Stability and Instability of Cycled Inverse Scattering Problems

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2. Cycled Inverse Scattering Problems
   - Cycled Tikhonov and 3dVar
   - Spectral Representation
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   - Spectral Representation

3 Convergence Analysis for Cycled Inversion 1: Range Arguments
   - Setup: Constant System
   - Convergence for Data $f \in R(H)$ / Divergence for $f \notin R(H)$
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5 Reading ↔ Rennes
Motivation and Introduction

Literature


Moodey, Lawless, P. and van Leeuwen: **Nonlinear error dynamics for cycled data assimilation methods** 2013 Inverse Problems 29 025002


Motivation and Introduction

Motivation I: Inverse Scattering in Applications ...

Medical Imaging via Scattering and Inverse Source Problems

Seismic Exploration
Motivation II: Inversion and Dynamics ...

Describe the system by a state \( \phi \in X \).

You apply an inverse scattering algorithm to reconstruct \( \phi_k \) at time \( t_k \) given data \( f_k \in Y \).

The system evolves from \( \phi_k \) to \( \phi(b)_{k+1} \).

You measure scattering data at \( t_k+1 \) to reconstruct \( \phi_{k+1} \) given measurements \( f_{k+1} \) and the first guess \( \phi(b)_{k+1} \).
Motivation II: Inversion and Dynamics ...

Inverse scattering +
  further knowledge
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Motivation II: Inversion and Dynamics ...

Inverse scattering +

- further knowledge

- Describe the system by a state $\varphi \in X$. 
Inverse scattering +

further knowledge

- Describe the system by a state $\varphi \in X$.
- The state at time $t_k$ is $\varphi_k = \varphi(t_k) \in X$. 
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You apply an inverse scattering algorithm to reconstruct $\varphi_k$ at $t_k$ given data $f_k \in Y$. 

Inverse scattering + further knowledge
Motivation II: Inversion and Dynamics ...

Inverse scattering +

further knowledge

- Describe the system by a **state** \( \varphi \in X \).
- The state at time \( t_k \) is \( \varphi_k = \varphi(t_k) \in X \).
- You apply an **inverse scattering** algorithm to reconstruct \( \varphi_k \) at \( t_k \) given data \( f_k \in Y \).
- The **system evolves** from \( \varphi_k \) to \( \varphi_{k+1}^{(b)} \).
Inverse scattering +

further knowledge

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Motivation II: Inversion and Dynamics ...

Inverse scattering + further knowledge

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- The system evolves from $\varphi_k$ to $\varphi_{k+1}^{(b)}$.
- You measure scattering data at $t_{k+1}$ to reconstruct $\varphi_{k+1}$ given measurements $f_{k+1}$ and...
Motivation II: Inversion and Dynamics ...

Inverse scattering +

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- The state at time $t_k$ is $\varphi_k = \varphi(t_k) \in X$.

- You apply an inverse scattering algorithm to reconstruct $\varphi_k$ at $t_k$ given data $f_k \in Y$.

- The system evolves from $\varphi_k$ to $\varphi_{k+1}$.

- You measure scattering data at $t_{k+1}$ to reconstruct $\varphi_{k+1}$ given
  - measurements $f_{k+1}$ and
  - the first guess $\varphi_{k+1}^{(b)}$. 

Roland Potthast
NWP: Measurements for State Determination ...

- Synop,
- TEMP,
- Radiosondes,
- Buoys,
- Airplanes (AMDAR),
- Radar,
- Wind Profiler,
- Scatterometer,
- Radiances,
- GPS/GNSS,
- Ceilometer,
- Lidar
Setting for reconstruction problem

Partial Differential Equation (Acoustic, Electromagnetic, Elastic)

Boundary Condition on Object

Incident Wave

Remote Measurements
Electromagnetic scattering problem

- Bounded scatterer in three dimensions with boundary of class $C^2$, incident field $E^i$
- Scattered field $E^s$ solves Maxwell equations

\[
\begin{align*}
\text{curl} E^s - i\kappa H^s &= 0 \\
\text{curl} H^s + i\kappa E^s &= 0
\end{align*}
\]

in $\mathbb{R}^3 \setminus \overline{D}$ and satisfies the Silver-Müller radiation condition

\[
E^s \times x + rH^s \to 0, \quad r = |x| \to \infty.
\]

- On the boundary $\Gamma := \partial D$ the tangential component of the total field $E = E^i + E^s$ vanishes, i.e. we have the perfect conductor boundary condition

\[

\nu \times E|_\Gamma = 0
\]
Measured data

Measured data are either the scattered field $E^s$ on some surface $\Lambda$ or the far field pattern $E^\infty$ defined by

$$E^s(x) = \frac{e^{ik|x|}}{|x|} \left\{ E^\infty(\hat{x}) + O\left(\frac{1}{x}\right) \right\}, \quad \hat{x} := x/|x|$$

(4)

uniformly on $\mathbb{S}$ for $|x| \to \infty$.

