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Diffraction from polygonal-conical screens : an operator approach

joint work with L.P. Castro and R. Duduchava

To the honour of Martin Costabel

on the occasion of his 65th birthday

Formulation of screen diffraction problems

Given an open subset $\Sigma \subset \mathbb{R}^2$, consider the domain Ω defined by

$$\Omega = \mathbb{R}^3 \setminus \Gamma$$
$$\Gamma = \overline{\Sigma} \times 0 = \{x = (x_1, x_2, 0) \in \mathbb{R}^3 : x' = (x_1, x_2) \in \overline{\Sigma}\}.$$

Problems of diffraction from a plane screen Γ (closed in $\mathbb{R}^3$) are often formulated in terms of (or reduced to) the solution of the three-dimensional Helmholtz equation (HE) in Ω with Dirichlet or Neumann conditions on Γ, briefly written as

$$\begin{aligned} \left(\Delta + k^2\right) u &= 0 \quad \text{in} \quad \Omega \\ Bu &= g \quad \text{on} \quad \Gamma = \partial\Omega. \end{aligned}$$

Herein k is the wave number and we assume that $\Im m\, k > 0$ throughout this paper (some parts are restricted to $\Re e k = 0$). B stands for the boundary operator, taking the trace or normal derivative of u on Γ.

Reformulation as interface problem

We think of the weak formulation looking for $u \in L^2(\Omega)$ with restrictions $u^\pm = u|_{\Omega^\pm}$ to the upper and lower half-space $\Omega^\pm = \{x \in \mathbb{R}^3 : \pm x_3 > 0\}$ that satisfy $u^\pm \in H^1(\Omega^\pm)$ and the common transmission conditions across the complement $\Sigma' = \mathbb{R}^2 \setminus \overline{\Sigma}$ of $\overline{\Sigma}$:

$$\begin{array}{rclcl} u_0^+ - u_0^- & = & [u^+ - u^-]|_{x_3=0} & = & 0 \\ u_1^+ - u_1^- & = & [\partial u^+/\partial x_3 - \partial u^-/\partial x_3]|_{x_3=0} & = & 0 \end{array} \quad \text{on} \quad \Sigma'$$

according to the trace theorem and by help of representation formulas, see HW08 for details. In a sense, this is equivalent to state that the HE holds across the complementary screen $\Sigma' = \mathbb{R}^2 \setminus \overline{\Sigma}$, see MS89. The boundary data g are arbitrarily given in the corresponding data space $H^{1/2}(\Sigma)$ or $H^{-1/2}(\Sigma)$, respectively (values of g in the boundary of Σ do not matter in this space setting).

The operator associated with the BVP

For convenience we study the (homogeneous) HE, since boundary value problems for the inhomogeneous HE $Au = (\Delta + k^2)u = f$ can be "equivalently reduced" under the present assumptions, see S12.

Hence the operator associated with the boundary value problem can be written as

$$B_0 \quad = \quad B|_{\ker A} \quad : \quad \mathcal{H}^1(\Omega) \to H^{\pm 1/2}(\Sigma)$$

where $\mathcal{H}^1(\Omega)$ denotes the space of weak solutions of the HE in Ω and B_0 denotes the restriction of B to this space. We are looking for the inverse B_0^{-1}, the so-called resolvent operator.

Generalization with different data on the faces

$$u \in \mathcal{H}^1(\Omega)$$

$$B_0 u = \begin{pmatrix} B^+ \\ B^- \end{pmatrix} u = g = \begin{pmatrix} g^+ \\ g^- \end{pmatrix} \quad \text{on} \quad \Gamma = \partial\Omega$$

where now (a) in case of the Dirichlet problem: g is given in the space $H^{1/2}(\Sigma)^2 = H^{1/2}(\Sigma) \times H^{1/2}(\Sigma)$ with the compatibility condition that $g^+ - g^-$ is extensible by zero from Σ to the full plane (corresponding to $x_3 = 0$) within $H^{1/2}(\mathbb{R}^2)$, respectively (b) in case of the Neumann problem: g is given in the space $H^{-1/2}(\Sigma)^2 = H^{-1/2}(\Sigma) \times H^{-1/2}(\Sigma)$ with the compatibility condition that $g^+ - g^-$ be extensible by zero from Σ to the full plane (corresponding to $x_3 = 0$) within $H^{-1/2}(\mathbb{R}^2)$, see HW08 for details. We abbreviate this by

$$g \in H^{1/2}(\Sigma)^2_\sim \quad \text{respectively} \quad g \in H^{-1/2}(\Sigma)^2_\sim \quad .$$

