On Poincaré's and Lions' lemma and on De Rham's theorem

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joint work with
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Outline

I. Weak versions of a classical theorem of Poincaré

In this work, we assume that **Ω** is a **bounded open connected** of $\mathbb{R}^3$ with **Lipschitz-continuous boundary**. The notation ${}_{X'}<,>_X$ denotes a duality pairing between a topological space X and its dual X'. We shall use bold characters for the vector fields or the vector spaces and the non-bold characters for the scalars. The letter C denotes a constant that not necessarily the same at its various occurrences.

We set

$$V = \{\boldsymbol{\varphi} \in \boldsymbol{H}_0^1(\Omega);\ \operatorname{div} \boldsymbol{\varphi} = 0\},$$

$$\boldsymbol{H}_0(\operatorname{div}, \Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega),\ \operatorname{div} \boldsymbol{v} \in L^2(\Omega),\ \ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \ \text{ on } \Gamma\}$$

and we note that

$$[\boldsymbol{H}_0(\operatorname{div}, \Omega)]' \hookrightarrow \boldsymbol{H}^{-1}(\Omega).$$

Theorem 1 (De Rham-Lions-Necas-Korn-Poincaré)

Let Ω be a bounded, connected open subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary. The followings are equivalent:
(i) div $: \boldsymbol{H}_0^1(\Omega)/V \mapsto L_0^2(\Omega)$ is an isomorphism.
(ii) **Lions lemma**: $L^2(\Omega) = \{q \in H^{-1}(\Omega);\ \nabla q \in \boldsymbol{H}^{-1}(\Omega)\}$.
(iii) **Necas inequality**:

$$\forall q \in L^2(\Omega), \quad \|q\|_{L^2(\Omega)} \leq C(\|q\|_{H^{-1}(\Omega)} + \|\nabla q\|_{\boldsymbol{H}^{-1}(\Omega)}) \tag{1}$$

(iv) **Korn's inequality**:

$$\forall \boldsymbol{v} \in \boldsymbol{H}^1(\Omega), \quad \|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)} \leq C(\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)} + \|e(\boldsymbol{v})\|_{\boldsymbol{L}^2(\Omega)}) \tag{2}$$

with $e(\boldsymbol{v})_{ij} = \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}$.
(v) **Weak lemma of Poincaré**: Given any distribution $\boldsymbol{f} \in H_0(\mathrm{div}, \Omega)'$ that satisfies

$$\mathbf{curl}\ \boldsymbol{f} = \mathbf{0} \text{ in } \boldsymbol{H}^{-2}(\Omega) \text{ and } {}_{H_0(\mathrm{div},\Omega)'}< \boldsymbol{f}\,,\ \boldsymbol{v} >_{H_0(\mathrm{div},\Omega)} = 0 \quad \text{for all } \boldsymbol{v} \in K_T(\Omega), \tag{3}$$

there exists a scalar potential χ in $L^2(\Omega)$ such that

$$\boldsymbol{f} = \mathbf{grad}\ \chi \quad \text{in } \Omega \quad \text{and} \quad \|\chi\|_{L^2(\Omega)} \leq C\|\boldsymbol{f}\,\|_{H_0(\mathrm{div},\Omega)'}. \tag{4}$$

Recall first the **original Poincaré Lemma**:

Let Ω a bounded open **simply connected** of $\mathbb{R}^3$ and let

$$\boldsymbol{f} \in \mathcal{C}^1(\Omega) \quad \text{such that} \quad \mathbf{curl}\ \boldsymbol{f} = \mathbf{0}.$$

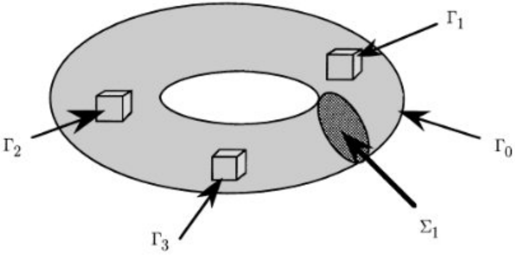
Then

$$\boldsymbol{f} = \mathbf{grad}\ \chi \quad \text{with} \quad \chi \in \mathcal{C}^2(\Omega).$$

We suppose that $\boldsymbol{\Omega}$ is an open set multiply connected with a boundary Γ non connected. We denote $\Gamma = \bigcup_{i=0}^{I} \Gamma_i$ with Γ_i the connected components of Γ

and $\boldsymbol{\Sigma} = \bigcup_{j=1}^{J} \Sigma_j$ and Σ_j a finite number of cuts.

$\boldsymbol{\Omega}^\circ = \Omega \setminus \cup_{j=1}^{J} \Sigma_j$ is simply connected.