For the above scattering problem the far field pattern can be calculated via integral equations of the second kind.

$$E^s(\partial D) = \mathcal{H}(\partial D) \quad \text{or} \quad f(\varphi) = \mathcal{H}(\varphi)$$

(5)

when $\varphi := \partial D(t)$. 
Mathematical Setup for Cycled Inversion

State Space:

- $\mathcal{X}$: state space, containing all state variables in one vector $\varphi$
- $\varphi$: state of the system
Mathematical Setup for Cycled Inversion

State Space:

- $X$: state space, containing all state variables in one vector $\varphi$
- $\varphi$: state of the system
- $t_k$: time discretization point
- $\varphi_k$: state at time $t_k$
Mathematical Setup for Cycled Inversion

State Space:

- **State space** \( X \), containing all state variables in one vector \( \varphi \)
- **State** \( \varphi \)
- **Time discretization point** \( t_k \)
- **State at time** \( t_k \) \( \varphi_k \)
- **Model operator at time** \( t_k \) \( M_k : X \rightarrow X \)

\[
\varphi_k \mapsto \varphi_{k+1} = M_k(\varphi_k)
\]
Mathematical Setup for Cycled Inversion

State Space:
- $X$ \textit{state space}, containing all state variables in one vector $\varphi$
- $\varphi$ state of the system
- $t_k$ time discretization point
- $\varphi_k$ state at time $t_k$
- $M_k : X \rightarrow X$ \textit{model operator} at time $t_k$, $\varphi_k \mapsto \varphi_{k+1} = M(\varphi_k)$

Observation Space
- $Y_k$ \textit{observation space} at time $t_k$
- $f_k$ observation vector at time $t_k$
- $H_k : X \rightarrow Y_k$ \textit{observation operator}
- $x, y$ points in physical space
Data Assimilation or Cycled Inversion Task

Definition (Data Assimilation Task)

Given measurements \( f_k \) at \( t_k \) for \( k = 1, 2, 3, \ldots \) determine the states \( \varphi_k \) from the equations

\[
H \varphi_k = f_k, \quad k = 1, 2, 3, \ldots \tag{6}
\]

taking care of the *model dynamics* given by \( M_k \).
Motivation and Introduction

Cycled Inverse Problems

Data Assimilation or Cycled Inversion Task

Definition (Data Assimilation Task)

Given measurements $f_k$ at $t_k$ for $k = 1, 2, 3, \ldots$ determine the states $\varphi_k$ from the equations

$$H\varphi_k = f_k, \quad k = 1, 2, 3, \ldots$$

(6)

taking care of the model dynamics given by $M_k$.

- Usually the measurement space is dynamic, i.e. changing in every time-step.
- In general $H$ is a non-linear operator, non-injective, ill-posed.
- The value $f_k$ contains significant data error with stochastic components and a dynamic bias.
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5 Reading ↔ Rennes
In every assimilation step $k \in \mathbb{N}$ we solve the variational minimization problem to find the minimum of

$$J(\varphi) := \alpha \| \varphi - \varphi_{k+1}^{(b)} \|^2 + \| f_{k+1} - H\varphi_{k+1}^{(b)} \|^2, \quad \varphi \in X,$$

where

$$\varphi_{k+1}^{(b)} := M\varphi_k^{(a)}, \quad k = 0, 1, 2, \ldots$$

For linear operators the minimum is given by the *normal equations*, which can be reformulated into the update formula

$$\varphi_{k+1}^{(a)} = \varphi_{k+1}^{(b)} + R_\alpha \left( f_{k+1} - H\varphi_{k+1}^{(b)} \right)$$

with $R_\alpha = \left( \alpha I + H^*H \right)^{-1} H^*$, or in terms of the analysis fields $\varphi_k^{(a)}$

$$\varphi_{k+1}^{(a)} = M\varphi_k^{(a)} + R_\alpha \left( f_{k+1} - HM\varphi_k^{(a)} \right)$$

for $k = 0, 1, 2, \ldots$.
Start with $\phi(a)_0$ and for $k = 1, 2, 3, ...$
do:

Calculate first guess $\phi(b)_k = M_k - 1 \phi(a)_{k-1}$

Assimilate data $f_k$ at time $t_k$ calculating $\phi(a)_k$.
Start with $\varphi_0^{(a)}$ and for $k = 1, 2, 3, \ldots$ do:
1 Start with $\varphi_0^{(a)}$ and for $k = 1, 2, 3, \ldots$ do:

2 Calculate first guess

$$\varphi_k^{(b)} = M_{k-1} \varphi_{k-1}^{(a)}$$ (11)
1. Start with $\phi_0^{(a)}$ and for $k = 1, 2, 3, \ldots$ do:

2. Calculate **first guess**

   $$\phi_k^{(b)} = M_{k-1} \phi_{k-1}^{(a)}$$  \hspace{1cm} (11)

3. **Assimilate data** $f_k$ at time $t_k$ calculating $\phi_k^{(a)}$. 

Roland Potthast
3dVAR - 2

Functional at time slice

\[ J(\varphi) = \|\varphi - \varphi^{(b)}\|_B^{-2} + \|f - H\varphi\|_R^{-2} \]  \hspace{1cm} (12)

Update Formula

\[ \varphi_k^{(a)} = \varphi_k^{(b)} + (B^{-1} + H^* R^{-1} H)^{-1} H^* R^{-1} (f_k - H(\varphi_k^{(b)})) \]

\[ = \varphi_k^{(b)} + BH^* (R + HBH^*)^{-1} \left( f_k - H(\varphi_k^{(b)}) \right). \]  \hspace{1cm} (13)
Nonlinear Observation Operators

In the case of nonlinear observation operators $\mathcal{H}$ we employ linearization

$$\mathcal{H}(\varphi^{(b)} + \delta \varphi) = \mathcal{H}(\varphi^{(b)}) + H\delta \varphi + O(||\delta \varphi||^2).$$

with $H$ being the linearization of $\mathcal{H}$ at $\varphi^{(b)}$. Then, we need to minimize

$$J(\delta \varphi) = ||\delta \varphi||^2 + ||(f - \mathcal{H}(\varphi^{(b)})) - H\delta \varphi||^2.$$

Nonlinear three-dimensional Variational Data Assimilation (3dVar)

The variational update formula of 3dVar is

$$\varphi^{(a)}_k = \varphi^{(b)}_k + (B^{-1} + H'R^{-1}H)^{-1}H'R^{-1}(f_k - \mathcal{H}(\varphi^{(b)}_k))$$

$$= \varphi_0 + BH'(R + HBH')^{-1}(f_k - \mathcal{H}\varphi^{(b)}_k).$$

(14)

$$\varphi^{(b)}_{k+1} = M_k \varphi^{(a)}_k, \ k = 1, 2, 3, \ldots$$

(15)
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5 Reading ↔ Rennes
Assume that $H : X \to Y$ is linear and compact. Then, there is a singular system \( \{ (\psi_n, g_n, \mu_n), \ n \in \mathbb{N} \} \) such that $H\psi_n = \mu_n g_n$, $H'g_n = \mu_n \psi_n$ and

$$H'H\psi_n = \mu_n^2 \psi_n,$$  \hspace{1cm} (16)
Assume that $H : X \rightarrow Y$ is linear and compact. Then, there is a **singular system**
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This yields
\[
(\alpha I + H' H)\psi_n = (\alpha + \mu_n^2)\psi_n, \ n \in \mathbb{N}.
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\[ H' H \psi_n = \mu_n^2 \psi_n, \] (16)

This yields
\[ (\alpha I + H' H) \psi_n = (\alpha + \mu_n^2) \psi_n, \quad n \in \mathbb{N}. \] (17)

With measurements
\[ f = \sum_{n=1}^{\infty} f_n g_n \in Y \] (18)

and denote the spectral coefficients of the Tikhonov solution $\varphi$ by $\gamma_n$. 3dVar or Tikhonov regularization, respectively, $(\alpha I + H' H) \varphi = H' f$ is equivalent to the spectral damping scheme
\[ \gamma_n = \frac{\mu_n}{\alpha + \mu_n^2} f_n, \quad n \in \mathbb{N}. \] (19)
The true Inverse is

$$\gamma_n^{\text{true}} = \frac{1}{\mu_n} f_n^{\text{true}}.$$  

(20)
The true Inverse is

$$\gamma_n^{true} = \frac{1}{\mu_n} f_n^{true}. \quad (20)$$

This inversion is unstable, if $\mu_n \to 0$, $n \to \infty$!
The true Inverse is

\[ \gamma_n^{true} = \frac{1}{\mu_n} f_n^{true}. \]  

(20)

This inversion is **unstable**, if \( \mu_n \to 0, \ n \to \infty \)!

