Weighted Korn Inequalites for General External Cusps

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$\Omega \subset \mathbb{R}^n, \ u \in W^{1,p}(\Omega)^n, \ 1 < p < \infty$

Korn's Inequality:

$$\|Du\|_{L^p(\Omega)^n} \leq C\Big\{\|u\|_{L^p(\Omega)^n} + \|\varepsilon(u)\|_{L^p(\Omega)}\Big\}.$$

$\Omega \subset \mathbb{R}^n, \ u \in W^{1,p}(\Omega)^n, \ 1 < p < \infty$

General Korn's Inequality:

$$\|Du\|_{L^p(\Omega)^n} \leq C\Big\{\|u\|_{L^p(\Omega)^n} + \|\varepsilon(u)\|_{L^p(\Omega)}\Big\}.$$

$\Omega \subset \mathbb{R}^n, \ u \in W^{1,p}(\Omega)^n, \ 1 < p < \infty$

The first case of Korn's Inequality:

$$\|Du\|_{L^p(\Omega)^n} \leq C\Big\{\|u\|_{L^p(\Omega)^n} + \|\varepsilon(u)\|_{L^p(\Omega)}\Big\}.$$

$\Omega \subset \mathbb{R}^n, \ u \in W^{1,p}(\Omega)^n, \ 1 < p < \infty$

$$\|Du\|_{L^p(\Omega)^n} \leq C\Big\{\|u\|_{L^p(B)^n} + \|\varepsilon(u)\|_{L^p(\Omega)}\Big\}.$$

$\Omega \subset \mathbb{R}^n, \ u \in W^{1,p}(\Omega)^n, \ 1 < p < \infty$

(Second Case)

$$\|Du\|_{L^p(\Omega)^n} \leq C\|\varepsilon(u)\|_{L^p(\Omega)}$$

$\Omega \subset \mathbb{R}^n, \ u \in W^{1,p}(\Omega)^n, \ 1 < p < \infty$

(Second Case)

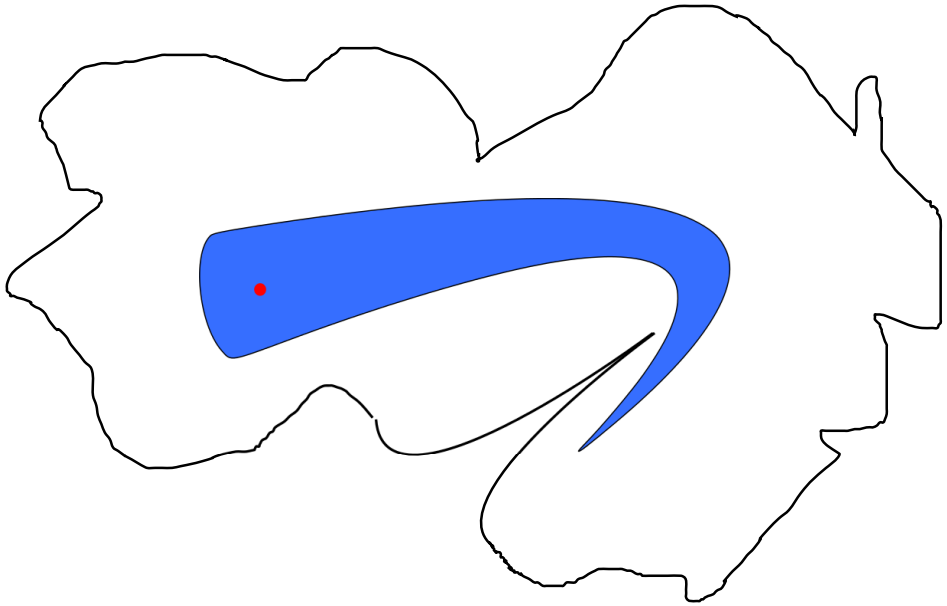
$$\|Du\|_{L^p(\Omega)^n} \leq C\|\varepsilon(u)\|_{L^p(\Omega)}$$

$$\int_\Omega \frac{Du - Du^t}{2} = 0$$

Definition

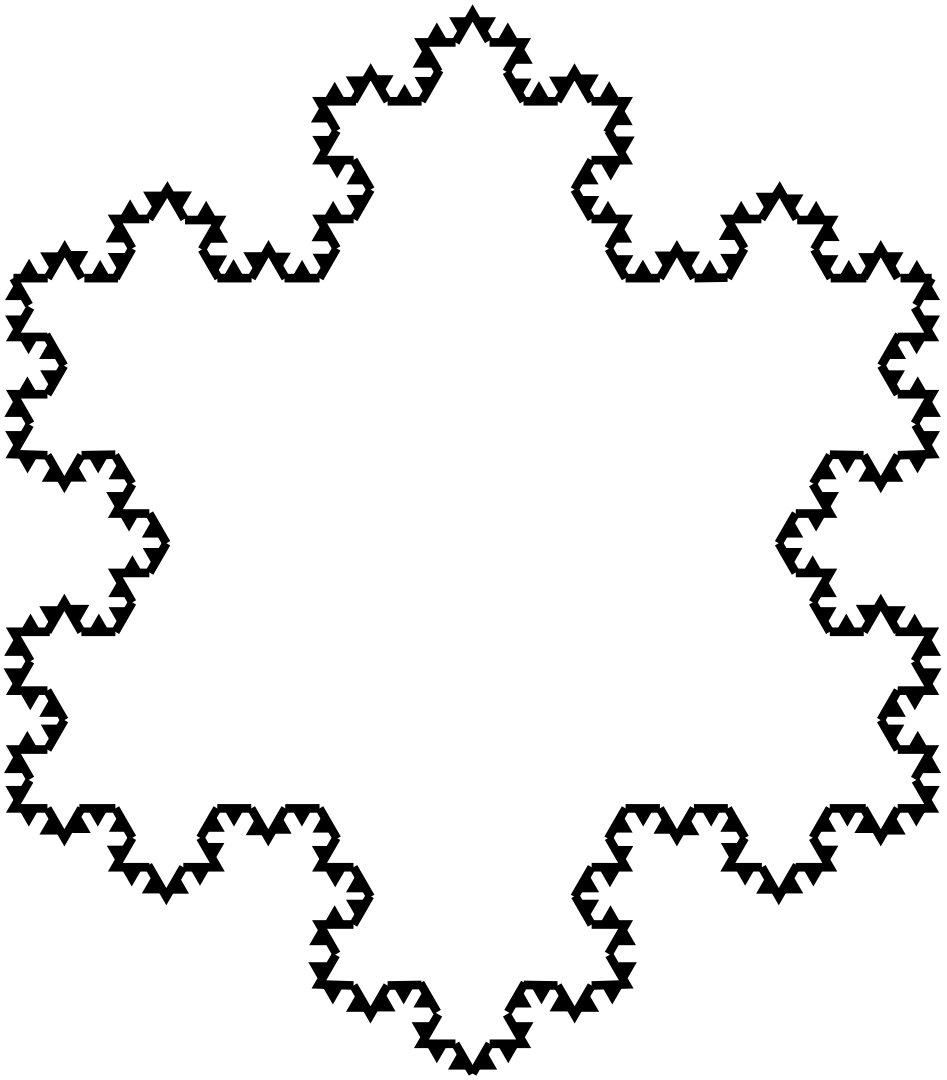
John Domain

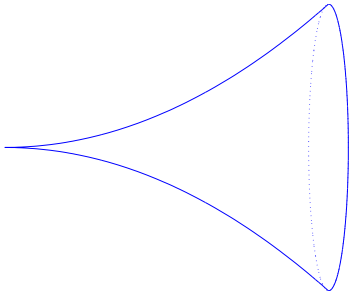
Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, and $x_0 \in \Omega$. We say that Ω is a C-John domain with respect to x_0 if for any $y \in \Omega$ there exists a curve $\rho : [0, l] \to \Omega$, parametrized by arclength such that $\rho(0) = y$ $\rho(l) = x_0$ and $d_\partial(\rho(t)) \geq Ct$.



Bogovskii's arguments for $u \in W_0^{1,p}$ such that $divu = f$, in domains star-shaped w.r.t. a ball can be generalized.

In particular general Korn's Inequality (and second case) holds for John domains (A.-Durán-Muschietti, 2006)





Counterexamples: Friedrich 1937, Geymonat-Gilardi 1998, Weck 1994, A.-Durán-López García 2012.