We recall also that $K_T(\Omega)$ is defined by

$$\boldsymbol{K}_T(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\,\boldsymbol{v} = 0,\ \mathbf{curl}\,\boldsymbol{v} = \mathbf{0}\ \text{ in } \Omega \text{ and } \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\}.$$

is spanned by $\widetilde{\mathbf{grad}}\, q_j^T$ for $1 \leq j \leq J$, where each q_j^T is solution in $H^1(\Omega^\circ)$ of the problem solution of:

$$\begin{cases} -\Delta q_j^T = 0 & \text{in } \Omega^\circ, \\ \partial_n\, q_j^T = 0 & \text{on } \Gamma, \\ \left[\, q_j^T \,\right]_k = \text{constant} \quad \text{and} \quad [\,\partial_n\, q_j^T\,]_k = 0,\ 1 \leq k \leq J, \\ \left\langle \partial_n\, q_j^T,\, 1 \right\rangle_{\Sigma_k} = \delta_{j\,k},\ 1 \leq k \leq J, \end{cases}$$

We note that $\boldsymbol{K}_T^p(\Omega) = \{0\}$ if $J = 0$, i.e Ω is simply connected.

Proof of Theorem 1. First, we have the following identity

$$\{q \in H^{-1}(\Omega);\, \nabla q \in \boldsymbol{H}^{-1}(\Omega)\} = \{q \in H^{-1}(\Omega);\, \nabla q \in H_0(\mathrm{div}, \Omega)'\},$$

because the density of $\mathcal{D}(\Omega)$ in the space

$$\boldsymbol{H}_0^1(\mathrm{div};\, \Omega) = \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega);\, \mathrm{div}\, \boldsymbol{v} \in H_0^1(\Omega)\}.$$

1. Implication (i) $\Longrightarrow$ (ii). Remark first that the isomorphism of point (i) is equivalent to the following

$$\textbf{grad} \;: L^2(\Omega)/\mathbb{R} \mapsto V^\circ \tag{5}$$

isomorphism, where the polar set V° is defined by

$$V^\circ = \{\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega);\, \forall \boldsymbol{v} \in V, < \boldsymbol{f}\,, \boldsymbol{v} >= 0\}.$$

Point (ii) is then consequence of the density of $\mathcal{D}(\Omega)$ in $\boldsymbol{H}_0^1(\mathrm{div};\, \Omega)$.

2. Implication (ii) $\Longrightarrow$ (iii). Point (iii) is a simple consequence of Banach's theorem, the continuity of the identity from $L^2(\Omega)$ into $H^{-1}(\Omega)$ and the continuity of the gradient operator from $L^2(\Omega)$ into $\boldsymbol{H}^{-1}(\Omega)$.

3. Implication (iii) $\Longrightarrow$ (i). Using the compactness imbedding of $L^2(\Omega)$ into $H^{-1}(\Omega)$ and Peetre-Tartar theorem, we deduce the isomorphism (5) and by duality the point (i).

4. Implication (ii) $\Longrightarrow$ (iv). As in Duvaut-Lions, we prove first the identity

$$\boldsymbol{H}^1(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega);\ e(\boldsymbol{v}) \in \boldsymbol{L}^2(\Omega)\}.$$

Using Banach's theorem, we deduce the Korn's inequality.

5. Implication (i) $\Longrightarrow$ (v). Let $\boldsymbol{f} \in H_0(\mathrm{div}, \Omega)'$ that satisfies (3). Hence, there exist $\boldsymbol{h} \in L^2(\Omega)^3$ and $\chi_0 \in L^2(\Omega)$ such that

$$\boldsymbol{f} = \boldsymbol{h} + \mathbf{grad}\ \chi_0 \quad \text{in } \Omega. \tag{6}$$

Observe that, thanks to the density of $\mathcal{D}(\Omega)$ in $H_0(\mathrm{div}, \Omega)$,

$$_{H_0(\mathrm{div},\Omega)'} < \mathbf{grad}\ \chi_0,\ \boldsymbol{v} >_{H_0(\mathrm{div},\Omega)} = 0 \quad \text{for all } \ \boldsymbol{v} \in K_T(\Omega).$$

Therefore, as for $\boldsymbol{f}$, the function $\boldsymbol{h} \in L^2(\Omega)^3$ satisfies relations (3). We will prove that for any vector field $\boldsymbol{v} \in H_0(\mathrm{div}, \Omega)$ such that div $\boldsymbol{v} = 0$ in Ω, there holds $\int_\Omega \boldsymbol{h} \cdot \boldsymbol{v}\ d\boldsymbol{x} = 0$. Let

$$\boldsymbol{z} = \sum_{j=1}^{J} < \boldsymbol{v} \cdot \boldsymbol{n},\ 1 >_{\Sigma_j}\ \widetilde{\mathbf{grad}}\ q_j^T$$

and $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{z}$. According to Amrouche-Bernardi-Dauge-Girault (M2AS, 1998), there exists a vector potential $\boldsymbol{\psi} \in L^2(\Omega)^3$ that satisfies $\boldsymbol{w} = \mathbf{curl}\ \boldsymbol{\psi}$, div $\boldsymbol{\psi} = 0$ in Ω and $\boldsymbol{\psi} \times \boldsymbol{n} = \mathbf{0}$ on Γ. Hence

$$\int_\Omega \boldsymbol{h} \cdot \boldsymbol{v}\ d\boldsymbol{x} = \int_\Omega \boldsymbol{h} \cdot \mathbf{curl}\ \boldsymbol{\psi}\ d\boldsymbol{x} = 0.$$

We deduce from (5) that $\boldsymbol{h} = \mathbf{grad}\ \chi$, with $\chi \in H^1(\Omega)$.

