Weighted Korn Inequalities for General External Cusps

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$\Omega \subset \mathbb{R}^n$, $u \in W^{1,p}(\Omega)^n$, $1 < p < \infty$

Korn’s Inequality:

$$\|Du\|_{L^p(\Omega)^n} \leq C\left\{\|u\|_{L^p(\Omega)^n} + \|\varepsilon(u)\|_{L^p(\Omega)}\right\}.$$
Let $\Omega \subset \mathbb{R}^n$, $u \in W^{1,p}(\Omega)^n$, $1 < p < \infty$.

General Korn's Inequality:

$$\|Du\|_{L^p(\Omega)^n} \leq C\left\{\|u\|_{L^p(\Omega)^n} + \|\varepsilon(u)\|_{L^p(\Omega)}\right\}.$$
\( \Omega \subset \mathbb{R}^n, \ u \in W^{1,p}(\Omega)^n, \ 1 < p < \infty \)

The first case of Korn’s Inequality:

\[
\| Du \|_{L^p(\Omega)^n} \leq C \left\{ \| u \|_{L^p(\Omega)^n} + \| \varepsilon(u) \|_{L^p(\Omega)} \right\}.
\]
\[ \Omega \subset \mathbb{R}^n, \quad u \in W^{1,p}(\Omega)^n, \quad 1 < p < \infty \]

\[ \| Du \|_{L^p(\Omega)^n} \leq C \left\{ \| u \|_{L^p(B)^n} + \| \varepsilon(u) \|_{L^p(\Omega)} \right\}. \]
$\Omega \subset \mathbb{R}^n$, $u \in W^{1,p}(\Omega)^n$, $1 < p < \infty$

(Second Case)

$\|Du\|_{L^p(\Omega)^n} \leq C\|\varepsilon(u)\|_{L^p(\Omega)}$
\[ \Omega \subset \mathbb{R}^n, \quad u \in W^{1,p}(\Omega)^n, \quad 1 < p < \infty \]

(Second Case)

\[ \| Du \|_{L^p(\Omega)^n} \leq C \| \varepsilon(u) \|_{L^p(\Omega)} \]

\[ \int_{\Omega} \frac{Du - Du^t}{2} = 0 \]
Definition

**John Domain**

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, and $x_0 \in \Omega$. We say that $\Omega$ is a C-John domain with respect to $x_0$ if for any $y \in \Omega$ there exists a curve $\rho : [0, l] \rightarrow \Omega$, parametrized by arclength such that $\rho(0) = y$, $\rho(l) = x_0$ and $d_\partial(\rho(t)) \geq Ct$. 
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Bogovskii’s arguments for $u \in W^{1,p}_0$ such that $\text{div} u = f$, in domains star-shaped w.r.t. a ball can be generalized.
In particular general Korn’s Inequality (and second case) holds for John domains (A.-Durán-Muschietti, 2006)
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Hölder $\alpha$ domains, $d = d_{\partial \Omega}$,

$$\|d^{1-\alpha} Du\|_{L^p(\Omega)} \leq C \left\{ \|u\|_{L^p(\Omega)}^n + \|\varepsilon(u)\|_{L^p(\Omega)} \right\}$$  \hspace{1cm} (1)

(A.-Durán-Lombardi 2006)
Hölder $\alpha$ domains, $\alpha \leq \beta \leq 1$

\[\| d^{1-\beta} Du \|_{L^p(\Omega)} \leq C \left\{ \| u \|_{L^p(\Omega)^n} + \| \varepsilon(u) d^{\alpha-\beta} \|_{L^p(\Omega)} \right\} \quad (1)\]

( A.-Durán-Lombardi 2006)
External Cusp $\Omega = \{(x, y) \in (0, 1) \times \mathbb{R}^{n-1} : \|y\| < x^\gamma\}$
Distance to the tip of the cusp $d = d_0$, if $(x, y) \in \Omega$ $d = d_0 \sim x$

$\forall B \subset \subset \Omega$

$$\| d^{\gamma-1} Du \|_{L^p(\Omega)} \leq C \left\{ \| u \|_{L^p(B)^n} + \| \varepsilon(u) \|_{L^p(\Omega)} \right\}$$