Hölder α domains, $d = d_{\partial\Omega}$,

$$\|d^{1-\alpha} Du\|_{L^p(\Omega)} \leq C\Big\{\|u\|_{L^p(\Omega)^n} + \|\varepsilon(u)\|_{L^p(\Omega)}\Big\} \tag{1}$$

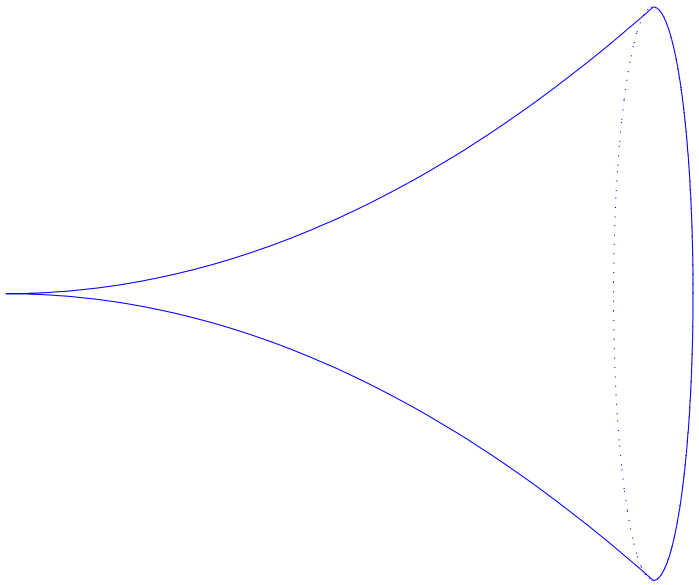
(A.-Durán-Lombardi 2006)

Hölder α domains, $\alpha \leq \beta \leq 1$

$$\|d^{1-\beta} Du\|_{L^p(\Omega)} \leq C\Big\{\|u\|_{L^p(\Omega)^n} + \|\varepsilon(u) d^{\alpha-\beta}\|_{L^p(\Omega)}\Big\} \qquad (1)$$

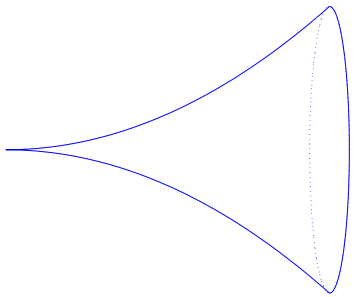
(A.-Durán-Lombardi 2006)

External Cusp $\Omega = \{(x, \mathbf{y}) \in (0,1) \times \mathbb{R}^{n-1} : \|\mathbf{y}\| < x^{\gamma}\}$



Distance to the tip of the cusp $d = d_0$, if $(x, \mathbf{y}) \in \Omega$ $d = d_0 \sim x$
$B \subset\subset \Omega$

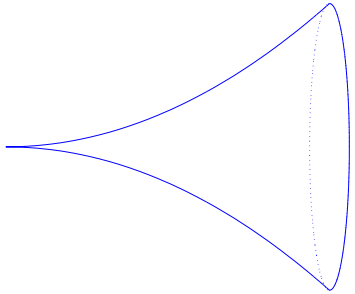
$$\|d^{\gamma-1} Du\|_{L^p(\Omega)} \leq C\Big\{\|u\|_{L^p(B)^n} + \|\varepsilon(u)\|_{L^p(\Omega)}\Big\}$$



(Durán-López García 2010)

Distance to the tip of the cusp $d = d_0$, if $(x, \mathbf{y}) \in \Omega$ $d = d_0 \sim x$
$B \subset\subset \Omega$

$$\|\phi'(x) Du\|_{L^p(\Omega)} \leq C \Big\{ \|u\|_{L^p(B)^n} + \|\varepsilon(u)\|_{L^p(\Omega)} \Big\}$$



(Durán-López García 2010)
Calling $\phi(x) = x^\gamma$

Distance to the tip of the cusp $d = d_0 \sim x$, $0 \leq \beta$,

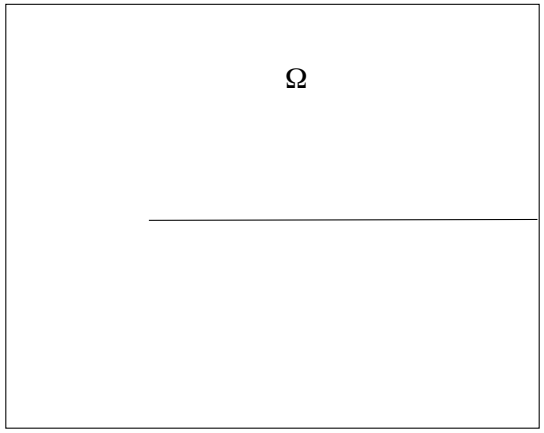
$$\|d^{\beta} Du\|_{L^p(\Omega)} \leq C\Big\{\|u\|_{L^p(B)^n} + \|\varepsilon(u) d^{1-\gamma+\beta}\|_{L^p(\Omega)}\Big\}$$

(Durán-López García 2010)

For $\beta = 0$,

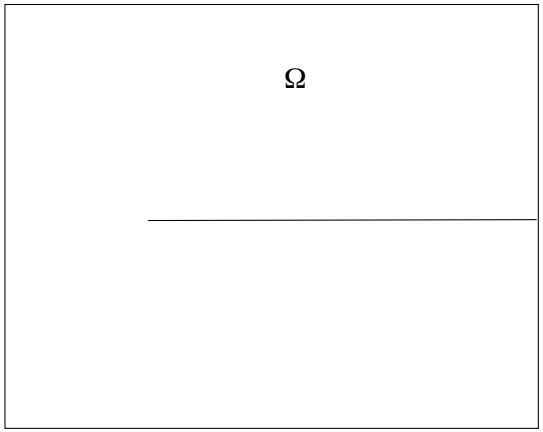
$$\|Du\|_{L^p(\Omega)} \leq C\Big\{\|u\|_{L^p(B)^n} + \|\varepsilon(u)\phi'^{-1}\|_{L^p(\Omega)}\Big\}$$

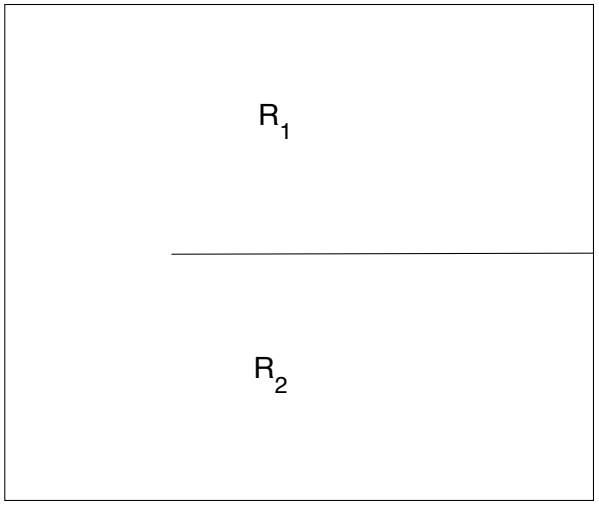
(Durán-López García 2010)

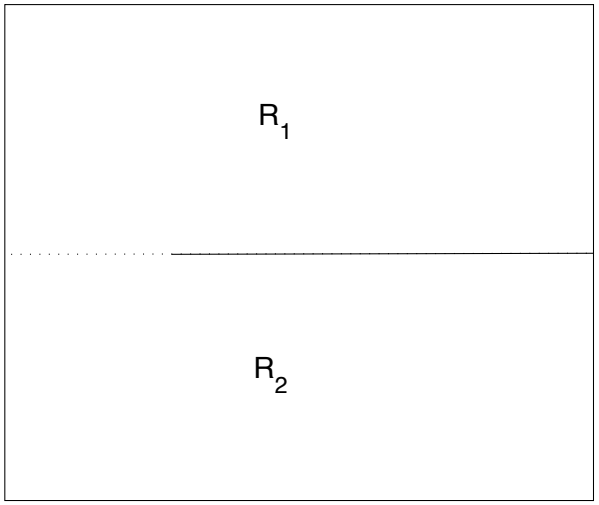

Ω

Inequality $\mathcal{I}$, $\int_\Omega w = 0$

w=(Du-Du^t)/2


Ω


R_1
R_2



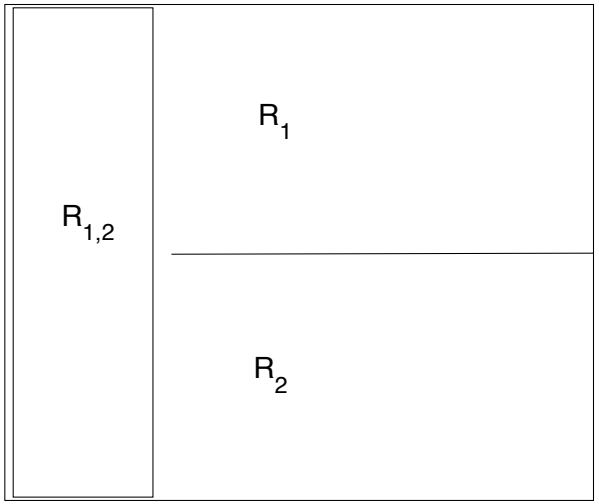
$$A_1 = \frac{1}{|R_1|}\int_{R_1} w \qquad A_2 = \frac{1}{|R_2|}\int_{R_2} w$$

$$\int_\Omega w^p = \int_{R_1} w^p + \int_{R_2} w^p \leq$$
$$C\left(\int_{R_1}(w - A_1)^p + \int_{R_2}(w - A_2)^p + \int_{R_1} A_1^p + \int_{R_2} A_2^p\right)$$

$$A_1|R_1| + A_2|R_2| = 0$$

$$\int_{R_1} A_1^p + \int_{R_2} A_2^p = A_1^p|R_1| + A_2^p|R_2| \leq C(A_1 - A_2)^p|R_2| = C\int_{R_2}(A_1 - A_2)^p$$

$$\int_{R_1} A_1^p + \int_{R_2} A_2^p = A_1^p|R_1| + A_2^p|R_2| \leq C(A_1 - A_2)^p|R_1| = C\int_{R_2}(A_1 - A_2)^p$$