6. Implication (v) $\Longrightarrow$ (ii). Let $q \in H^{-1}(\Omega)$ such that $\nabla q \in H_0(\mathrm{div}, \Omega)'$. We will prove that $q \in L^2(\Omega)$. Setting $\boldsymbol{f} = \nabla q$, we observe that $\boldsymbol{f}$ satisfies (3), so that we have

$$\nabla q = \nabla \chi$$

for some $\chi \in L^2(\Omega)$ and then $q = \chi$ up to additive constant.

7. Implication (iv) $\Longrightarrow$ (ii). Let $q \in H^{-1}(\Omega)$ such that $\nabla q \in H_0(\mathrm{div}, \Omega)'$. We will prove thah $q \in L^2(\Omega)$. Suppose that property (iv) holds. We prove first that

$$Inf_{\boldsymbol{d} \in \boldsymbol{R}(\Omega)} \| \boldsymbol{v} + \boldsymbol{d} \|_{\boldsymbol{L}^2(\Omega)} \leq C \| e(\boldsymbol{v}) \|_{\boldsymbol{L}^2(\Omega)} \tag{7}$$

for any $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ with $\boldsymbol{v} \cdot \boldsymbol{n} = 0$ on Γ. The space $\boldsymbol{R}(\Omega)$ contains the elements $\boldsymbol{d} = \boldsymbol{b} \times \boldsymbol{x}$ with $\boldsymbol{b}$ an arbitrary constant vector of $\mathbb{R}^3$. Using Lax-Milgram lemma, there exists a unique $\boldsymbol{u} \in \boldsymbol{H}^1(\Omega)$, up an additive element of $\boldsymbol{R}(\Omega)$ when Ω is axisymmetric, satisfying

$$
(\mathcal{E}_T) \begin{cases} -\Delta\, \boldsymbol{u} - \nabla \mathrm{div}\, \boldsymbol{u} = \nabla q \quad \text{in} \quad \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma, \\ [\mathbf{e}(\boldsymbol{u})\boldsymbol{n}]_{\boldsymbol{\tau}} = \mathbf{0} \quad \text{on} \quad \Gamma. \end{cases}
$$

We deduce now that $\Delta\, \boldsymbol{u} \in H_0(\mathrm{div}, \Omega)'$ and then we have

$$
\Delta\, \boldsymbol{u} = \boldsymbol{\psi} + \nabla\theta \quad \text{with} \quad \boldsymbol{\psi} \in \boldsymbol{L}^2(\Omega) \quad \text{and} \quad \theta \in L^2(\Omega).
$$

As $\boldsymbol{\psi}$ is also a gradient, then

$$
\boldsymbol{\psi} = \nabla\chi \quad \text{with} \quad \chi \in H^1(\Omega).
$$

Finally, as Ω is connected and

$$
\nabla(q + \mathrm{div}\, \boldsymbol{u} + \theta + \chi) = \mathbf{0},
$$

we deduce that $q \in L^2(\Omega)$.

II. The divergence operator

Theorem 2

Let Ω be a bounded, connected open subset of $\mathbb{R}^n$, with $n \geq 2$ and with a Lipschitz-continuous boundary. The following operator

$$\operatorname{div} \quad : \boldsymbol{H}_0^1(\Omega)/V \mapsto L_0^2(\Omega) \tag{8}$$

is an isomorphism.

Proof. To prove (8), we will prove that for any $f \in L_0^2(\Omega)$, there exists a vector field $\boldsymbol{u} \in \boldsymbol{H}_0^1(\Omega)$ such that div $\boldsymbol{u} = f$ and satisfying the inequality

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \tag{9}$$

First step. We suppose that Ω is starlike with respect to some open ball B contained it in. In this case, the vector $\boldsymbol{u} := \boldsymbol{R}f$ is constructed explicitly.

i) Construction of $\boldsymbol{R}$. Suppose $f \in \mathcal{D}(\Omega)$ and let $\tilde{f} \in \mathcal{D}(\mathbb{R}^n)$ its extension by 0 outside of Ω, with $n \geq 2$. Let θ a fixed function satisfying

$$\theta \in \mathcal{D}(\mathbb{R}^n), \quad 0 \leq \theta \leq 1, \quad \text{with } \operatorname{supp}\theta \subset B \quad \text{and} \quad \int_{\mathbb{R}^n} \theta = 1.$$

Consider the function $t \mapsto t^n \tilde{f}(y + t(x - y))$, with $t \in \mathbb{R}$ and $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$. Then

$$\frac{d}{dt}(t^n \tilde{f})(y + t(x - y)) = t^{n-1} \nabla_x \cdot ((x - y)\tilde{f}(y + t(x - y)).$$

Multiplying this relation by $\theta(y)$ and integrating with respect y in Ω and with respect t on $]1, \infty[$. Then,

$$f(x) = -\int_\Omega \int_1^\infty \theta(y) t^{n-1} \nabla_x \cdot ((x - y)\tilde{f}(y + t(x - y))dt\, dy.$$