Screen properties 1

The following domain properties are crucial in what follows.
First we shall assume the strong extension property, see HW08, i.e., for any $s \in \mathbb{R}$, there exists a continuous extension operator which is left invertible by restriction:

$$\begin{aligned} \ell_\Sigma^s \quad &: \quad H^s(\Sigma) \to H^s(\mathbb{R}^2) \\ r_\Sigma \ell_\Sigma^s \quad &= \quad I_{H^s(\Sigma)}. \end{aligned}$$

Lipschitz domains (that are bounded and characterized by fulfilling the uniform cone property Gri85,HW08) and (unbounded) special Lipschitz domains in the sense of Stein (of the form $\Sigma = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > \varphi(x_1)\}$ where φ is uniformly Lipschitz continuous and rotations of this kind of domain) fulfil the strong extension property.

The existence of a continuous extension operator guarantees the equivalence of B_0 to an operator which has the form of a general Wiener-Hopf operator (explained later).

Screen properties 2

Second we confine our considerations to domains with the property

$$\text{int clos } \Sigma = \Sigma$$

which is needed for a relaxed use of Sobolev spaces. Note that this excludes "cracks" in the screen (also called "slit domains") with discontinuities across the cracks, which could be considered using more complicated notation than $H^s(\Sigma)$ (in general, the notion of $H^s(\Sigma)$ with Lipschitz domains Σ is not suitable for that case, see HW08, p. 110, for instance).

A domain Σ with these two properties is said to be an E-domain. The properties are actually needed only for $s = \pm 1/2$ in the basic results.

Further we shall work with an algebra $\mathcal{A}_2$ of open subsets $\Sigma \subset \mathbb{R}^2$ which contains open half-planes, finite intersections and the interior of complements of elements of $\mathcal{A}_2$.

Polygonal-conical domains

A convex polygonal-conical domain (convex PCD) in $\mathbb{R}^2$ is given by

$$\Sigma \quad = \quad \bigcap_{j=1,\ldots,m} \Sigma_j \quad \text{where} \quad \Sigma_j \quad \text{are open half-planes.}$$

A polygonal-conical domain (PCD) in $\mathbb{R}^2$ is given by

$$\Sigma \quad = \quad \operatorname{int} \bigcup_{j=1,\ldots,m} \operatorname{clos} \Sigma_j \quad \text{where} \quad \Sigma_j \quad \text{are convex PCDs.}$$

1. Convex PCDs are simply connected, PCDs may be multiply connected, both possibly unbounded (cones are included).
2. PCDs are E-domains.
3. The set of PCDs (including $\Sigma = \emptyset$ and $\Sigma = \mathbb{R}^2$) coincides with the minimal set algebra $\mathcal{A}_2$ described above, since those sets

$$\Sigma \quad = \quad \mathbb{R}^2 \setminus \left(\bigcap_{j=1,\ldots,m} (\mathbb{R}^2 \setminus \Sigma_j) \right) \text{ where } \Sigma_j \text{ are convex PCDs.}$$

Sobolev spaces notation

In order to describe the spaces for the boundary data in more detail, we recall the definition of the usual Sobolev spaces $H^s = H^s(\mathbb{R}^n)$ (sometimes named Bessel potential or fractional Sobolev spaces) and of the Sobolev spaces $H^s(\Sigma)$, H^s_Σ, $\tilde{H}^s(\Sigma)$, as well (see, e.g., Esk81, HW08). Thus let

$$H^s = H^s(\mathbb{R}^n) = \left\{ f \in \mathcal{S}' : \|f\| = \left(\int_{\mathbb{R}^n} \left| \hat{f}(\xi) \right|^2 (\xi^2 + 1)^s d\xi \right)^{1/2} < \infty \right\}$$

where $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ denotes the Schwartz distribution space and $\hat{f}(\xi) = \mathcal{F}_{x \mapsto \xi} f(x) = \int_{\mathbb{R}^n} e^{ix\xi} f(x) dx$ the Fourier transform of $f \in \mathcal{S}$ extended to distributions $f \in \mathcal{S}'$. H^s is a Hilbert space with inner product

$$\langle \varphi, \psi \rangle_s = \int_{\mathbb{R}^n} \hat{\varphi}(\xi) \, \overline{\hat{\psi}(\xi)} \, (\xi^2 + 1)^s d\xi \quad , \quad \varphi, \psi \in H^s(\mathbb{R}^n).$$