R_1
$R_{1,2}$
R_2

$$\int_{R_2}(A_1 - A_2)^p \leq C(\int_{R_2}(A_1 - A_{1,2})^p + \int_{R_2}(A_{1,2} - A_2)^p)$$

$$\int_{R_2}(A_1 - A_2)^p \leq C\left(\int_{R_1 \cap R_{1,2}} (A_1 - A_{1,2})^p + \int_{R_2 \cap R_{1,2}} (A_{1,2} - A_2)^p\right)$$

$$\int_{R_1 \cap R_{1,2}} (A_1 - A_{1,2})^p$$

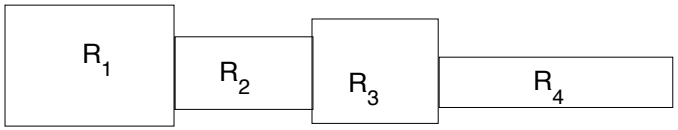
$$\int_{R_1 \cap R_{1,2}} (A_1 - A_{1,2})^p \leq C\left(\int_{R_1 \cap R_{1,2}} (A_1 - u)^p + \int_{R_1 \cap R_{1,2}} (u - A_{1,2})^p\right)$$

$$\int_{R_1 \cap R_{1,2}} (A_1 - A_{1,2})^p \leq$$
$$C\left(\int_{R_1 \cap R_{1,2}} (A_1 - u)^p + \int_{R_1 \cap R_{1,2}} (u - A_{1,2})^p\right) \leq$$
$$C\left(\int_{R_1} (A_1 - u)^p + \int_{R_{1,2}} (u - A_{1,2})^p\right)$$

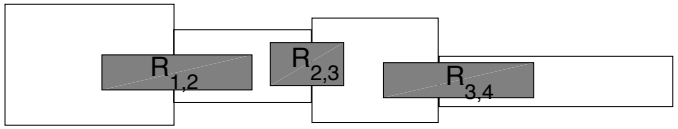
The final constant depends on the ratios $\frac{|R_i|}{|R_j|}, \frac{|R_i|}{|R_{i,j}|}$ with $i, j \in \{1, 2\}$

The final constant depends on the ratios $\frac{|R_i|}{|R_j|}, \frac{|R_i|}{|R_{i,j}|}$ with $i, j \in \{1, 2\}$ and on the individual constants of the inequality $\mathcal{I}$ for each sub-domain $R_1, R_2, R_{1,2}$.

$$\cup_i R_i \subset \Omega$$



$$\cup_i R_{i,i+1} \subset \Omega$$



$$|\Omega \setminus \cup_i R_i| = 0$$

C_i the constant of the inequality $\mathcal{I}$ on R_i.

the ratios $\frac{|R_i|}{|R_j|}, \frac{|R_j|}{|R_{i,i+1}|} \leq C$

you can bound the constant of the inequality $\mathcal{I}$ in Ω by

$$\sharp(rect)max\{C_i\}$$

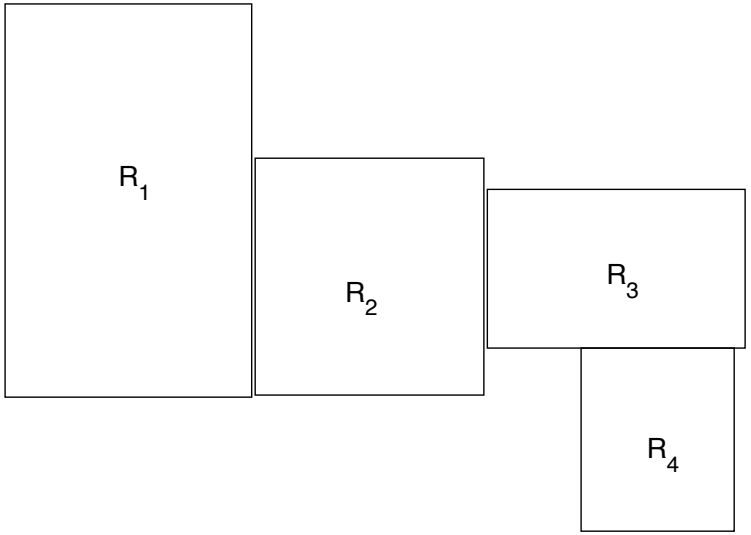
a and b are *C-comparable*, and we write $a \underset{C}{\sim} b$, if $\frac{1}{C}a \leq b \leq Ca$.

With $R \subset \mathbb{R}^n$ we denote an open rectangle with edges parallels to the coordinate axis. $len_i(R)$ is the length of the R's i-th edge.

R_1, R_2 two rectangles, we write $R_1 \underset{C}{\sim} R_2$ if $len_i(R_1) \underset{C}{\sim} len_i(R_2)$

Definition

A countable collection of rectangles $\mathcal{C} = \{R_i\}$ for which $\sum_i |R_i| < \infty$, is called a *Chain of Rectangles* if a) $\overline{R}_i \cap \overline{R}_j = \emptyset$ for $|i-j| > 1$, b) for any i, R_i and R_{i+1} are touching, and c) there exists a constant C such that $R_i \underset{C}{\sim} R_{i+1}$, for any i.


R_1
R_2
R_3
R_4

Remark

there exists a rectangle $R_{i,i+1} \subset \overline{R}_i \cup \overline{R}_{i+1}$ *and a constant* $\tilde{C}$ *depending only on* C*, such that*

$$R_{i,i+1} \underset{\tilde{C}}{\sim} (R_{i,i+1} \cap R_i) \underset{\tilde{C}}{\sim} R_i \underset{\tilde{C}}{\sim} (R_{i,i+1} \cap R_{i+1}) \underset{\tilde{C}}{\sim} R_{i+1}$$

Remark

thanks to that

$$|R_{i,i+1}| \underset{\tilde{C}}{\sim} |(R_{i,i+1} \cap R_i)| \underset{\tilde{C}}{\sim} |R_i| \underset{\tilde{C}}{\sim} |(R_{i,i+1} \cap R_{i+1})| \underset{\tilde{C}}{\sim} |R_{i+1}|$$

with $\tilde{C}$ depending only on C.

Definition

The collection of intermediate rectangles $R_{i,i+1}$ is denoted with $\mathcal{C}_I = \{R_{i,i+1}\}$.

Theorem (Second Case of Korn's Inequality for Chains of Rectangles)

- *Let $\mathcal{C} = \{R_i\}$ be a chain of rectangles, and C_i the constants for the second case of Korn's inequality on R_i.*
- *A domain Ω, such that for any i, $R_i, R_{i,i+1} \subset \Omega$ and $|\Omega \setminus \cup\mathcal{C}| = 0$.*

Then for any $u \in W^{1,p}(\Omega)^n$ such that $\frac{1}{|\Omega|}\int_\Omega \frac{Du - Du^t}{2} = 0$ we have

$$\|Du\|_{L^p(\Omega)^{n\times n}} \leq C(1+\mathbf{A})\|\varepsilon(u)\|_{L^p_\sigma(R)^{n\times n}},$$

$$\mathbf{A} = \sup_{k>0}\left(\sum_{j=k}^{\infty}|R_j|\right)^{\frac{1}{p}}\left(\sum_{j=0}^{k}|R_j|^{1-p'}\right)^{\frac{1}{p'}}$$

and the weight σ is constant on each R_i being $\sigma|_{R_i} = C_i$.

If we assume (for instance) that for any k, $|R_{k+1}| \leq \alpha_k |R_k|$ with $\alpha_k \leq \alpha < 1$: we get

$$\mathbf{A} \leq \left(\frac{1}{1-\alpha}\right)^{1/p} \left(\frac{1}{1-\alpha^{p'-1}}\right)^{1/p'},$$

and therefore

$$\|Du\|_{L^p(\Omega)^{n\times n}} \leq C\|\varepsilon(u)\|_{L^p_\sigma(R)^{n\times n}}.$$

the weight σ is constant on each R_i being $\sigma|_{R_i} = C_i$.

Taking into account that $C_i \leq C\frac{L_{M_i}}{L_{m_i}}$ (a particular case for convex domains Durán 2012) we get

$$\|Du\|_{L^p(\Omega)^{n\times n}} \leq C\|\varepsilon(u)\|_{L^p_\sigma(R)^{n\times n}}.$$

the weight σ is constant on each R_i being $\sigma|_{R_i} = \frac{L_{M_i}}{L_{m_i}}$.