The following vector field

$$x \in \Omega, \quad \boldsymbol{u}(x) = -\int_{\mathbb{R}^n} (x-y)\theta(y) \int_1^\infty t^{n-1} \tilde{f}(y + t(x-y)) dt\, dy. \tag{10}$$

satisfies, as we will prove later,

$$\operatorname{div} \boldsymbol{u} = f \quad \text{in } \Omega.$$

Setting successively $z = y + t(x-y)$, $t = (r-1)/r$, $s = 1-r$ and after changing z into y and s into t, we obtain

$$x \in \Omega, \quad \boldsymbol{u}(x) = \int_{\mathbb{R}^n} (x-y) f(y) \int_1^\infty t^{n-1} \theta(y + t(x-y)) dt\, dy. \tag{11}$$

Setting finally $r = t|x-y|$, we have also

$$\boldsymbol{u}(x) := \boldsymbol{R}f(x) = \int_\Omega \frac{x-y}{|x-y|^n} f(y) \int_{|x-y|}^\infty \theta(y + r\frac{x-y}{|x-y|}) r^{n-1} dr\, dy. \tag{12}$$

We verify then that supp $\boldsymbol{R}f$ is compact and $\boldsymbol{R}f$ is $\mathcal{C}^\infty(\Omega)$.

ii) Proof of div $Rf = f$. We observe first that

$$\boldsymbol{u}(x) = \lim_{\varepsilon \to 0} \int_{|x-y| \geq \varepsilon} \tilde{f}(y) \boldsymbol{K}(x,y)\, dy,$$

where

$$\boldsymbol{K}(x,y) = (x-y) \int_1^\infty t^{n-1} \theta(y+t(x-y)) dt = \frac{x-y}{|x-y|^n} \int_{|x-y|}^\infty \theta(y+r\frac{x-y}{|x-y|}) r^{n-1} dr.$$

Then

$$\frac{\partial u_i}{\partial x_j}(x) = \lim_{\varepsilon \to 0} [\int_{|x-y| \geq \varepsilon} \tilde{f}(y) \frac{\partial K_i}{\partial x_j}(x,y)\, dy + \int_{|x-y| = \varepsilon} \tilde{f}(y) \frac{x_j - y_j}{|x-y|} K_i(x,y)\, d\sigma_y] \tag{13}$$

and

$$\begin{aligned} & \int_{|x-y|=\varepsilon} \tilde{f}(y) \frac{x_j - y_j}{|x-y|} K_i(x,y)\, d\sigma_y - \tilde{f}(x) \int_{\Omega} \frac{(x_j - y_j)(x_i - y_i)}{|x-y|^2} \theta(y)\, d\sigma_y \\ = \quad & \int_{|z|=1} \tilde{f}(x - \varepsilon z) |z|^2 \int_0^\infty \theta(x + sz)(s + \varepsilon)^{n-1} ds\, d\sigma_z \\ - \quad & \tilde{f}(x) \int_{|z|=1} z_i z_j \int_0^\infty \theta(x + sz) s^{n-1} ds\, d\sigma_z. \end{aligned}$$

Consequently

$$\lim_{\varepsilon\to 0}\int_{|x-y|=\varepsilon}\tilde{f}(y)\frac{x_j-y_j}{|x-y|}K_i(x,y)\,d\sigma_y=\tilde{f}(x)\int_\Omega\frac{(x_j-y_j)(x_i-y_i)}{|x-y|^2}\theta(y)\,d\sigma_y \quad (14)$$

Besides, we have

$$\begin{aligned}\frac{\partial K_i}{\partial x_j}(x,y) &= \delta_{ij}\int_1^\infty t^{n-1}\theta(y+t(x-y))dt+(x_i-y_i)\int_1^\infty t^n\frac{\partial\theta}{\partial x_j}(y+t(x-y))dt\\ &= \frac{\delta_{ij}}{|x-y|^n}\int_0^\infty\theta(x+r\frac{x-y}{|x-y|})(|x-y|+r)^{n-1}dr\\ &+ \frac{x_i-y_i}{|x-y|^n}\int_0^\infty\frac{\partial\theta}{\partial x_j}(x+r\frac{x-y}{|x-y|})(|x-y|+r)^n dr. \qquad (15)\end{aligned}$$

From this relations, we deduce that

$$\begin{aligned}\nabla_x \cdot \boldsymbol{K}(x,y) &= n\int_1^{\infty} t^{n-1}\theta(y+t(x-y)) + \int_1^{\infty} t^n \frac{\partial\theta}{\partial t}\theta(y+t(x-y) \\ &= -\theta(x)\end{aligned}$$

and

$$\operatorname{div} \boldsymbol{u}(x) = -\theta(x)\int_{\Omega} f(y)dy + f(x) = f(x)$$

because by assumption the integral of f is equal to 0.

iii) We will prove now that

$$\boldsymbol{R} : \; L^2_0(\Omega) \mapsto \boldsymbol{H}^1_0(\Omega)$$

is continuous, that means that

$$\|\nabla \boldsymbol{R} f\|_{\boldsymbol{L}^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}. \tag{16}$$