(Durán-López García 2010)
Distance to the tip of the cusp $d = d_0$, if $(x, y) \in \Omega \ d = d_0 \sim x$

$B \subset \subset \Omega$

$$\| \phi'(x)Du \|_{L^p(\Omega)} \leq C\left\{ \| u \|_{L^p(B)^n} + \| \varepsilon(u) \|_{L^p(\Omega)} \right\}$$

(Durán-López García 2010)

Calling $\phi(x) = x^\gamma$
Distance to the tip of the cusp $d = d_0 \sim x$, $0 \leq \beta$, 

$$\| d^{\beta} Du \|_{L^p(\Omega)} \leq C \left\{ \| u \|_{L^p(B)^n} + \| \varepsilon(u) d^{1-\gamma+\beta} \|_{L^p(\Omega)} \right\}$$

(Durán-López García 2010)
For $\beta = 0$,

$$\|Du\|_{L^p(\Omega)} \leq C\left\{\|u\|_{L^p(B)^n} + \|\varepsilon(u)\phi^{-1}\|_{L^p(\Omega)}\right\}$$

(Durán-López García 2010)
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Inequality $\mathcal{I}$, $\int_{\Omega} w = 0$
$w = (Du - Du^t) / 2$
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$\Omega$

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$R_1$

$R_2$
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\[ A_1 = \frac{1}{|R_1|} \int_{R_1} w \quad A_2 = \frac{1}{|R_2|} \int_{R_2} w \]
\[ \int_{\Omega} w^p = \int_{R_1} w^p + \int_{R_2} w^p \leq C(\int_{R_1} (w - A_1)^p + \int_{R_2} (w - A_2)^p + \int_{R_1} A_1^p + \int_{R_2} A_2^p) \]
A_1 |R_1| + A_2 |R_2| = 0
\[ \int_{R_1} A_1^p + \int_{R_2} A_2^p = A_1^p |R_1| + A_2^p |R_2| \leq C (A_1 - A_2)^p |R_2| = C \int_{R_2} (A_1 - A_2)^p \]
\[ \int_{R_1} A_1^p + \int_{R_2} A_2^p = A_1^p |R_1| + A_2^p |R_2| \leq C (A_1 - A_2)^p |R_1| = C \int_{R_2} (A_1 - A_2)^p \]
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\[ \int_{R_2} (A_1 - A_2)^p \leq C (\int_{R_2} (A_1 - A_{1,2})^p + \int_{R_2} (A_{1,2} - A_2)^p) \]
\[ \int_{R_2} (A_1 - A_2)^p \leq C \left( \int_{R_1 \cap R_{1,2}} (A_1 - A_{1,2})^p + \int_{R_2 \cap R_{1,2}} (A_{1,2} - A_2)^p \right) \]
\[ \int_{R_1 \cap R_{1,2}} (A_1 - A_{1,2})^p \]
\[ \int_{R_1 \cap R_{1,2}} (A_1 - A_{1,2})^p \leq C \left( \int_{R_1 \cap R_{1,2}} (A_1 - u)^p + \int_{R_1 \cap R_{1,2}} (u - A_{1,2})^p \right) \]
$\int_{R_1 \cap R_{1,2}} (A_1 - A_{1,2})^p \leq C(\int_{R_1 \cap R_{1,2}} (A_1 - u)^p + \int_{R_1 \cap R_{1,2}} (u - A_{1,2})^p) \leq C(\int_{R_1} (A_1 - u)^p + \int_{R_{1,2}} (u - A_{1,2})^p)$
The final constant depends on the ratios $\frac{|R_i|}{|R_j|}, \frac{|R_i|}{|R_{i,j}|}$ with $i, j \in \{1, 2\}$. 
The final constant depends on the ratios $\frac{|R_i|}{|R_j|}, \frac{|R_i|}{|R_{i,j}|}$ with $i, j \in \{1, 2\}$ and on the individual constants of the inequality $\mathcal{I}$ for each sub-domain $R_1, R_2, R_{1,2}$. 
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\[ \bigcup_i R_i \subset \Omega \]