Consider a weight ω

$$\omega(x) \underset{C}{\sim} \omega_{R_i} \underset{C}{\sim} \omega_{R_{i+1}} \ \forall i, \forall x \in R_i.$$

being ω_{R_i} appropriate constants. The same proof:

$$\|Du\|_{L^p_\omega(\Omega)^{n\times n}} \leq C\|\varepsilon(u)\|_{L^p_{\omega\sigma(R)}}.$$

(for any k, $\omega(R_{k+1}) \leq \alpha_k \omega(R_k)$ with $\alpha_k \leq \alpha < 1$)

Everything done as far for the second case of Korn's inequality for chains of rectangles can be done for Poincaré. Since the constant in the Poincaré inequality for rectangles (and in general for convex domains) depends only on the diameter of the rectangle, the weight involved in the inequality can be weakened as it is stated below.

Theorem (Poincaré inequality for Chains of Rectangles)

Same hypotheses:
Then for any $u \in W^{1,p}_{\omega}(\Omega)^n$ *such that* $\frac{1}{\omega(\Omega)} \int_\Omega u\omega = 0$, *we have*

$$\|u\|_{L^p_\omega(\Omega)} \leq C\|Du\|_{L^p_{\omega\sigma}(\Omega)},$$

where the weight σ *is constant on each* R_i *and can be taken as* $\sigma|_{R_i} = L_{M_i}$.

Second Case of Korn's Inequality + Poincaré Inequality yields Korns inequality of the form

$$\|Du\|_{L^p_\omega(\Omega)^n} \leq C\Big\{\|u\|_{L^p(B)^n} + \|\varepsilon(u)\|_{L^p_{\omega\sigma}(\Omega)}\Big\}.$$

The job done for chains of rectangles can be easily generalized as follows.

Definition

A collection of open sets $\mathcal{W} = \{\Omega_i\}$ such that $\Omega_i \cap \Omega_j = \emptyset$ is called a chain of quasi-rectangles if a) exists a chain of rectangles $\mathcal{C} = \{R_i\}$ such that $R_i \subset \Omega_i \subset CR_i$, and b) there is a fixed constant C such that $C_{K_i} \leq C\frac{L_{M_i}}{L_{m_i}}$ and $C_{P_i} \leq CL_{M_i}$ being C_{K_i} and C_{P_i} the constants for the Korn's second inequality and Poincaré inequality for Ω_i, respectively.

Theorem (Second Case of Korn's Inequality for Chains of Quasi-Rectangles)

- *Let* $\mathcal{W} = \{\Omega_i\}$ *be a chain of quasi-rectangles, and* C_i *the constants for the second case of Korn's inequality on* Ω_i.
- *A domain* Ω*, such that for any* i, $\Omega_i, R_{i,i+1} \subset \Omega$ *and* $|\Omega \setminus \cup\mathcal{W}| = 0$.
- *If* $|\Omega_{k+1}| \leq \alpha_k |\Omega_k|$ k, $\alpha_k \leq \alpha < 1$.

Then for any $u \in W^{1,p}(\Omega)^n$ *such that* $\frac{1}{|\Omega|}\int_\Omega \frac{Du - Du^t}{2} = 0$ *we have*

$$\|Du\|_{L^p(\Omega)^{n\times n}} \leq C\|\varepsilon(u)\|_{L^p_\sigma(R)^{n\times n}},$$

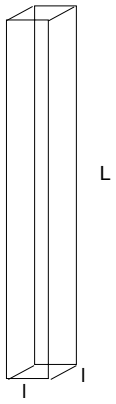
where the weight σ *is constant on each* Ω_i *being* $\sigma|_{R_i} = C_i = \frac{L_{M_i}}{L_{m_i}}$.

The weighted version also follows in the same fashion.

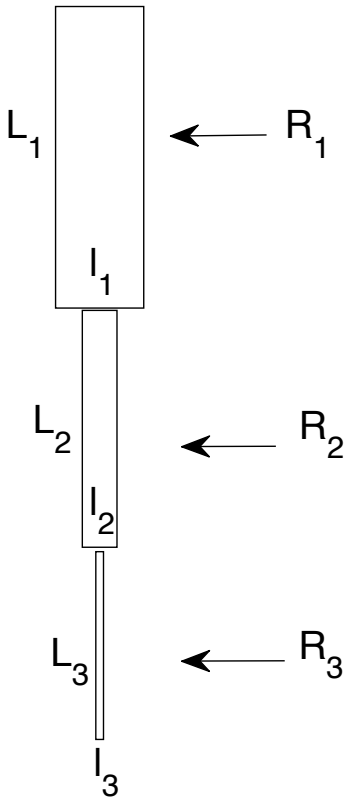
Application to external cusps

Consider a chain of quasi-rectangles $\mathcal{W} = \{\Omega_i\}$, such that each R_i of the associated chain of rectangles has $n-1$ short edges of equal size that we denote $l(R_i) = l_i$ and a long edge (the vertical, i.e. along the x_n axis) $L(R_i) = L_i$.