Bessel potential operators

The function $\lambda(\xi) = (\xi^2+1)^{1/2}$, $\xi \in \mathbb{R}^n$, will play a special role in what follows, since it can be considered as a particular case of the square root of the "Helmholtz symbol" $t(\xi) = \lambda_k(\xi) = (\xi^2 - k^2)^{1/2}$ for $k = i$ (the double notation has historical reasons). We shall always choose branches, continuous in $\mathbb{R}^n$, such that $\lambda_k(\xi) \to +\infty$ as $|\xi| \to +\infty$. It may be useful to consider the spaces H^s as the isometric images of the Bessel potential operators

$$\Lambda^{-s} = \mathcal{F}^{-1}\lambda^{-s} \cdot \mathcal{F} : L^2 \to H^s \quad , \quad s \in \mathbb{R}.$$

Sobolev spaces on Σ

The restriction operator which restricts a function or distribution on $\mathbb{R}^n$ to a measurable subset Σ will be denoted by r_Σ. Thus $H^s(\Sigma) = r_\Sigma(H^s)$, and the norm in $H^s(\Sigma)$ is defined by

$$\|f\|_{H^s(\Sigma)} = \inf_{\ell} \|\ell f\|_{H^s}$$

where ℓf stands for any extension of f to a distribution in H^s. Furthermore, we denote by H^s_Σ the (closed) subspace of H^s which consists of all distributions with support in the closure of Σ. Further $\widetilde{H}^s(\Sigma) = r_\Sigma(H^s_\Sigma)$ and

$$\|f\|_{\widetilde{H}^s(\Sigma)} = \inf_{\ell_0} \|\ell_0 f\|_{H^s}$$

where $\ell_0 f$ stands for any extension of f to a distribution in H^s_Σ (which is unique only for $s \geq -1/2$, see ENS1, in which case the last infimum is redundant). Notice that while $\widetilde{H}^s(\Sigma)$ is always continuously embedded in $H^s(\Sigma)$, these two spaces coincide for $s \in]-1/2, 1/2[$.

Main Theorem (sketch)

Let Σ be a PCD. Then the resolvent operator B_0^{-1} for the Dirichlet or Neumann problem is explicitly given in terms of infinite operator products (presented later) which strongly converge in the common (Bessel potential) norm of $H^{\pm 1/2}(\mathbb{R}^2)$ for $k = i$ and in a modified equivalent norm for $k \in i\mathbb{R}_+$, respectively. In the remaining cases of $k \in \mathbb{C}$, $\Im m k > 0$ the resolvent operator can be explicitly represented by (additional) use of Neumann series.

Sketch of the proof

The principle steps are:
(1) to show operator equivalence of B_0 with a boundary pseudo-differential operator that has the form of a general Wiener-Hopf operator (WHO),
(2) to represent B_0^{-1} in terms of a certain projector acting in $H^{\pm 1/2}(\mathbb{R}^2)$, which depends heavily on the form of Σ,
(3) for screens which are convex PCDs, to give an explicit formula for these kind of projectors in case of $k \in i\mathbb{R}_+$, choosing a topology where they are orthogonal and using a result of Halmos for the representation of the orthogonal projector onto the intersection of closed Hilbert subspaces,
(4) to reduce in case of arbitrary k with $\Im m k > 0$ to the previous by approximation, and finally
(5) to reduce the case of arbitrary PCDs to the case of convex PCDs by matrical coupling of associated WHOs and a so-called geometric perspective of Ferreira dos Santos San88, San89 for general WHOs.