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\[ \bigcup_{i} R_{i,i+1} \subset \Omega \]
\[ |\Omega \setminus \bigcup_i R_i| = 0 \]
\( C_i \) the constant of the inequality \( \mathcal{I} \) on \( R_i \).
the ratios \( \frac{|R_i|}{|R_j|}, \frac{|R_j|}{|R_{i,i+1}|} \leq C \)
you can bound the constant of the inequality $I$ in $\Omega$ by

$$\#(\text{rect}) \max \{C_i\}$$
a and b are $C$-comparable, and we write $a \sim_{C} b$, if $\frac{1}{C} a \leq b \leq Ca$. 
With $R \subset \mathbb{R}^n$ we denote an open rectangle with edges parallels to the coordinate axis. $len_i(R)$ is the length of the $R$’s $i$-th edge.
$R_1, R_2$ two rectangles, we write $R_1 \sim_C R_2$ if $\text{len}_i(R_1) \sim_C \text{len}_i(R_2)$
Definition
A countable collection of rectangles $\mathcal{C} = \{R_i\}$ for which $\sum_i |R_i| < \infty$, is called a *Chain of Rectangles* if a) $\overline{R}_i \cap \overline{R}_j = \emptyset$ for $|i - j| > 1$, b) for any $i$, $R_i$ and $R_{i+1}$ are touching, and c) there exists a constant $C$ such that $R_i \sim_C R_{i+1}$, for any $i$. 
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Remark

there exists a rectangle $R_{i,i+1} \subset \overline{R_i} \cup \overline{R_{i+1}}$ and a constant $\tilde{C}$ depending only on $C$, such that

$$R_{i,i+1} \sim \tilde{C} (R_{i,i+1} \cap R_i) \sim \tilde{C} R_i \sim \tilde{C} (R_{i,i+1} \cap R_{i+1}) \sim \tilde{C} R_{i+1}$$
Remark

thanks to that

\[ |R_{i,i+1}| \sim_{\tilde{C}} |(R_{i,i+1} \cap R_i)| \sim_{\tilde{C}} |R_i| \sim_{\tilde{C}} |(R_{i,i+1} \cap R_{i+1})| \sim_{\tilde{C}} |R_{i+1}| \]

with \( \tilde{C} \) depending only on \( C \).
Definition
The collection of intermediate rectangles $R_{i,i+1}$ is denoted with $\mathcal{C}_I = \{R_{i,i+1}\}$.
Theorem (Second Case of Korn’s Inequality for Chains of Rectangles)

Let $C = \{ R_i \}$ be a chain of rectangles, and $C_i$ the constants for the second case of Korn’s inequality on $R_i$.

A domain $\Omega$, such that for any $i$, $R_i, R_{i+1} \subset \Omega$ and $|\Omega \setminus \bigcup C| = 0$.

Then for any $u \in W^{1,p}(\Omega)^n$ such that $\frac{1}{|\Omega|} \int_{\Omega} \frac{Du - Du^t}{2} = 0$ we have

$$
\|Du\|_{L^p(\Omega)^{n \times n}} \leq C(1 + A)\|\varepsilon(u)\|_{L^p_\sigma(R)^{n \times n}},
$$

$$
A = \sup_{k > 0} \left( \sum_{j=k}^{\infty} |R_j|^{1/p} \right)^{1/p} \left( \sum_{j=0}^{k} |R_j|^{1-p'} \right)^{1/p'},
$$

and the weight $\sigma$ is constant on each $R_i$ being $\sigma|_{R_i} = C_i$. 

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If we assume (for instance) that for any $k$, $|R_{k+1}| \leq \alpha_k |R_k|$ with $\alpha_k \leq \alpha < 1$: we get

$$A \leq \left( \frac{1}{1 - \alpha} \right)^{1/p} \left( \frac{1}{1 - \alpha^{p'}} \right)^{1/p'},$$

and therefore

$$\| Du \|_{L^p(\Omega)^{n \times n}} \leq C \| \varepsilon(u) \|_{L^p_{\sigma}(R)^{n \times n}}.$$

the weight $\sigma$ is constant on each $R_i$ being $\sigma|_{R_i} = C_i$. 