Application to external cusps



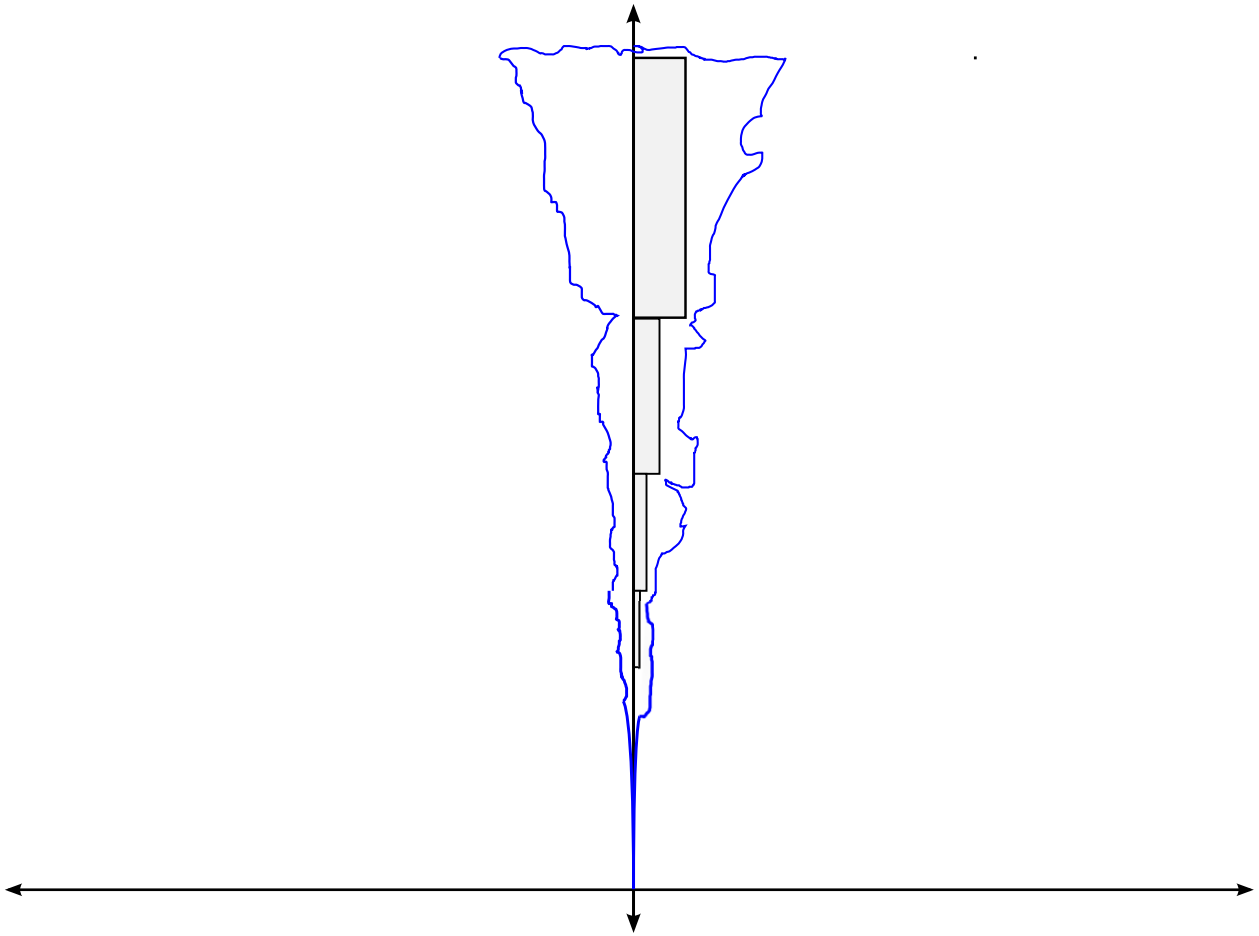
Application to external cusps



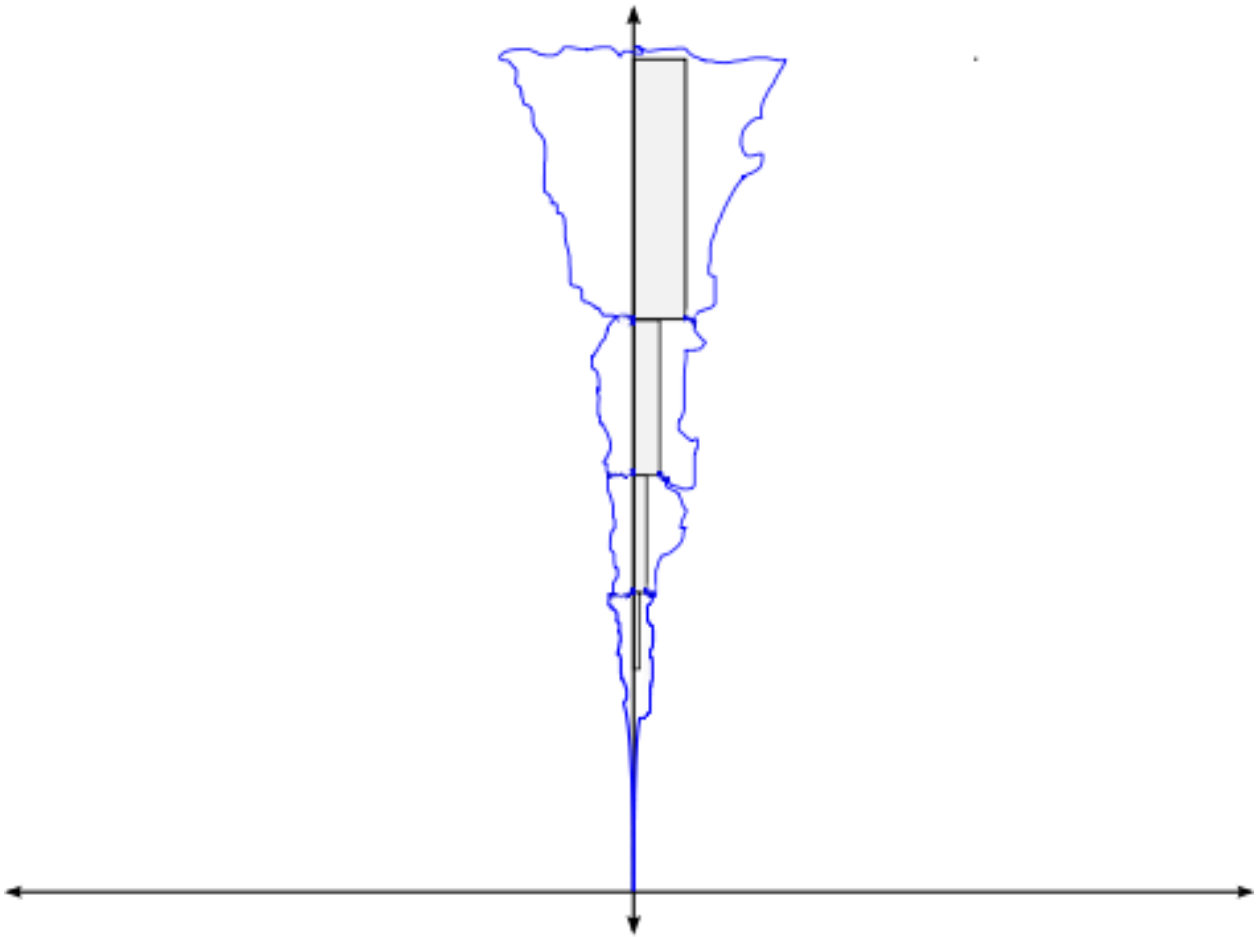
Application to external cusps

Consider a domain Ω, such that for any i, $\Omega_i, R_{i,i+1} \subset \Omega$ and $|\Omega \setminus \cup\mathcal{W}| = 0$.

Application to external cusps



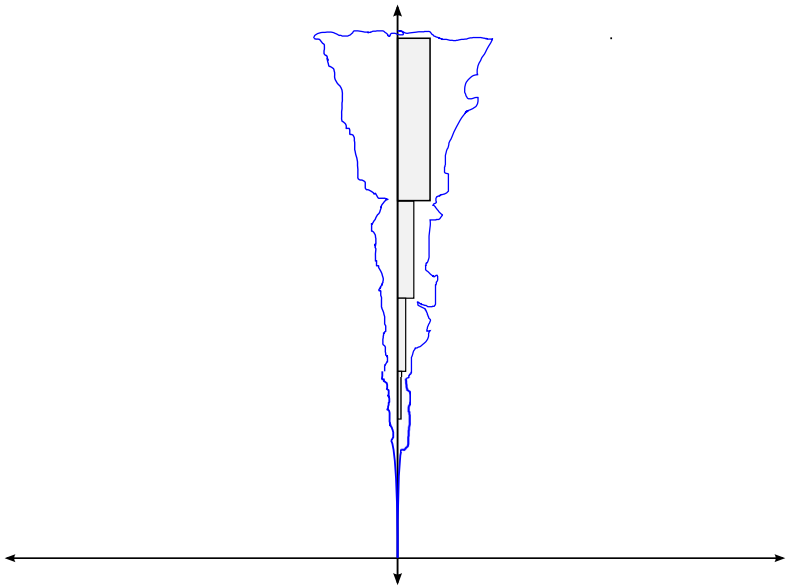
Application to external cusps



Then for any $u \in W^{1,p}(\Omega)^n$ such that $\frac{1}{|\Omega|}\int_\Omega \frac{Du-Du^t}{2} = 0$ we have

$$\|Du\|_{L^p(\Omega)} \leq C\|\varepsilon(u)\|_{L^p_\sigma(\Omega)},$$

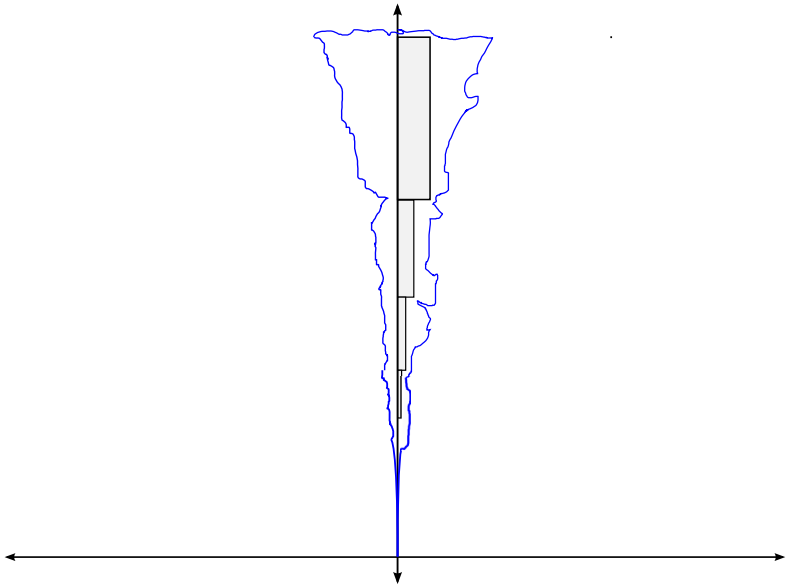
where the weight σ is constant on each Ω_i being $\sigma|_{R_i} = C_i = \frac{L_i}{l_i}$.

