 0$.

Acoustic boundary conditions on $\partial\mathbb{R}^3_+$

$$\partial_\nu u - \alpha \partial_t u = 0 \qquad (\alpha \geq 0) .$$

simple: Dirichlet boundary conditions $u = f$ on $\Gamma = \partial\{$ tire $\}$:

$$\rightsquigarrow \dot{f} = \mathcal{V}\dot{\phi}(t,x) = \int_\Gamma \left\{ \frac{\dot{\phi}(x',t-|x-x'|)}{4\pi|x-x'|} + \frac{\dot{\phi}(x',t-|x-\tilde{x}'|)}{4\pi|x-\tilde{x}'|} + \Sigma(\alpha)\dot{\phi}(t,x) \right\} d\Gamma_{x'}$$

Galerkin approximation $\rightsquigarrow \phi_{\Delta t,h}$ s.t.

$$\int_0^T \int_\Gamma \mathcal{V}\dot{\phi}_{\Delta t,h}(t,x)\, \psi_{\Delta t,h}(t,x)\, d\Gamma_x\, dt = \int_0^T \int_\Gamma \dot{f}(t,x)\, \psi_{\Delta t,h}(t,x)\, d\Gamma_x\, dt \quad \forall \psi_{\Delta t,h}$$

First step: $\mathbb{R}^3 \setminus \Omega$ – no street.

What about the wave equation? – Motivation

$u = u(t,x)$ sound pressure

$$\partial_t^2 u - \Delta u = 0 \quad \text{in } \mathbb{R}_t \times \Omega_x\,, \quad \Omega = \mathbb{R}^3_+ \setminus \{\text{ tire }\}$$

$$u = 0 \quad \text{for } t \leq 0\,.$$

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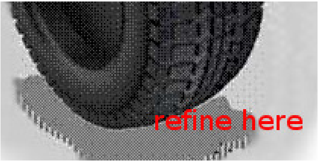
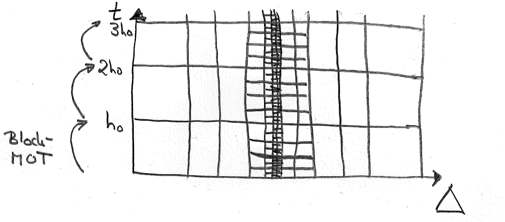
First step: $\mathbb{R}^3 \setminus \Omega$ – no street.

A first adaptive method: singular geometry





1. Start with coarse space-time grid: $(\Delta t)_i \simeq (\Delta x)_i \simeq h_0 \; \forall \Delta_i$
2. Solve discretisation of $\mathcal{V}\dot{\phi} = \dot{f}$.
3. Compute time-integrated error indicator $\eta(\Delta_i)$
4. $\sum_i \eta(\Delta_i) < \varepsilon \implies$ STOP
5. $\eta(\Delta_i) > \delta\eta_{max} \implies \Delta_i \to \Delta/4,\ (\Delta t)_i \to \frac{(\Delta t)_i}{2}$
6. GO TO 2.

Boundary integral operators for wave equation

$$\mathcal{V}\phi(t,x) = \int_\Gamma \left\{ \frac{\phi(x',t-|x-x'|)}{4\pi|x-x'|} + \frac{\phi(x',t-|x-\tilde{x}'|)}{4\pi|x-\tilde{x}'|} + \Sigma(\alpha)\phi(t,x) \right\} d\Gamma_{x'} \quad (x \in \Gamma)$$

space–time anisotropic Sobolev spaces: $H^{r,s}_\sigma(\mathbb{R}_+, \mathbb{R}^2)$ defined using Fourier–Laplace transform ($\sigma > 0$):
$\{\psi : \text{supp}\,\psi \subset \overline{\mathbb{R}_+} \times \mathbb{R}^2,\ \int_{\mathbb{R}+i\sigma} d\omega \int_{\mathbb{R}^2} d\xi |\omega|^{2r}(|\omega|^2 + |\xi|^2)^s |\mathcal{F}\psi(\omega,\xi)|^2 < \infty\}$

$$\rightsquigarrow \quad H^{r,s}_\sigma(\mathbb{R}_+, \Gamma)\ .$$

Bamberger–Ha Duong: $\mathcal{V}\partial_t$ coercive with loss,

$$\|\phi\|^2_{H^{1,-\frac{1}{2}}_\sigma(\mathbb{R}_+,\Gamma)} \gtrsim \langle \mathcal{V}\partial_t\phi, \phi\rangle \gtrsim \|\phi\|^2_{H^{0,-\frac{1}{2}}_\sigma(\mathbb{R}_+,\Gamma)}$$