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Taking into account that $C_i \leq C \frac{L_{M_i}}{L_{m_i}}$ (a particular case for convex domains Durán 2012) we get

$$\|Du\|_{L^p(\Omega)^n} \leq C \|\varepsilon(u)\|_{L^p_{\sigma}(R)^n}.$$ 

the weight $\sigma$ is constant on each $R_i$ being $\sigma|_{R_i} = \frac{L_{M_i}}{L_{m_i}}$. 
Consider a weight $\omega$

$$\omega(x) \sim \omega_{R_i} \sim \omega_{R_{i+1}} \forall i, \forall x \in R_i.$$ 

being $\omega_{R_i}$ appropriate constants. The same proof:

$$\|Du\|_{L^p(\Omega)^n} \leq C\|\varepsilon(u)\|_{L^p_{\omega_\sigma(R)}}.$$

(for any $k$, $\omega(R_{k+1}) \leq \alpha_k \omega(R_k)$ with $\alpha_k \leq \alpha < 1$)
Everything done as far for the second case of Korn’s inequality for chains of rectangles can be done for Poincaré. Since the constant in the Poincaré inequality for rectangles (and in general for convex domains) depends only on the diameter of the rectangle, the weight involved in the inequality can be weakened as it is stated below.
Theorem (Poincaré inequality for Chains of Rectangles)

Same hypotheses:
Then for any \( u \in W^{1,p}_\omega(\Omega)^n \) such that \( \frac{1}{\omega(\Omega)} \int_{\Omega} u \omega = 0 \), we have

\[
\|u\|_{L^p_\omega(\Omega)} \leq C \|Du\|_{L^p_{\omega \sigma}(\Omega)},
\]

where the weight \( \sigma \) is constant on each \( R_i \) and can be taken as \( \sigma|_{R_i} = L_{M_i} \).
Second Case of Korn’s Inequality + Poincaré Inequality yields
Korn’s inequality of the form

$$\| Du \|_{L^p(\Omega)^n} \leq C \left\{ \| u \|_{L^p(B)^n} + \| \varepsilon(u) \|_{L^p_{\omega\sigma}(\Omega)} \right\}.$$
The job done for chains of rectangles can be easily generalized as follows.
Definition
A collection of open sets $\mathcal{W} = \{\Omega_i\}$ such that $\Omega_i \cap \Omega_j = \emptyset$ is called a chain of quasi-rectangles if a) exists a chain of rectangles $\mathcal{C} = \{R_i\}$ such that $R_i \subset \Omega_i \subset CR_i$, and b) there is a fixed constant $C$ such that $C_{K_i} \leq C \frac{L_{M_i}}{L_{m_i}}$ and $C_{P_i} \leq C L_{M_i}$ being $C_{K_i}$ and $C_{P_i}$ the constants for the Korn’s second inequality and Poincaré inequality for $\Omega_i$, respectively.
Theorem (Second Case of Korn’s Inequality for Chains of Quasi-Rectangles)

Let \( \mathcal{W} = \{\Omega_i\} \) be a chain of quasi-rectangles, and \( C_i \) the constants for the second case of Korn’s inequality on \( \Omega_i \).

A domain \( \Omega \), such that for any \( i \), \( \Omega_i, R_{i,i+1} \subset \Omega \) and \( |\Omega \setminus \bigcup \mathcal{W}| = 0 \).

If \( |\Omega_{k+1}| \leq \alpha_k |\Omega_k| \) \( k \), \( \alpha_k \leq \alpha < 1 \).