Thanks to the Newton's binôme formula, we have the following decomposition:

$$\frac{\partial K_i}{\partial x_j}(x,y) = K_{ij}(x, x-y) + G_{ij}(x,y),$$

where

$$\begin{aligned} K_{ij}(x, x-y) &= \frac{\delta_{ij}}{|x-y|^n}\int_0^\infty \theta(x + r\frac{x-y}{|x-y|})r^{n-1}dr + \\ &+ \frac{x_i - y_i}{|x-y|^{n+1}}\int_0^\infty \frac{\partial \theta}{\partial x_j}(x + r\frac{x-y}{|x-y|})r^n dr \\ &:= \frac{k_{ij}(x, x-y)}{|x-y|^n} \end{aligned}$$

and G_{ij} satisfies the following estimate:

$$|G_{ij}(x,y)| \leq C\delta(\Omega)^{n-1}/|x-y|^{n-1},$$

with $C = C(\theta, n)$.

We deduce from (13)-(15) that

$$\begin{aligned}
\frac{\partial (Rf)_i}{\partial x_j}(x) &= \int_\Omega K_{ij}(x, x-y) f(y) dy + \int_\Omega G_{ij}(x,y) f(y) dy + \\
&+ f(x) \int_\Omega \frac{(x_j - y_j)(x_i - y_i)}{|x-y|^2} \theta(y)\, dy \\
:&= J_1 f(x) + J_2 f(x) + J_3 f(x).
\end{aligned}$$

It is clear that $|J_3 f(x)| \leq |f(x)|$ and then

$$\|J_3 f\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}.$$

We have also

$$|J_2 f(x)| \leq C\delta(\Omega)^{n-1} \Big| \int_{\mathbb{R}^n} \frac{\tilde{f}(y)}{|x-y|^{n-1}} dy \Big|.$$

That means that

$$|J_2 f(x)| \leq C\delta(\Omega)^{n-1} |I_1 \tilde{f}(x)|,$$

where $I_1 \tilde{f}$ is the Riesz potential of order 1. Hence

$$\|J_2 f\|_{L^2(\Omega)} \leq C\delta(\Omega)^{n-1} \|I_1 \tilde{f}\|_{L^2(\mathbb{R}^n)}.$$

Recall now that for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$,

$$\|\nabla I_1 \varphi\|_{\boldsymbol{L}^2(\mathbb{R}^n)} \leq C\|\varphi\|_{L^2(\mathbb{R}^n)}$$

and by duality, for any $\varphi \in \mathcal{D}(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \varphi = 0$,

$$\|I_1 \varphi\|_{L^2(\mathbb{R}^n)} \leq C\|\varphi\|_{W_0^{-1,2}(\mathbb{R}^n)}$$

where $W_0^{-1,2}(\mathbb{R}^n)$ is the dual space of

$$W_0^{1,2}(\mathbb{R}^n) = \{v \in \mathcal{D}'(\mathbb{R}^n);\ \frac{v}{\omega} \in L^2(\mathbb{R}^n)\ \nabla v \in \boldsymbol{L}^2(\mathbb{R}^n)\}$$

with

$$\omega = 1+|x| \quad \text{if} \quad n \geq 3 \quad \text{and} \quad \omega = (1+|x|)\ln(2+|x|) \quad \text{if} \quad n = 2.$$

Applying the previous inequality to $\tilde{f}$, because Ω is bounded, we get

$$\|I_1 \tilde{f}\|_{L^2(\mathbb{R}^n)} \leq C\|\tilde{f}\|_{W_0^{-1,2}(\mathbb{R}^n)} \leq C\|f\|_{L^2(\Omega)}.$$

Finally, concerning the estimate of $J_1 f$, we need the following lemma due to Calderon-Zygmund.

Lemma 3 (Calderon-Zygmund)

Let be $K(x,y) = N(x, x-y)$, where N is homogeneous of degre $-n$ in y and
i) for any x,

$$\int_{|y|=1} N(x,y)dy = 0,$$

ii) there exist $q > 0$ and $C > 0$ such that

$$\forall x, \quad \int_{|y|=1} |N(x,y)|^q dy \leq C,$$

then the operator defined by

$$Kf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy$$

is continuous from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$, for any p such that $q/(q-1) \leq p < \infty$.

Applying this lemma with $N(x,y) = K_{ij}(x,y)$. It is clear that $K_{ij}(x,y)$ is homogeneous of degre $-n$ in y and

$$
\begin{aligned}
\int_{|z|=1} K_{ij}(x,z)dz &= \int_{|z|=1}\int_0^{\infty} \big(\delta_{ij}\theta(x+rz)r^{n-1}drd\sigma_z + \\
&+ z_i\frac{\partial\theta}{\partial x_j}(x+rz)r^n\big)drd\sigma_z \\
&= \int_{\mathbb{R}^n} \big(\delta_{ij}\theta(x+y) + y_i\frac{\partial\theta}{\partial x_j}(x+y)\big)dy = 0
\end{aligned}
$$

Finally, the property ii) is satisfied by using the fact that supp $\theta \subset B$ and Ω is bounded, that finishes the proof of the continuity property (16) and the first step of the proof of Theorem 1.