Representation Theorem for Dirichlet problems

Assume that $\Sigma \subset \mathbb{R}^2$ be any (proper) open subset of $\mathbb{R}^2$, Ω be given as before and $\Omega^{\pm} = \{x \in \mathbb{R}^3 : \pm x_3 > 0\}$. Then the Dirichlet problem in Ω is well-posed if and only if the following WHO is invertible:

$$W_{t^{-1},\Sigma} \quad = \quad r_\Sigma\, A_{t^{-1}} \quad : \quad H_\Sigma^{-1/2} \to H^{1/2}(\Sigma).$$

In this case, the solution of the Dirichlet problem is given by

$$u = \mathcal{K}_{D,\Omega}(g_1, g_2) = \begin{cases} \mathcal{K}_{D,\Omega^+}\, u_0^+ & \text{in} \quad \Omega^+ \\ \mathcal{K}_{D,\Omega^-}\, u_0^- & \text{in} \quad \Omega^- \end{cases}$$

$$\mathcal{K}_{D,\Omega^+} u_0^+(x) = \mathcal{F}^{-1}_{\xi'\mapsto(x')} e^{-t(\xi')x_3}\widehat{u_0^+}(\xi') = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{-t(\xi_1,\xi_2)x_3}\widehat{u_0^+}(\xi_1,\xi_2)d$$

$$\mathcal{K}_{D,\Omega^-} u_0^-(x) = \mathcal{F}^{-1}_{\xi'\mapsto(x')} e^{t(\xi')x_3}\widehat{u_0^-}(\xi') = \frac{1}{(2\pi)^2}\int_{\mathbb{R}^2} e^{t(\xi_1,\xi_2)x_3}\widehat{u_0^-}(\xi_1,\xi_2)d\xi'$$

$$\begin{pmatrix} u_0^+ \\ u_0^- \end{pmatrix} = \Upsilon_D^{-1}\begin{pmatrix} \ell_0 & 0 \\ 0 & A_{t^{-1}}W^{-1}_{t^{-1},\Sigma} \end{pmatrix}\Upsilon_D\begin{pmatrix} g^+ \\ g^- \end{pmatrix}, \quad \Upsilon_D = \begin{pmatrix} I & -I \\ I & I \end{pmatrix}$$

abbreviating $\xi' = (\xi_1, \xi_2) \in \mathbb{R}^2$, $d\xi' = d\xi_1 d\xi_2$ and $x' = (x_1, x_2) \in \mathbb{R}^2$.

Representation Theorem for Neumann problems

Assume that $\Sigma \subset \mathbb{R}^2$ be any (proper) open subset of $\mathbb{R}^2$, Ω and $\Omega^{\pm}$ be given as before . Then the Neumann problem in Ω is well-posed if and only if the following WHO is invertible:

$$W_{t,\Sigma} \quad = \quad r_\Sigma \, A_t \quad : \quad H^{1/2}_{\Sigma} \to H^{-1/2}(\Sigma).$$

In this case, the solution of the Neumann problem is given by

$$u = \mathcal{K}_{N,\Omega}(g_1, g_2) = \begin{cases} \mathcal{K}_{N,\Omega^+} \; u_1^+ & \text{in} \quad \Omega^+ \\ \mathcal{K}_{N,\Omega^-} \; u_1^- & \text{in} \quad \Omega^- \end{cases}$$

$$\mathcal{K}_{N,\Omega^+} u_1^+(x) = \mathcal{F}^{-1}_{\xi' \mapsto (x')} e^{-t(\xi') x_3} \widehat{u_1^+}(\xi')$$

$$\mathcal{K}_{N,\Omega^-} u_1^-(x) = \mathcal{F}^{-1}_{\xi' \mapsto (x')} e^{t(\xi') x_3} \widehat{u_1^-}(\xi')$$

$$\begin{pmatrix} u_1^+ \\ u_1^- \end{pmatrix} = \Upsilon_N^{-1} \begin{pmatrix} A_t W^{-1}_{t,\Sigma} & 0 \\ 0 & \ell_0 \end{pmatrix} \Upsilon_N \begin{pmatrix} g^+ \\ g^- \end{pmatrix}, \quad \Upsilon_N = \begin{pmatrix} I & I \\ I & -I \end{pmatrix}.$$