Then for any \( u \in W^{1,p}(\Omega)^n \) such that \( \frac{1}{|\Omega|} \int_{\Omega} \frac{Du - Du^t}{2} = 0 \) we have

\[
\| Du \|_{L^p(\Omega)^{n \times n}} \leq C \| \varepsilon(u) \|_{L^p_{\sigma}(R)^{n \times n}},
\]

where the weight \( \sigma \) is constant on each \( \Omega_i \) being \( \sigma|_{R_i} = C_i = \frac{L_{Mi}}{L_{mi}} \).
The weighted version also follows in the same fashion.
Consider a chain of quasi-rectangles $\mathcal{W} = \{\Omega_i\}$, such that each $R_i$ of the associated chain of rectangles has $n-1$ short edges of equal size that we denote $l(R_i) = l_i$ and a long edge (the vertical, i.e. along the $x_n$ axis) $L(R_i) = L_i$. 
Application to external cusps
Application to external cusps
Consider a domain $\Omega$, such that for any $i$, $\Omega_i, R_{i,i+1} \subset \Omega$ and $|\Omega \setminus \bigcup \mathcal{W}| = 0$. 
Application to external cusps
Application to external cusps
Then for any \( u \in W^{1,p}(\Omega)^n \) such that \( \frac{1}{|\Omega|} \int_{\Omega} \frac{Du - Du^t}{2} = 0 \) we have

\[
\|Du\|_{L^p(\Omega)} \leq C \|\varepsilon(u)\|_{L^p_{\sigma}(\Omega)},
\]

where the weight \( \sigma \) is constant on each \( \Omega_i \) being \( \sigma|_{R_i} = C_i = \frac{L_i}{l_i} \).
Then for any $u \in W^{1,p}(\Omega)^n$ such that $\frac{1}{|\Omega|} \int_{\Omega} \frac{Du-Du^t}{2} = 0$ we have

$$\|Du\|_{L^p(\Omega)} \leq C \|\varepsilon(u)\phi^{-1}\|_{L^p(\Omega)},$$

being $\phi$ the profile of the chain of rectangles.
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The extension approach

\[ \phi = x^2 \]

\[ L_3 \]

\[ L_2 \]

\[ L_1 \]

\[ l_1 l_2 \]

\[ l_3 \]

\[ \phi' = l/L \]

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\[ \#(\text{quasirect}) \max \{ C_i \} \]
$4 \max \{ C_i \}$
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\[ \frac{L}{\ell} \max \{ C_i \} \]
Definition

**Uniform Domain** Let $\Omega$ be a domain in $\mathbb{R}^n$, we say that $\Omega$ is a uniform domain if for every pair of points $x, y$ in $\Omega$ there is a rectifiable curve $\gamma \subset \Omega$ joining $x$ and $y$:

$$\ell(\gamma) \leq \frac{1}{\varepsilon}|x - y|$$

$$d_\partial(z) \leq \varepsilon \frac{|x - z||z - y|}{|x - y|} \quad \forall z \in \gamma.$$
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Uniform $\subset$ John
\[ \mathcal{W} = \mathcal{W}(\Omega) \quad \mathcal{W}^c = \mathcal{W}((\Omega^c)^c) \]
\[ \mathcal{W} = \mathcal{W}(\Omega) \quad \mathcal{W}^c = \mathcal{W}((\Omega^c)^0) \]
\[ \mathcal{W} = \mathcal{W}(\Omega) \quad \mathcal{W}^c = \mathcal{W}((\Omega^c)^c) \]
$\mathcal{W} = \mathcal{W}(\Omega) \quad \mathcal{W}^c = \mathcal{W}((\Omega^c)^o)$
\[ \mathcal{W} = \mathcal{W}(\Omega) \quad \mathcal{W}^c = \mathcal{W}((\Omega^c)^o) \]
$\mathcal{W} = \mathcal{W}(\Omega) \quad \mathcal{W}^c = \mathcal{W}((\Omega^c)^o)$
$\mathcal{W} = \mathcal{W}(\Omega) \quad \mathcal{W}^c = \mathcal{W}((\Omega^c)^0)$

$Q, Q_1 \in \mathcal{W}$, Finite Chain Between $Q^*, Q_1^*$
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For every cube $T \in \mathcal{W}(\Omega)$, let us define:

$$P_T(x) = a + M(x - x_T)$$

(2)

where $a \in \mathbb{R}^n$ and $M = (m_{ij}) \in \mathbb{R}^{n \times n}$ are defined by:

$$a = \frac{1}{|T|} \int_T u \quad M_{i,j} = \frac{1}{2|T|} \int_T \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)$$

and $x_T$ is the center of $T$. 
Thank you!