Second step. We know that every bounded Lipschitz domain Ω is the union of a finite number of bounded domains, each of which is starlike with respect to an open ball. Thanks to a partition of unity subordinated to this covering, we extend the result of the step 1 to the case where Ω is a bounded Lipschitz domain.

Third step. We have prove that

$$\forall f \in \mathcal{D}(\Omega) \cap L^2_0(\Omega), \quad \exists \boldsymbol{u} \in \boldsymbol{\mathcal{D}}(\Omega) \quad \text{such that} \quad \text{div } \boldsymbol{u} = f$$

with the inequality

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

Let us consider now $f \in L^2_0(\Omega)$ only. And let $\psi \in \mathcal{D}(\Omega)$ fixed with $\int_\Omega \psi = 1$ and

$$f_k \in \mathcal{D}(\Omega) \quad \text{such that} \quad f_k \to f \quad \text{in } L^2(\Omega).$$

Setting $F_k = f_k - \psi \int_\Omega f_k$, then $F_k \in \mathcal{D}(\Omega) \cap L^2_0(\Omega)$ and there exists $\boldsymbol{u}_k \in \mathcal{D}(\Omega)^3$ satisfying div $\boldsymbol{u}_k = F_k$, with

$$\boldsymbol{u}_k \to \boldsymbol{u} \in \boldsymbol{H}^1_0(\Omega) \quad \text{and} \quad \text{div } \boldsymbol{u} = f.$$

Moreover

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

Corollary 4 (De Rham in $\boldsymbol{H}^{-1}(\Omega)$ First Version-Necas)

The operator

$$\textbf{grad} \ : L^2(\Omega)/\mathbb{R} \mapsto V^\circ \tag{17}$$

is an isomorphism and we have the following properties:
i) For any $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ satisfying

$$\forall \boldsymbol{v} \in V, \quad < \boldsymbol{f} \, , \boldsymbol{v} >= 0,$$

there exists $\pi \in L^2(\Omega)$, unique up to an additive constant, such that $f = \nabla\pi$.
ii) We have the following inequality:

$$\forall q \in L^2(\Omega), \quad Inf_{k\in\mathbb{R}} \| \, q + k \, \|_{L^2(\Omega)} \leq C(\Omega) \| \nabla q \, \|_{\boldsymbol{H}^{-1}(\Omega)}. \tag{18}$$

In particular, we have also the inequality:

$$\forall q \in L^2(\Omega), \quad \| \, q \, \|_{L^2(\Omega)} \leq C(\Omega) (\| \, q \, \|_{H^{-1}(\Omega)} + \| \nabla q \, \|_{\boldsymbol{H}^{-1}(\Omega)}). \tag{19}$$

Proof. The isomorphism's property of the gradient operator is a simple consequence of the isomorphism (8) of his dual operator.

The property i) is nothing that the surjectivity of the gradient operator.

The inequality (18) of the property ii) means that the inverse of the gradient operator (17) is continuous.

Finally, the inequality (19) can be deduce by the following decomposition:

$$u = u - \frac{1}{|\Omega|}\int_\Omega u + \frac{1}{|\Omega|}\int_\Omega u$$

and the following inequality which is equivalent to (18):

$$\forall v \in L^2_0(\Omega), \quad \| \, v \, \|_{L^2(\Omega)} \leq C(\Omega) \| \nabla v \, \|_{\boldsymbol{H}^{-1}(\Omega)} \tag{20}$$

Corollary 5.

For any open set ω of Ω, with $meas(\omega) > 0$, there exists $C_\omega > 0$ depending also on Ω such that

$$\forall u \in L^2(\Omega), \quad \| u \|_{L^2(\Omega)} \leq C_\omega(\| u \|_{L^2(\omega)} + \|\nabla u \|_{\boldsymbol{H}^{-1}(\Omega)}).$$

Proof. This is an immediate consequence of Peetre-Tartar theorem, the inequality (19) and the compactness of $L^2(\Omega)$ into $H^{-1}(\Omega)$.

As in Girault-Raviart (1986), we can prove the following corollary.

Corollary 6.

We have the algebraical and topological following indentity:

$$X =: \left\{ u \in L^2_{\mathrm{loc}}(\Omega);\ \nabla u \in \boldsymbol{H}^{-1}(\Omega) \right\} = L^2(\Omega).$$

As in Amrouche-Girault (Czech. Math. Journ., 1994), we can prove the following both results.

Theorem 7. (De Rham in $\boldsymbol{H}^{-1}(\Omega)$ Second Version)

Let $\boldsymbol{f} \in \boldsymbol{H}^{-1}(\Omega)$ satisfying

$$\forall \boldsymbol{v} \in \mathcal{V}, \quad < \boldsymbol{f}\,, \boldsymbol{v} >= 0,$$

where

$$\mathcal{V} = \{\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega);\, \operatorname{div} \boldsymbol{v} = 0\}\,.$$

Then there exists $\pi \in L^2(\Omega)$, unique up to an additive constant, such that $f = \nabla\pi$. Moreover $\mathcal{V}$ is dense in V.

Corollary 8.