General Wiener-Hopf operators

A *general Wiener-Hopf operator* is given by

$$W = P_2 A|_{P_1 X}$$

where $A : X \to Y$ is a bounded linear operator acting in Banach spaces and $P_1 \in \mathcal{L}(X), P_2 \in \mathcal{L}(Y)$ are projectors.
By convention, W is regarded as an operator from $P_1 X = \operatorname{im} P_1$ into $P_2 Y = \operatorname{im} P_2$, although, P_2 acts into Y, i.e., strictly speaking, not into $P_2 Y$ (cf. MikPro80). This *convention* will be applied and referred to in the sequel for convenience (and following the tradition).
For practical reasons we enlarge the convention by writing in some formulas W instead of $P_2\,A\,P_1$. Also the notation W^{-1} is used in both senses, which makes some formulas more compact.
The convention could be avoided by restriction of W in the domain *and* in the image writing $W = \operatorname{Rst} P_2 A : \operatorname{im} P_1 \to \operatorname{im} P_2$. Conversely, extension by zero on a complement of the domain, linear extension to the full space and embedding in the image space helps to extend W to an operator between the full spaces written as $\operatorname{Ext} W : X \to Y$.

Identification of general Wiener-Hopf operators

The connection between classical WHOs and general WHOs is given via a continuous extension operator

$$E^s_\Sigma \quad : \quad H^s(\Sigma) \to H^s(\mathbb{R}^n)$$

provided it exists for $\Sigma \subset \mathbb{R}^n$ (see Def. of E-domains), namely by the identification

$$\begin{aligned} X &= H^r \quad , \quad Y = H^s \\ P_1 X &= H^r_\Sigma \quad , \quad P_2 = E^s_\Sigma \, r_\Sigma \\ A = \mathcal{F}^{-1} \phi \cdot \mathcal{F} &\quad : \quad H^r \to H^s. \end{aligned}$$

We observe, in the identification of a general WHO, that not the full definitions of P_1 and P_2 are relevant but only $\operatorname{im} P_1$ and $\ker P_2$.

This yields:

Equivalent WHOs

Proposition Let $W = P_2 A|_{P_1 X}$ be a general WHO where X, Y are Hilbert spaces. Then

$$W \quad \sim \quad \widetilde{W} \quad = \quad \Pi A|_{PX}$$

where P and Π are orthogonal projectors (or any other projectors) onto $P_1 X$ and along $(I - P_2)Y$, respectively.

Proposition Let W, $\widetilde{W}$ be general WHOs related as before and let $\widetilde{W}^- : \Pi Y \to PX$ be a generalized inverse of $\widetilde{W}$. Then a generalized inverse of W is given by

$$W^- \quad = \quad P_1|_{PX} \widetilde{W}^- \Pi|_{P_2 Y}.$$

A geometric perspective (dos Santos 1988)

Proposition Let $W = P_2 A|_{P_1 X}$ be a general WHO. Then W is invertible if and only if

$$AP_1X \dotplus Q_2Y \quad = \quad Y.$$

If A is invertible we equivalently have

$$P_1X \dotplus A^{-1}Q_2Y \quad = \quad X.$$

Corollary In this case, the inverse of W is represented by

$$\begin{aligned} W^{-1} &= \quad A^{-1}\Pi \quad \text{where } \Pi \text{ projects onto } AP_1X \text{ along } Q_2Y \\ &= \quad PA^{-1} \quad \text{where } P \text{ projects onto } P_1X \text{ along } A^{-1}Q_2Y \end{aligned}$$

as an operator $W^{-1} : P_2Y \to P_1X$. Conversely: If W is invertible, then

$$\Pi \;=\; AW^{-1} \quad , \quad P \;=\; W^{-1}A$$

are projectors as operators acting in X and Y, respectively (more precisely $\Pi = AW^{-1}P_2$, $P = P_1W^{-1}A$).