Theorem (an a priori estimate)

If $\phi \in H^{2,-\frac{1}{2}}_\sigma(\mathbb{R}_+, \Gamma)$,

$$\|\phi - \phi_{h,\Delta t}\|_{\sigma,0,-\frac{1}{2}} \lesssim \|f - f_{h,\Delta t}\|_{\sigma,1,\frac{1}{2}} + \inf_{\psi_{h,\Delta t}} \|\phi - \psi_{h,\Delta t}\|_{\sigma,2,-\frac{1}{2}}\ .$$

Costabel–like approach for single–layer potential

$$\mathcal{V}\phi(t,x) = \int_\Gamma \frac{\phi(x', t - |x - x'|)}{4\pi|x - x'|} \quad (x \in \Gamma)$$

- A posteriori estimates need mapping properties of $\mathcal{V}, \mathcal{K}, \mathcal{K}', \mathcal{W}$ for general (r, s), $\sigma > 0$.
- Much studied for energy spaces, $s = 0$ (Bamberger–Ha Duong, ...), and/or $\sigma = 0$ (scattering/Helmholtz).

Theorem

For $s \in [-\frac{1}{2}, \frac{1}{2}]$: $\mathcal{V} : H_\sigma^{r+1, s-\frac{1}{2}}(\mathbb{R}_+, \Gamma) \to H_\sigma^{r, s+\frac{1}{2}}(\mathbb{R}_+, \Gamma)$

The loss of $+1$ derivative is also incurred by variational/energy arguments for $s = 0$.

A posteriori estimate for Dirichlet bvp

- Does elliptic a posteriori analysis generalize to wave equation?
- see also M. Gläfke (Ph.D. Brunel, '13), S. Sauter – A. Veit.
- $H^{r,s}_\sigma([0,T],\Gamma)$, $\partial_t f = \dot{f}$ etc.

Theorem

Assume $\mathcal{R} = \dot{f} - \mathcal{V}\dot{\phi}_{h,\Delta t} \in H^{0,1}([0,T],\Gamma) \Longrightarrow$

$$\begin{aligned} \|\phi - \phi_{h,\Delta t}\|^2_{H^{0,-\frac{1}{2}}} &\lesssim \|\mathcal{R}\|_{H^{0,1}([0,T],\Gamma)} \big(\Delta t \|\partial_t \mathcal{R}\|_{L^2([0,T],L^2(\Gamma))} \\ &\quad + \|h \cdot \nabla \mathcal{R}\|_{L^2([0,T],L^2(\Gamma))}\big) \\ &\lesssim \max\{\Delta t, h\} \|\mathcal{R}\|^2_{H^{0,1}([0,T],\Gamma)} \ . \end{aligned}$$

Estimate holds for $\mathbb{R}^3 \setminus \Omega$ and half–space $\mathbb{R}^3_+ \setminus \Omega$.

Proof of upper bound

$$
\begin{aligned}
\|\phi - \phi_{h,\Delta t}\|^2_{H^{0,-\frac{1}{2}}([0,T],\Gamma)} \\
&\lesssim \int_0^T dt \int_0^t ds \int_\Gamma d\Gamma \; \mathcal{V}(\dot{\phi} - \dot{\phi}_{h,\Delta t})(\phi - \phi_{h,\Delta t}) \\
&= \int_0^T dt \int_0^t ds \int_\Gamma d\Gamma \; (\dot{f} - \mathcal{V}\dot{\phi}_{h,\Delta t})(\phi - \phi_{h,\Delta t}) \\
&\lesssim_T \; \|R\|_{H^{0,\frac{1}{2}}([0,T],\Gamma)} \; \|\phi - \phi_{h,\Delta t}\|_{H^{0,-\frac{1}{2}}([0,T],\Gamma)} \; .
\end{aligned}
$$

- interpolation inequality:

$$\|\mathcal{R}\|^2_{H^{0,\frac{1}{2}}} \lesssim \|\mathcal{R}\|_{H^{0,1}} \|\mathcal{R}\|_{L^2L^2} \; .$$

- residual orthogonal: $\mathcal{R} \perp \psi_{h,\Delta t}$.
- interpolation $\rightsquigarrow$ $h, \Delta t$.