Let be $u \in \mathcal{D}'(\Omega)$ such that $\nabla u \in \boldsymbol{H}^{-1}(\Omega)$. Then $u \in L^2(\Omega)$.

We are now in position to give some extensions.

Theorem 9.

For any integer $m \geq 1$, the following operator

$$\text{div} \quad : \boldsymbol{H}_0^{m+1}(\Omega)/V_{m+1} \mapsto H_0^m(\Omega) \cap L_0^2(\Omega) \tag{21}$$

is an isomorphism, where

$$V_{m+1} = \left\{ \boldsymbol{v} \in \boldsymbol{H}_0^{m+1}(\Omega);\ \text{div}\, \boldsymbol{v} = 0 \right\}.$$

Proof. We give here a sketch of the proof and we consider only the case $m = 1$, the same raisoning being avalaible for $m \geq 2$. We will use here the same arguments as in the point iii) of the first theorem concerning the divergence operator defined on $\boldsymbol{H}_0^1(\Omega)$ and we would like to prove the following estimate:

$$\forall f \in H_0^1(\Omega) \cap L_0^2(\Omega), \quad \| \frac{\partial^2 \boldsymbol{R} f}{\partial x_k \partial x_j} \|_{\boldsymbol{L}^2(\Omega)} \leq C \| f \|_{H^1(\Omega)}$$

for any $1 \leq j, k \leq n$.

Rewriting the relation (12) under the form

$$\boldsymbol{R} f(x) = \int_{\mathbb{R}^n} \frac{z}{|z|^n} \tilde{f}(x - z) \int_0^\infty \theta(x + s \frac{z}{|z|})(|z| + s)^{n-1} ds\, dy,$$

we obtain

$$\begin{aligned} \frac{\partial \boldsymbol{R} f}{\partial x_j}(x) &= \int_{\mathbb{R}^n} \frac{z}{|z|^n} \frac{\partial \tilde{f}}{\partial x_j}(x - z) \int_0^\infty \theta(x + s \frac{z}{|z|})(|z| + s)^{n-1} ds\, dy \\ &+ \int_{\mathbb{R}^n} \frac{z}{|z|^n} \tilde{f}(x - z) \int_0^\infty \frac{\partial \theta}{\partial x_j}(x + s \frac{z}{|z|})(|z| + s)^{n-1} ds\, dy \\ &=: \boldsymbol{g}_1(x) + \boldsymbol{g}_2(x). \end{aligned}$$

Estimate of $\|\frac{\partial \boldsymbol{g}_1}{\partial x_k}\|_{L^2(\Omega)^n}$

As below, we prove that

$$\|\frac{\partial \boldsymbol{g}_1}{\partial x_k}\|_{\boldsymbol{L}^2(\Omega)} \leq C\|\frac{\partial f}{\partial x_j}\|_{L^2(\Omega)}.$$

Estimate of $\|\frac{\partial \boldsymbol{g}_2}{\partial x_k}\|_{L^2(\Omega)^n}$

We remark that $\boldsymbol{g}_2$ is the same form as $\boldsymbol{R}f$, with the difference that θ is replaced by $\frac{\partial \theta}{\partial x_j}$. Note that we does'nt use the property $\int_B \theta = 1$ to find the estimate of the point iv). That means that with the same raisonning, we obtain

$$\|\frac{\partial \boldsymbol{g}_2}{\partial x_k}\|_{\boldsymbol{L}^2(\Omega)} \leq C\|f\|_{L^2(\Omega)}.$$

Hence we have established the result for $f \in \mathcal{D}(\Omega) \cap L^2_0(\Omega)$ and proceeding as in the step 3 of the proof of the first theorem concerning the divergence operator defined on $\boldsymbol{H}^1_0(\Omega)$, we extend this one to the case where $f \in H^1_0(\Omega) \cap L^2_0(\Omega)$.

Corollary 10. (De Rham in $\boldsymbol{H}^{-m-1}(\Omega)$ First Version)

Let m a positve integer and $\boldsymbol{f} \in \boldsymbol{H}^{-m-1}(\Omega)$ satisfying

$$\forall \boldsymbol{v} \in V_{m+1}, \quad < \boldsymbol{f} \, , \boldsymbol{v} >= 0.$$

Then there exists $\pi \in H^{-m}(\Omega)$, unique up to an additive constant, such that $f = \nabla \pi$.

Using then the densité of $\mathcal{V}$ in V_{m+1}, we can prove the following theorem:

Theorem 11. (De Rham in $\boldsymbol{H}^{-m-1}(\Omega)$ Second Version)

Let m a positve integer and $\boldsymbol{f} \in \boldsymbol{H}^{-m-1}(\Omega)$ satisfying the following property:

$$\forall \boldsymbol{v} \in \mathcal{V}, \quad < \boldsymbol{f} \, , \boldsymbol{v} >= 0.$$

Then there exists $\pi \in H^{-m}(\Omega)$, unique up an additive constant, such that $\boldsymbol{f} = \nabla \pi$.

As application, we can give a new proof of De Rham's theorem.

Corollary 12 (Original De Rham).