Two operator relations (BGK 1984)

Two bounded linear operators in Banach spaces $S \in \mathcal{L}(X_1, Y_1)$ and $T \in \mathcal{L}(X_2, Y_2)$ are said to be matrically coupled, if there exists an invertible operator matrix (with suitable entries $*$) such that

$$\begin{pmatrix} S & * \\ * & * \end{pmatrix} = \begin{pmatrix} * & * \\ * & T \end{pmatrix}^{-1}.$$

Two bounded linear operators in Banach spaces S and T are said to be equivalent after extension, in brief

$$S \overset{*}{\sim} T,$$

if there exist Banach spaces Z_1, Z_2 and linear homeomorphisms E, F such that

$$\begin{pmatrix} S & 0 \\ 0 & I_{Z_1} \end{pmatrix} = E \begin{pmatrix} T & 0 \\ 0 & I_{Z_2} \end{pmatrix} F.$$

Matrical coupling vs. equivalence after extension

Theorem Bart-Tsekanovsky 1991 Let S and T be bounded linear operators in Banach spaces. Then $S \overset{*}{\sim} T$ if and only if S and T are matrically coupled.

Remark The importance of this theorem for us consists in the consequence, that an inverse of T can be computed from an inverse of S (and vice versa, if E^{-1} and F^{-1} are known). This is obvious from an EAER but not from a MCR – and was a celebrated fact in the 1980s, see BGK84.

Theorem Speck 1983/85 Let S and T be bounded linear operators in Banach spaces which are matrically coupled, i.e., $S = W = P_2 A|_{P_1 X}$ and $T = W_* = Q_1 A^{-1}|_{Q_2 Y}$ in the above notation. Further let V be a generalized inverse of W, i.e., $WVW = W$. Then a generalized inverse of W_* is given by

$$V_* \quad = \quad Q_2(A - AP_1 V P_2 A)|_{Q_1 X}.$$

Construction of the projectors P and Π onto/along $H_{\Sigma}^{\pm 1/2}$

Abstract setting of projectors

$$\Pi \quad \text{onto} \quad AP_1X \quad \text{along} \quad Q_2Y$$
$$P \quad \text{onto} \quad P_1X \quad \text{along} \quad A^{-1}Q_2Y$$

where $A \in \mathcal{L}(X,Y)$ is boundedly invertible, $P_1 \in \mathcal{L}(X)$, $P_2 \in \mathcal{L}(Y)$ arbitrary projectors and $Q_1 = I_X - P_1$, $Q_2 = I_Y - P_2$.

Concrete realization where Σ is an E-domain, $t(\xi) = (\xi^2 - k^2)^{1/2}$

$$A \;=\; A_t \;=\; \mathcal{F}^{-1}\, t \cdot \mathcal{F} \quad : \quad H^{1/2}(\mathbb{R}^2) \to H^{-1/2}(\mathbb{R}^2)$$

P_1 is any projector in $H^{1/2}(\mathbb{R}^2)$ onto $H_{\Sigma}^{1/2}$

P_2 is any projector in $H^{-1/2}(\mathbb{R}^2)$ along $H_{\Sigma'}^{-1/2}$.

This notation is easily generalized to Sobolev spaces of arbitrary order. Let us see it in the context of a half-plane.

Example: the half-plane case

Consider $\Sigma = \mathbb{R}^2_{1+} = \{x \in \mathbb{R}^2 : x_1 > 0\}$, the (orthogonal) projectors $P_+ = \ell_0 r_+ : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ onto $L^2_\Sigma = L^2_{\mathbb{R}^2_{1+}}$, $P_- = I - P_+$, then the Bessel potential operators of order $s \in \mathbb{R}$ for Σ and Σ' are:

$$\begin{aligned} \Lambda^s_+ &= A_{\lambda^s_+} \quad , \quad \lambda^s_+(\xi) = \left(\xi_1 + i\sqrt{\xi_2^2 + 1}\right)^s , \xi \in \mathbb{R}^2 \\ \Lambda^s_- &= A_{\lambda^s_-} \quad , \quad \lambda^s_-(\xi) = \left(\xi_1 - i\sqrt{\xi_2^2 + 1}\right)^s , \xi \in \mathbb{R}^2. \end{aligned}$$

For any $s \in \mathbb{R}$ we find the orthogonal projectors

$$\begin{aligned} P^s_+ &= \Lambda^{-s}_+ P_+ \Lambda^s_+ \quad \text{onto} \quad H^s_\Sigma \\ P^s_- &= \Lambda^{-s}_- P_- \Lambda^s_- \quad \text{onto} \quad H^s_{\Sigma'} \\ \Pi^s_+ &= \Lambda^{-s}_- P_+ \Lambda^s_- \quad \text{along} \quad H^s_{\Sigma'} \\ \Pi^s_- &= \Lambda^{-s}_+ P_- \Lambda^s_+ \quad \text{along} \quad H^s_\Sigma. \end{aligned}$$

Hence $P^s_+ + \Pi^s_- = I_{H^s}$ and $P^s_- + \Pi^s_+ = I_{H^s}$.