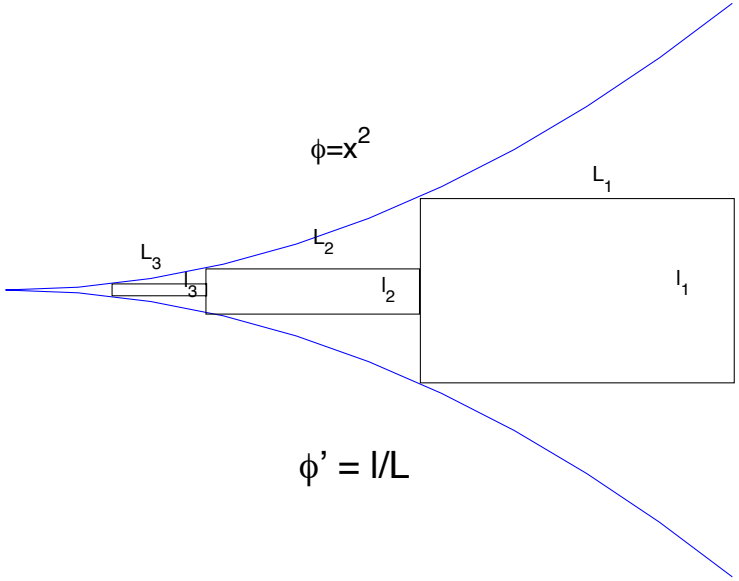


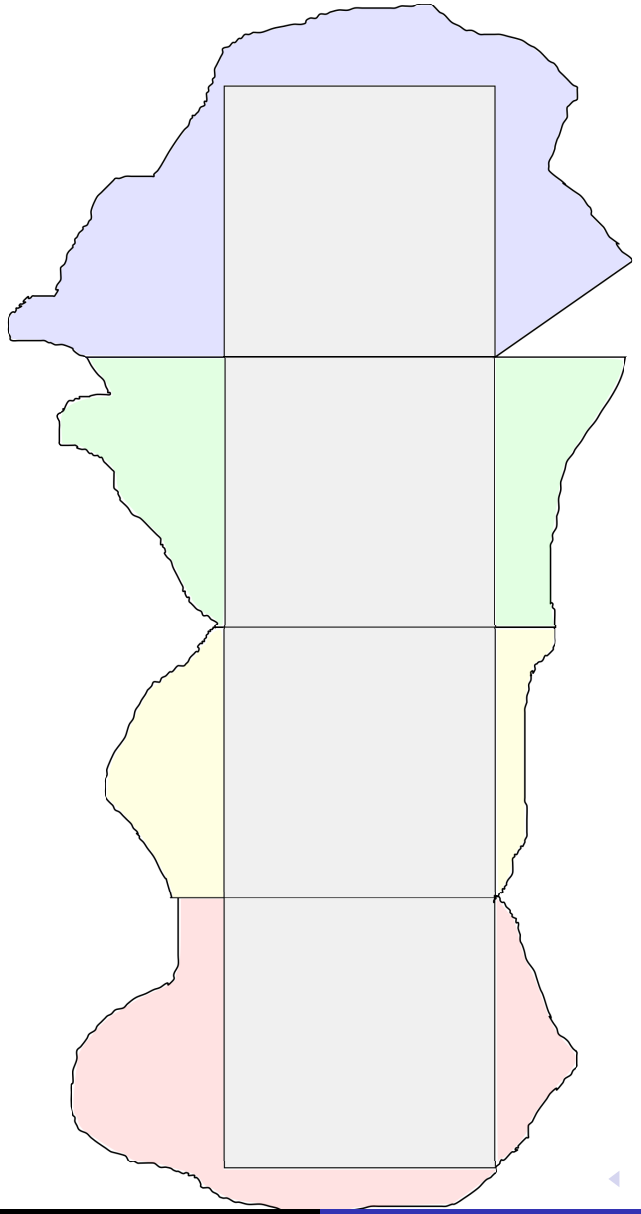
Then for any $u \in W^{1,p}(\Omega)^n$ such that $\frac{1}{|\Omega|}\int_\Omega \frac{Du - Du^t}{2} = 0$ we have

$$\|Du\|_{L^p(\Omega)} \leq C\|\varepsilon(u)\phi'^{-1}\|_{L^p(\Omega)},$$

being ϕ the profile of the chain of rectangles.