Lower bound for Dirichlet bvp in $\mathbb{R}^3 \setminus \Omega$

- dumb estimate

$$\|\mathcal{R}\|_{H^{r-1,s+\frac{1}{2}}} = \|\mathcal{V}(\dot{\phi} - \dot{\phi}_{h,\Delta t})\|_{H^{r-1,s+\frac{1}{2}}} \lesssim \|\phi - \phi_{h,\Delta t}\|_{H^{r+1,s-\frac{1}{2}}} \ .$$

$$\max\{\Delta t, h\}^{-\frac{1}{2}} \|\phi - \phi_{h,\Delta t}\|_{H^{0,-\frac{1}{2}}} \lesssim \|\mathcal{R}\|_{H^{0,1}} \lesssim \|\phi - \phi_{h,\Delta t}\|_{H^{2,0}} \ .$$

- elliptic ideas $\rightsquigarrow$

$$\|\phi - \phi_{h,\Delta t}\|_{H^{2,0}} \lesssim \max\{h, \Delta t\}^{-\alpha} \|\phi - \phi_{h,\Delta t}\|_{H^{2,-\frac{1}{2}}}$$

Theorem

Let $\mathcal{R} \in H^{0,1}$ and assume $\|\phi - \psi_{h,\Delta t}\|_{H^{2,0}} \gtrsim \max\{h, \Delta t\}^{\beta} \quad \forall \psi_{h,\Delta t} \Longrightarrow$

$$\|\mathcal{R}\|_{H^{0,1}} \lesssim \max\{h, \Delta t\}^{-\alpha} \|\phi - \phi_{h,\Delta t}\|_{H^{2,-\frac{1}{2}}} \ .$$

Lower bound for Dirichlet bvp in $\mathbb{R}^3 \setminus \Omega$

- dumb estimate

$$\|\mathcal{R}\|_{H^{r-1,s+\frac{1}{2}}} = \|\mathcal{V}(\dot\phi - \dot\phi_{h,\Delta t})\|_{H^{r-1,s+\frac{1}{2}}} \lesssim \|\phi - \phi_{h,\Delta t}\|_{H^{r+1,s-\frac{1}{2}}} .$$

$$\max\{\Delta t, h\}^{-\frac{1}{2}} \|\phi - \phi_{h,\Delta t}\|_{H^{0,-\frac{1}{2}}} \lesssim \|\mathcal{R}\|_{H^{0,1}} \lesssim \|\phi - \phi_{h,\Delta t}\|_{H^{2,0}} .$$

- elliptic ideas $\rightsquigarrow$

$$\|\phi - \phi_{h,\Delta t}\|_{H^{2,0}} \lesssim \max\{h, \Delta t\}^{-\alpha} \|\phi - \phi_{h,\Delta t}\|_{H^{2,-\frac{1}{2}}}$$

Theorem

Let $\mathcal{R} \in H^{0,1}$ and assume $\|\phi - \psi_{h,\Delta t}\|_{H^{2,0}} \gtrsim \max\{h, \Delta t\}^{\beta} \quad \forall \psi_{h,\Delta t} \Longrightarrow$

$$\max\{h, \Delta t\}^{-\frac{1}{2}} \|\phi - \phi_{h,\Delta t}\|^2_{H^{0,-\frac{1}{2}}} \lesssim \|\mathcal{R}\|_{H^{0,1}} \lesssim \max\{h, \Delta t\}^{-\alpha} \|\phi - \phi_{h,\Delta t}\|_{H^{2,-\frac{1}{2}}} .$$

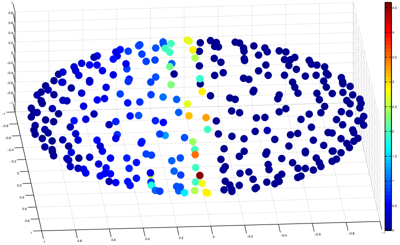
Proof of lower bound

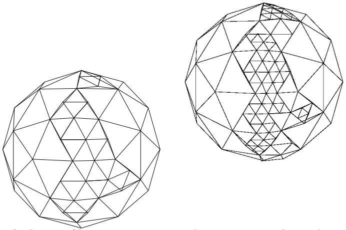
Elliptic estimates for Helmholtz, ω–dependent, $\mathrm{Im} \geq \sigma > 0$ (Bamberger, Ha Duong + . . .)

- Mapping properties $\mathcal{V} : H_\sigma^{r+1,s-\frac{1}{2}}(\Gamma) \to H_\sigma^{r,s+\frac{1}{2}}(\Gamma)$
- approximation: $\frac{\|\phi-\phi_h\|_{H^{-1}(\Gamma)}}{\|\phi-\phi_h\|_{L^2(\Gamma)}}$ ($\looparrowleft$ problem: no elliptic regularity)
- inverse estimates
- stable interpolation operators, etc.