The infimum of two or m orthogonal projectors

Lemma Halmos 1982 Given any Hilbert space H and orthogonal projectors $p_1, p_2, ..., p_m \in \mathcal{L}(H)$, the orthogonal projector onto $\operatorname{im} p_1 \cap \operatorname{im} p_2$ is given by the so-called infimum of the two projectors:

$$p_1 \wedge p_2 \quad = \quad \prod_{j=1}^{\infty} (p_1 p_2)^j \quad = \quad \lim_{N \to \infty} \prod_{j=1}^{N} (p_1 p_2)^j$$

which converges strongly.

The orthogonal projector p onto $\operatorname{im} p_1 \cap ... \cap \operatorname{im} p_m$ is given by

$$p \quad = \quad p_1 \wedge ... \wedge p_m \quad = \quad \wedge_{j=1}^{m} p_j$$

that is defined by iteration and represents an associative operation.

Transformation of half-planes

For any open half-plane $\Sigma \subset \mathbb{R}^2$ let

$$M_\Sigma \quad : \quad \Sigma \;\to\; \mathbb{R}^2_{1+} = \{x \in \mathbb{R}^2 : x_1 > 0\}$$

be the canonical linear transformation that transforms Σ onto $\mathbb{R}^2_{1+}$, by a minimal dilation plus a rotation in the mathematical positive sense. Furthermore

$$J_\Sigma f(x) \quad = \quad f(M_\Sigma\, x) \quad , \quad x \in \Sigma \text{ or } x \in \mathbb{R}^2$$

for functions and distributions defined on Σ or defined on $\mathbb{R}^2$, as well.

Case $k = i$, convex PCDs

Theorem MS 87 Let Σ be a convex PCD, i.e., $\Sigma = \cap_{j=1}^{m} \Sigma_j$ with half-planes $\Sigma_j \subset \mathbb{R}^2, j = 1, ..., m$ and $s \in \mathbb{R}$. Then the orthogonal projector P_Σ^s onto $P_1 X = H_\Sigma^s$ projects along $\Lambda^{-2s} H_{\Sigma'}^{-s}$ and is given by

$$\begin{aligned} P_\Sigma^s &= \wedge_{j=1}^{m} P_{\Sigma_j}^s \\ P_{\Sigma_j}^s &= J_{\Sigma_j}^{-1} \, P_+^s \, J_{\Sigma_j} \quad , \quad j = 1, ..., m. \end{aligned}$$

The orthogonal projector Π onto $\Lambda^{2s} \, H_\Sigma^s$ projects along $Q_2 Y = H_{\Sigma'}^{-s}$ and is given by

$$\Pi_\Sigma^{-s} = \Lambda^{2s} \, P_\Sigma^s \, \Lambda^{-2s}.$$

Case $k = i$, arbitrary PCDs

Theorem Let $\Sigma \subset \mathcal{A}_2$, i.e., $\Sigma = \text{ int} \bigcup_{j=1,\ldots,n} \operatorname{clos} \Sigma_j$ where Σ_j are convex PCDs, and assume that $s \in \mathbb{R}$. Then the orthogonal projector P^s_Σ onto H^s_Σ projects along $\Lambda^{-2s} H^{-s}_{\Sigma'}$, i.e., $P^s_\Sigma = I - \Pi^s_{\Sigma'}$ and is given by

$$P^s_\Sigma \quad = \quad I - \wedge_{j=1}^m \Pi^s_{\Sigma'_j} \quad = \quad I - \wedge_{j=1}^m (I - P^s_{\Sigma_j})$$

with $P^s_{\Sigma_j}$ taken from the previous Theorem (representing each Σ_j as intersection of half-planes).

Case $k \in i\mathbb{R}_+$, i.e., $\Re e k = 0$, $\Im m k > 0$

The previous results remain valid for $k \in i\mathbb{R}_+$ if we change the topology to another equivalent one. I.e., we remain in the same Hilbert spaces but infinite series and infinite products converge in a different sense, with respect to a modified norm.