ϕ=x^2
L1
L2
L3
l3
l2
l1
ϕ' = l/L



$$\sharp(quasirect)max\{C_i\}$$

$$4max\{C_i\}$$

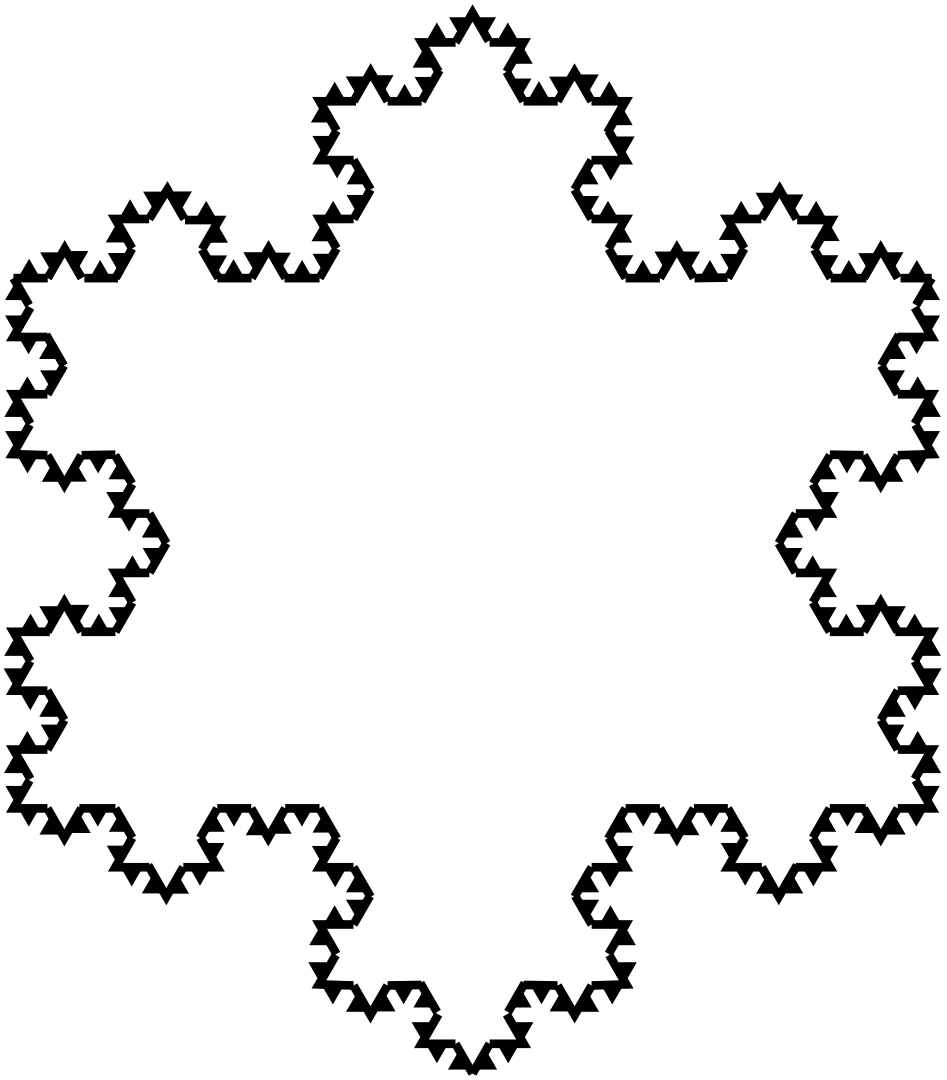
$$\frac{L}{l} max\{C_i\}$$

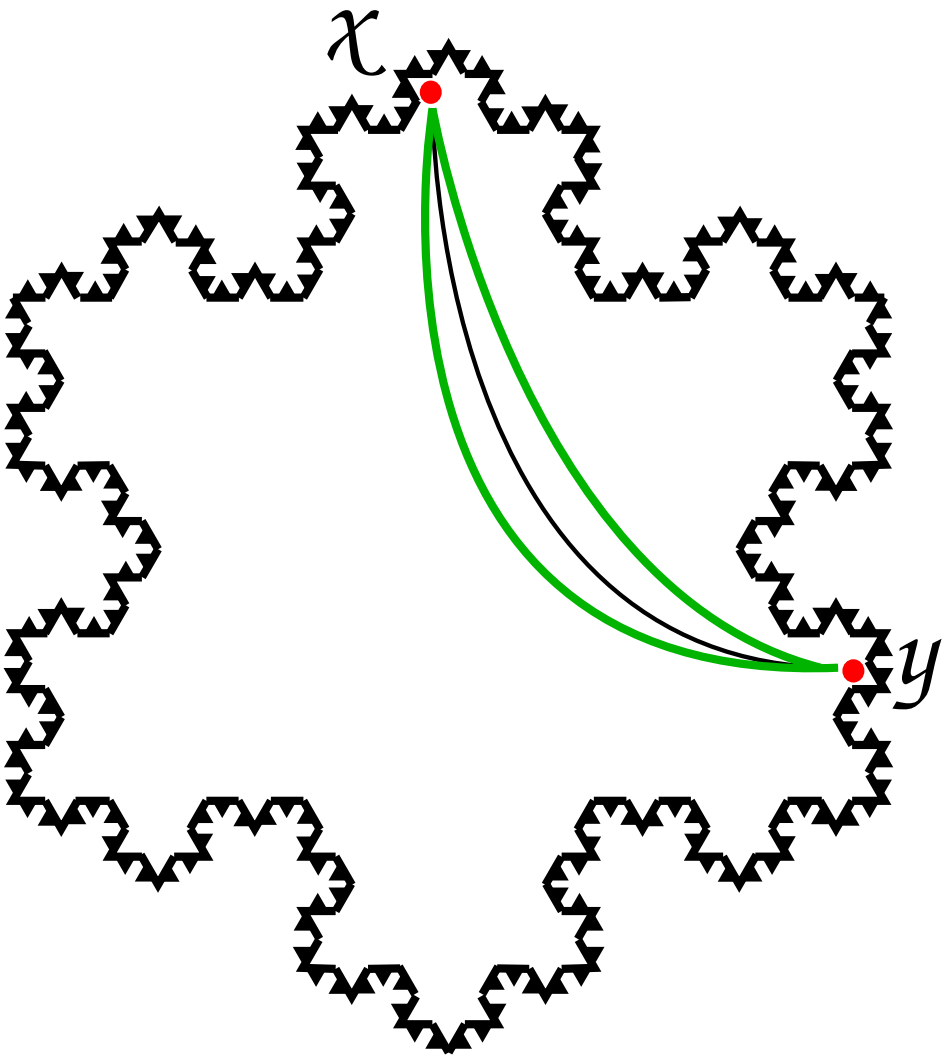
Definition

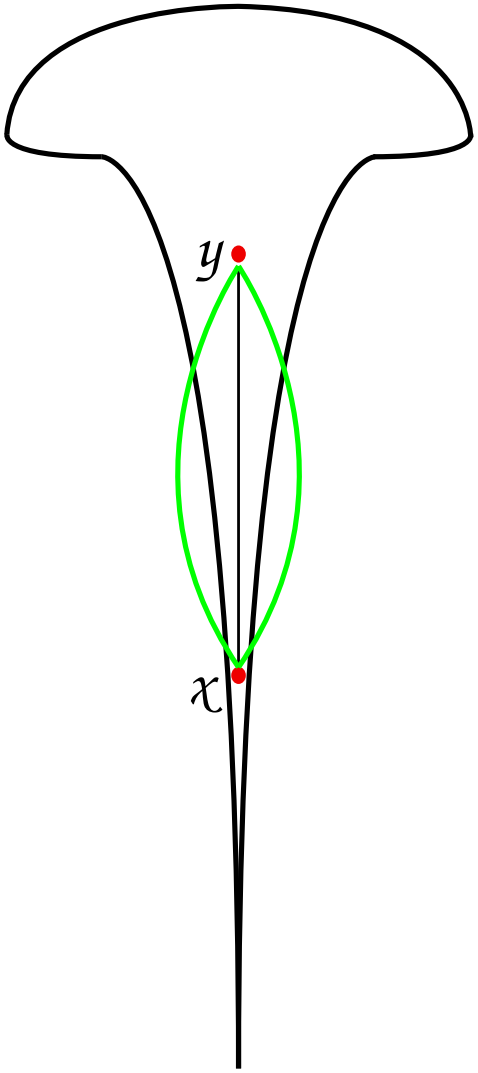
Uniform Domain Let Ω be a domain in $\mathbb{R}^n$, we say that Ω is a uniform domain if for every pair of points x, y in Ω there is a rectifiable curve $\gamma \subset \Omega$ joining x and y:

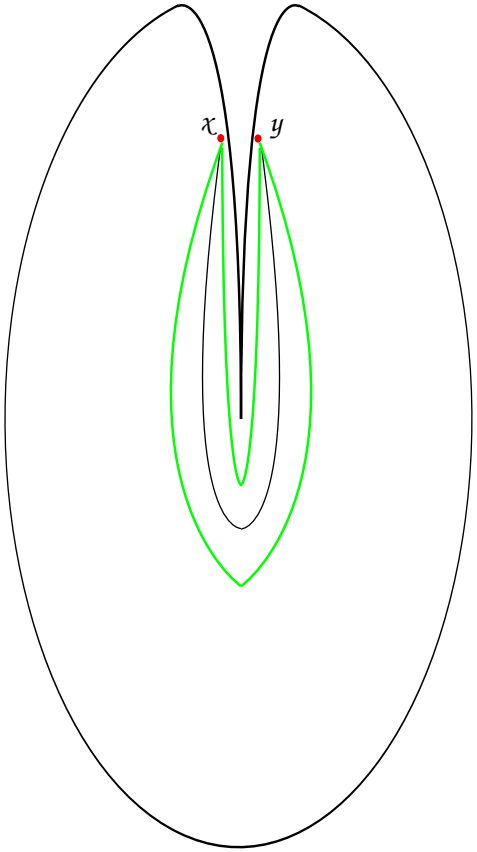
$$\ell(\gamma) \leq \frac{1}{\varepsilon}|x-y|$$

$$d_\partial(z) \leq \varepsilon \frac{|x-z||z-y|}{|x-y|} \quad \forall z \in \gamma.$$



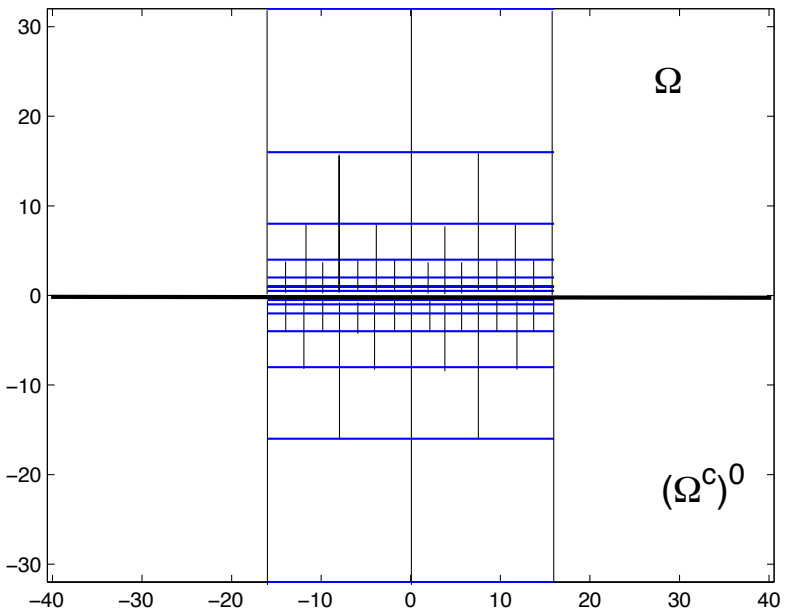

x
y


y
x

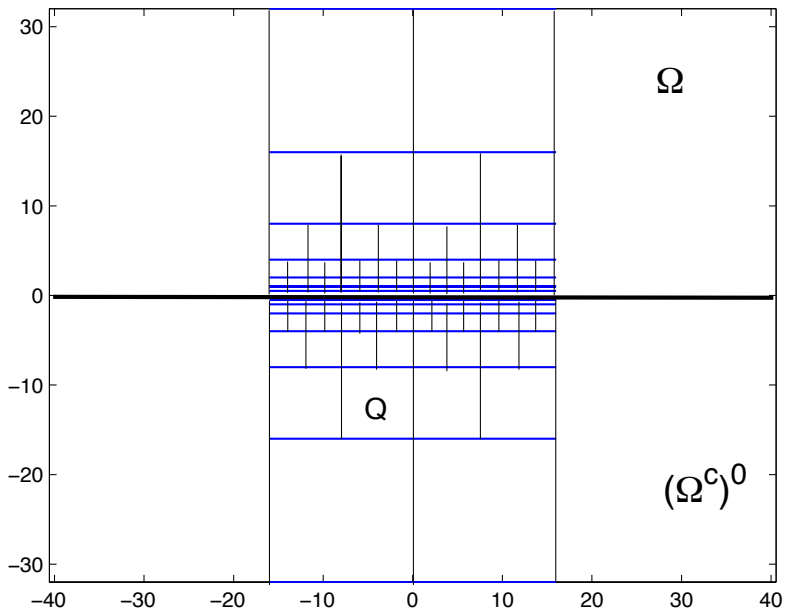

x
y

Uniform $\subset$ *John*

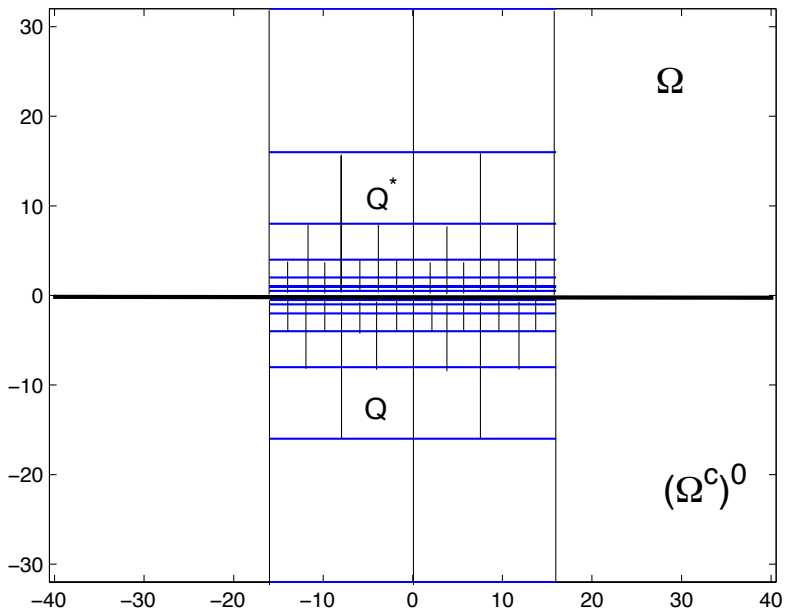
$$\mathcal{W} = \mathcal{W}(\Omega) \qquad \mathcal{W}^c = \mathcal{W}((\Omega^c)^o)$$