Numerical example

- $\Gamma = \mathbb{S}^2$ in $\mathbb{R}^3$ (no street)
- Dirichlet problem $\mathcal{V}\dot{\phi} = \dot{f}$, $\dot{f} = \begin{cases} 2, & x_1 > 0 \\ 0, & x_1 \leq 0 \end{cases}$.
- uniformly refined icosahedral mesh with 320 nodes, 10 time steps





$\eta_\nabla(\Delta_i) = \|\nabla(\mathcal{V}\dot{\phi} - \dot{f})\|_{L^2([0,T],L^2(\Delta_i))}$ and grid refinement (80 nodes)

Costabel's approach for $\mathcal{K}$, $\mathcal{K}'$, $\mathcal{W}$

Theorem

$$\mathcal{V} : H_\sigma^{r+1,s-\frac{1}{2}}(\mathbb{R}_+,\Gamma) \to H^{r,s+\frac{1}{2}}(\mathbb{R}_+,\Gamma)\ ,$$
$$\mathcal{K} : H_\sigma^{r+2,s-\frac{1}{2}}(\mathbb{R}_+,\Gamma) \to H_\sigma^{r,s-\frac{1}{2}}(\mathbb{R}_+,\Gamma)\ ,$$
$$\mathcal{K}' : H_\sigma^{r+2,s+\frac{1}{2}}(\mathbb{R}_+,\Gamma) \to H_\sigma^{r,s+\frac{1}{2}}(\mathbb{R}_+,\Gamma)\ ,$$
$$\mathcal{W} : H_\sigma^{r+3,s+\frac{1}{2}}(\mathbb{R}_+,\Gamma) \to H_\sigma^{r,s-\frac{1}{2}}(\mathbb{R}_+,\Gamma)\ .$$

- Identities like $\mathcal{K} = \gamma_0(-1+\mathcal{S}\gamma_1)T \rightsquigarrow$ unnecessary losses.
- Key for improvements: $\mathrm{Im}\,\omega \geq \sigma > 0 \quad \Longrightarrow \quad$ Rellich identities don't have contribution from ∞ as in scattering theory.

A posteriori estimate for acoustic bvp

Weak formulation of boundary problem:

$$\langle 2(\mathcal{K}p - \mathcal{W}\phi) + \alpha\dot{\phi}, \dot{\psi}\rangle_\sigma = \langle F, \dot{\psi}\rangle_\sigma$$
$$\langle \alpha^{-1}p + 2(\mathcal{V}\dot{p} - \mathcal{K}'\dot{\phi}), q\rangle_\sigma = \langle G, q\rangle_\sigma \ .$$

Theorem

$$R_1 = F - \alpha\dot{\varphi}_{h,\Delta t} - 2\mathcal{K}p_{h,\Delta t} + 2\mathcal{W}\varphi_{h,\Delta t} \in L^2([0,T], L^2(\Gamma)) \ ,$$
$$R_2 = G - \alpha^{-1}p_{h,\Delta t} - 2\mathcal{V}\dot{p}_{h,\Delta t} + 2\mathcal{K}'\dot{\varphi}_{h,\Delta t} \in L^2([0,T], L^2(\Gamma)) \ ,$$

$\Longrightarrow$ following a posteriori estimate holds:

$$\|p - p_{h,\Delta t}\|_{0,0} + \|\varphi - \varphi_{h,\Delta t}\|_{0,\frac{1}{2}} + \|\varphi - \varphi_{h,\Delta t}\|_{1,0} \lesssim \|R_1\|_{L^2L^2} + \|R_2\|_{L^2L^2} \ .$$

powers of $h, \underset{2}{\Delta} t$ for more regular R_1, R_2:

$$LHS \lesssim \sum \left\{ \Delta t\|\partial_t R_i\|_{L^2L^2} + \|h \cdot \nabla R_i\|_{L^2L^2} + \Delta t\|h \cdot \nabla \partial_t R_i\|_{L^2L^2} \right\} \ .$$

Conclusions & Outlook

- Elliptic analysis of boundary integral operators + a posteriori estimates partly generalizes to wave equation
- (well-known) “Loss” of time derivatives compared to Laplace
- Adaptive TDBEM for geometric singularities
- Optimal estimates? Full space–time adaptivity?