Definition Let $H^{s,k}(\mathbb{R}^n)$ be the space that coincides with $H^s(\mathbb{R}^n)$ as a linear space equipped with the form

$$\begin{aligned}\langle \varphi, \psi \rangle_{s,k} &= \langle A_t^{-s}\varphi\,,\, A_t^{-s}\psi \rangle_0 \\ &= \int_{\mathbb{R}^n} A_t^{-s}(x)\varphi(x) \cdot \overline{A_t^{-s}(x)\psi(x)}\, dx \\ &= \int_{\mathbb{R}^n} \widehat{\varphi}(\xi)\, \widehat{\psi}(\xi)\, |\xi^2 - k^2|^{-s}\, d\xi.\end{aligned}$$

Case $\Re e k \neq 0$, $\Im m k > 0$

Idea: Present the (non-orthogonal) projectors $\Pi_{\Sigma}^{\pm 1/2}$ etc. by Neumann series using the orthogonal projectors constructed before.

Notation

$$\Pi^s_{\Sigma,k} \quad - \quad \text{the projector onto} \quad A_{t^{-2s}} H^{-s}_{\Sigma} \quad \text{along} \quad H^s_{\Sigma'}$$

$$\Pi^s_{\Sigma,i} \quad - \quad \text{the projector onto} \quad \Lambda^{-2s} H^{-s}_{\Sigma} \quad \text{along} \quad H^s_{\Sigma'}$$

where the latter coincides with the first for $k = i$ and is orthogonal. Further projectors can be defined and treated by analogy. So we have:

$$P^s_{\Sigma,k} \quad - \quad \text{the projector onto} \quad H^s_{\Sigma} \quad \text{along} \quad A_{t^{-2s}} H^{-s}_{\Sigma'}$$

$$P^s_{\Sigma,k} \quad = \quad I - \Pi^s_{\Sigma',k}$$

which we employ basically for $s = \pm 1/2$.

Case $\Re e k \neq 0\,, \Im m k > 0$ ctd.

Proposition Put $\Pi = \Pi^{1/2}_{\Sigma,i}$. Then the projector $\Pi^{1/2}_{\Sigma,k}$ is given by

$$\Pi^{1/2}_{\Sigma,k} \quad = \quad A_{t^{-1}}\,\Lambda\,W_0^{-1} \qquad , \qquad W_0 \,=\, \Pi\,A_{t^{-1}}\,\Lambda|_{\Pi H^{1/2}}$$

(using the convention) where the inverse W_0^{-1} is given by a Neumann series.

Proposition Put $\Pi = \Pi^{-1/2}_{\Sigma,i}$. Then the projector $\Pi^{-1/2}_{\Sigma,k}$ is given by

$$\Pi^{-1/2}_{\Sigma,k} \quad = \quad A_t\,\Lambda^{-1}\,W_0^{-1} \qquad , \qquad W_0 \,=\, \Pi\,A_t\,\Lambda^{-1}|_{\Pi H^{-1/2}}$$

(using the convention) where the inverse W_0^{-1} is given by a Neumann series.

Explicit solution of the BVPs

Theorem Let $\Sigma \subset \mathbb{R}^2$ be a PCD and $k \in \mathbb{C}$, $\Im m k > 0$. The resolvent operators for the Dirichlet and Neumann problems are given by the corresponding representation formulas where

$$\begin{aligned} W_{t^{-1},\Sigma}^{-1} &= A_t \, \Pi_\Sigma^{1/2} &= P_\Sigma^{-1/2} A_t \\ W_{t,\Sigma}^{-1} &= A_t^{-1} \, \Pi_\Sigma^{-1/2} &= P_\Sigma^{1/2} A_t^{-1} \end{aligned}$$

and the projectors are also explicitly given in the corresponding cases

- for convex PCDs and $k = i$,
- for arbitrary PCDs and $k = i$,
- for arbitrary PCDs and $k \in i\mathbb{R}_+$,
- for arbitrary PCDs, $\Im m k > 0$, the Dirichlet problem,
- for arbitrary PCDs, $\Im m k > 0$, the Neumann problem.

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Et bonne chance pour Martin !!!