Let $\boldsymbol{f} \in \boldsymbol{\mathcal{D}}'(\Omega)$ satisfying the following property:

$$\forall \boldsymbol{v} \in \mathcal{V}, \quad < \boldsymbol{f} \,, \boldsymbol{v} >= 0.$$

Then there exists $\pi \in \mathcal{D}'(\Omega)$, unique up an additive constant, such that $\boldsymbol{f} = \nabla\pi$.

Proof. It is an immediate consequence of the fact that we have prove that the divergence operator

$$\mathrm{div} \;:\; \boldsymbol{\mathcal{D}}(\Omega)/\mathcal{V} \mapsto \mathcal{D}(\Omega) \cap L^2_0(\Omega).$$

is bijective.

III. The curl operator

Let us introduce the following space:

$$\mathcal{G} = \{\boldsymbol{v} \in \boldsymbol{\mathcal{D}}(\Omega);\ \mathbf{curl}\,\boldsymbol{v} = \mathbf{0}\},$$

where we suppose here that Ω is a Lipschitzian bounded domain of $\mathbb{R}^3$.

Theorem 13.

Let $\boldsymbol{f} \in \boldsymbol{\mathcal{D}}'(\Omega)$ satisfying the following property:

$$\forall \boldsymbol{v} \in \mathcal{G}, \quad < \boldsymbol{f}\,, \boldsymbol{v} > = 0. \tag{22}$$

Then there exists $\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}'(\Omega)$ with $\mathrm{div}\ \boldsymbol{\psi} = 0$ and such that $\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{\psi}$.

Proof. Note that the condition (22) implies that div $\boldsymbol{f} = 0$, which is a necessary condition for that $\boldsymbol{f} = \mathbf{curl}\,\boldsymbol{\psi}$. To prove the result is equivalent to prove that the **curl** operator is bijective from $\boldsymbol{\mathcal{D}}(\Omega)/\mathcal{G}$ onto $\mathcal{V} \perp \boldsymbol{K}_T(\Omega)$, where

$$\boldsymbol{K}_T(\Omega) = \{\boldsymbol{v} \in \boldsymbol{L}^p(\Omega);\ \mathrm{div}\,\boldsymbol{v} = 0,\ \mathbf{curl}\,\boldsymbol{v} = \mathbf{0} \text{ in } \Omega \quad \text{and} \quad \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma\}.$$

Recall that $\boldsymbol{K}_T(\Omega)$ is reduced to zero when Ω is simply connected.

We will use here the same ideas as in the proof of Theorem 2.

Construction of the operator T.

Let $\boldsymbol{f} \in \mathcal{V}$ satisfying the following property:

$$\forall \boldsymbol{v} \in \boldsymbol{K}_T(\Omega), \quad \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} = 0.$$

Instead of the operator $\boldsymbol{R}$, we set

$$\boldsymbol{T}\boldsymbol{f}(x) := \int_\Omega \boldsymbol{f}(y) \times \frac{x-y}{|x-y|^3} \int_{|x-y|}^{\infty} \theta(y + r\frac{x-y}{|x-y|}) r^{n-1} dr\, dy, \tag{23}$$

where Ω is starlike with respect to some open ball B contained it in and supp $\theta \subset B$. Then, we verify that $\boldsymbol{f} = \textbf{curl}\, \boldsymbol{T}\boldsymbol{f}$ and $\boldsymbol{f} = \textbf{curl}\, \boldsymbol{T}\boldsymbol{f} \in \boldsymbol{\mathcal{D}}(\Omega)$. And we finish the proof for general Lipschitzian bounded domain as in Theorem 2.

For Further Reading

- C. Amrouche, C. Bernardi, M. Dauge, V. Girault, Vector potentials in three dimensional nonsmooth domains, *Math. Meth. Appl. Sci.* **21** (1998), 823–864.
- Amrouche, C., Girault, V., *Decomposition of vector spaces and application to the Stokes problem in arbitrary dimension*, Czechoslovak Mathematical Journal **44**-119, 109–140, (1994).
- Amrouche, C., Ciarlet, P. G., Ciarlet, P. Jr., *Weak vector and scalar potentials: applications to PoincarÃ©'s theorem and Korn's inequality in Sobolev spaces with negative exponents*, Anal. Appl. (Singap.) **8**-1, 1–17, (2010)
- Bogovskii, M. E., *Solution of the first boundary value problem for an equation of continuity of an incompressible medium*, (Russian) Dokl. Akad. Nauk SSSR **248**-5, 1037–1040, (1979)

- Costabel, M., McIntosh, A., *On Bogovskii and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains*, Math. Z. **265**-2, 297–320, (2010)
- Duvaut, G.; Lions, J.-L., *Les inÃ©quations en mécanique et en physique*, (French) Travaux et Recherches Mathématiques, No. 21. Dunod, Paris, (1972).
- Girault, V., Raviart, P.-A., *Finite element methods for Navier-Stokes equations. Theory and algorithms*, Springer Series in Computational Mathematics, 5. Springer-Verlag, Berlin, (1986).
- J. Necas Equations aux Dérivées Partielles, *Presses de l'Université de Montréal*, Montréal, 1965.

Thank you for your attention!