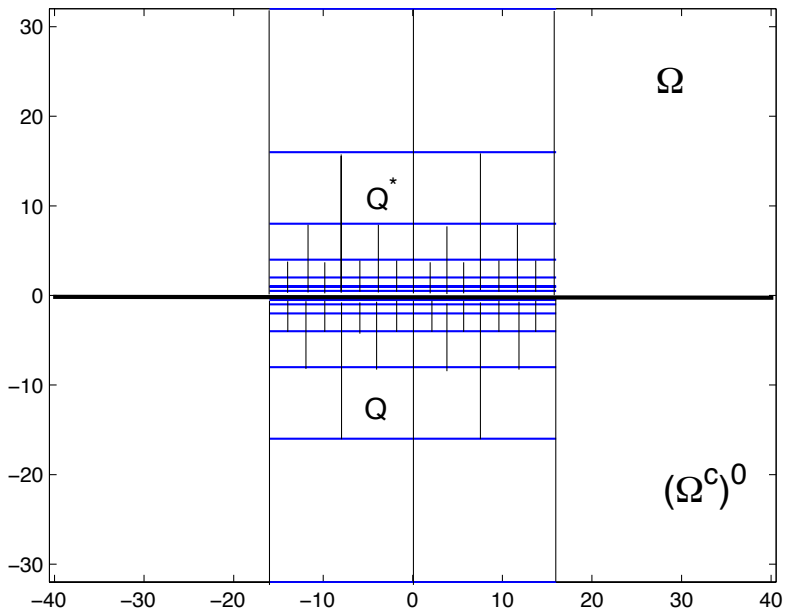
$$\mathcal{W} = \mathcal{W}(\Omega) \qquad \mathcal{W}^c = \mathcal{W}((\Omega^c)^o)$$



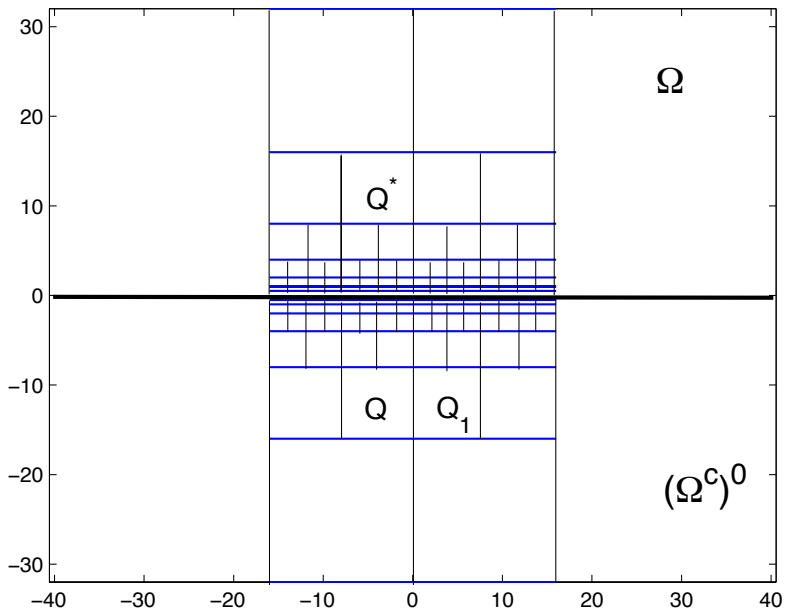
$$\mathcal{W} = \mathcal{W}(\Omega) \qquad \mathcal{W}^c = \mathcal{W}((\Omega^c)^o)$$



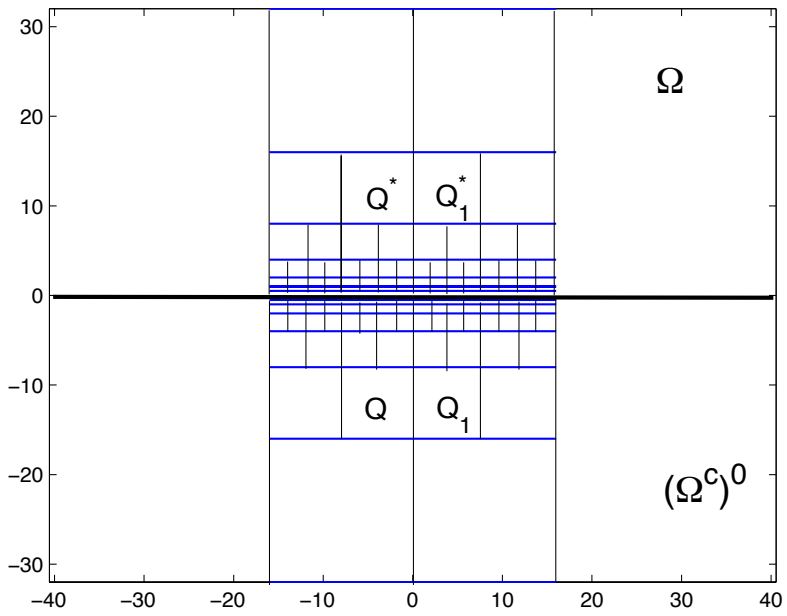
$$\mathcal{W} = \mathcal{W}(\Omega) \qquad \mathcal{W}^c = \mathcal{W}((\Omega^c)^o)$$



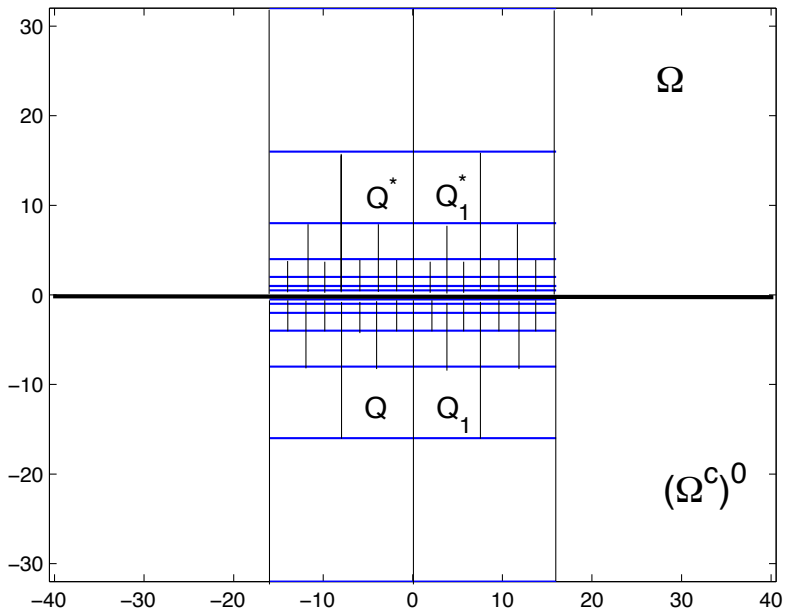
$$\mathcal{W} = \mathcal{W}(\Omega) \qquad \mathcal{W}^c = \mathcal{W}((\Omega^c)^o)$$



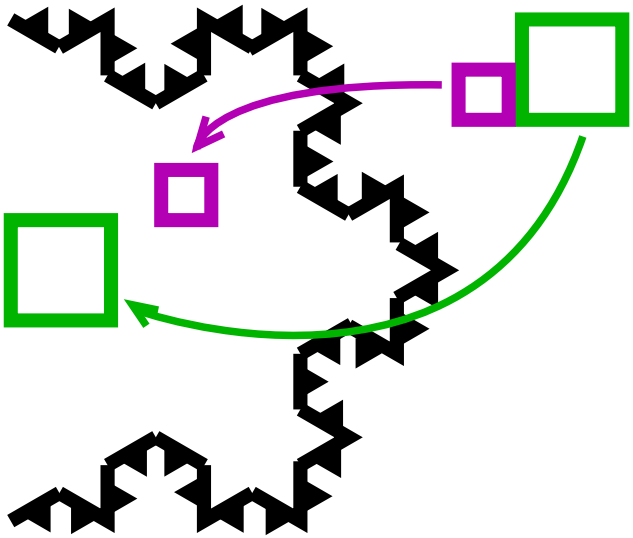
$$\mathcal{W} = \mathcal{W}(\Omega) \qquad \mathcal{W}^c = \mathcal{W}((\Omega^c)^o)$$



$$\mathcal{W} = \mathcal{W}(\Omega) \qquad \mathcal{W}^c = \mathcal{W}((\Omega^c)^o)$$



$Q, Q_1 \in \mathcal{W}$, Finite Chain Between Q^*, Q_1^*



For every cube $T \in \mathcal{W}(\Omega)$, let us define:

$$P_T(x) = a + M(x - x_T) \tag{2}$$

where $a \in \mathbb{R}^n$ and $M = (m_{ij}) \in \mathbb{R}^{n \times n}$ are defined by:

$$a = \frac{1}{|T|} \int_T u \qquad M_{i,j} = \frac{1}{2|T|} \int_T \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)$$

and x_T is the center of T.

Thank you!