**Tikhonov regularization** is **stable** for \( \alpha > 0 \)

\[ \gamma_n = \frac{\mu_n}{\alpha + \mu_n^2} f_n, \quad n \in \mathbb{N}. \]  

(21)

Tikhonov **shifts the eigenvalues** of \( H' H \) by \( \alpha \).
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5 Reading ↔ Rennes
A system with constant dynamics

As a simple model system for study we use constant dynamics $M = \text{Identity}$, i.e

$$\varphi_{k+1}^{(b)} = \varphi_k^{(a)}, \quad k = 1, 2, 3, \ldots$$

(22)

for 3dVar. Also, we employ identical measurements $f_k \equiv f, \ k \in \mathbb{N}$. 
A system with constant dynamics

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\]

for 3dVar. Also, we employ \textbf{identical measurements} \( f_k \equiv f, \; k \in \mathbb{N} \).

Then, \textbf{3dVar} is given by the iteration

\[
\varphi_k = \varphi_{k-1} + (\alpha I + H' H)^{-1} H' (f - H \varphi_{k-1}), \quad k = 1, 2, 3, \ldots \tag{23}
\]

(This coincides with work of Engl on 'iterated Tikhonov regularization'!)
A system with constant dynamics

As a simple model system for study we use constant dynamics $M = \text{Identity}$, i.e

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Then, 3dVar is given by the iteration

\[ \varphi_k = \varphi_{k-1} + (\alpha I + H'H)^{-1}H'(f - H\varphi_{k-1}), \quad k = 1, 2, 3, \ldots \]  

(23)

(This coincides with work of Engl on 'iterated Tikhonov regularization'!)

For the spectral coefficients $\gamma_{n,k}$ of $\varphi_k$ this leads to the iteration

\[ \gamma_{n,k} = \gamma_{n,k-1} + \frac{\mu_n}{\alpha + \mu_n^2}(f_{n,k} - \mu_n \gamma_{n,k-1}) \]  

(24)
Spectral Formula I

We employ

\[ f = H \varphi^{(true)} + \delta, \quad f_{n,k} = \mu_n \gamma_n^{(true)} + \delta_n. \]  \hspace{1cm} (25)

and obtain

\[ \gamma_{n,k} = \gamma_{n,k-1} + \frac{\mu_n^2}{\alpha + \mu_n^2} (\gamma_n^{(true)} - \gamma_{n,k-1}) + \frac{\mu_n^2}{\alpha + \mu_n^2} \delta_n. \]

**Theorem (Spectral Formula I)**

The 3dVar cycling for a constant dynamics with identical measurements \( f = H \varphi^{(true)} + \delta \) lead to the spectral update formula

\[ \gamma_{n,k} = (1 - q_n) \gamma_n^{(true)} + q_n \gamma_{n,k-1} + \frac{(1 - q_n)}{\mu_n} \delta_n \]  \hspace{1cm} (26)

using

\[ q_n = \frac{\alpha}{\alpha + \mu_n^2} = 1 - \frac{\mu_n^2}{\alpha + \mu_n^2}. \]  \hspace{1cm} (27)
Theorem (Spectral Formula II)

The 3dVar cycling for a constant dynamics with identical measurements $f = H\varphi^{(true)} + \delta$ can be carried out explicitly. The development of its spectral coefficients is given by

$$
\gamma_{n,k} = (1 - q_n^k)\gamma_n^{(true)} + q_n^k\gamma_{n,0} + \frac{(1 - q_n^k)}{\mu_n}\delta_n
$$

(28)

using

$$
q_n = \frac{\alpha}{\alpha + \mu_n^2} = 1 - \frac{\mu_n^2}{\alpha + \mu_n^2}.
$$

(29)

Proof. Induction over $k$. □
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5 Reading \( \leftrightarrow \) Rennes
Convergence Analysis for Cycled Inversion 1: Range Arguments

Convergence for $f \in R(H)$

Theorem (Convergence for $f \in R(H)$)

Cycled 3dVar for a constant dynamics and identical measurements $f^{(true)} + \delta \in R(H)$ tends to the true solution $\varphi^{(true)} + \sigma$ with $H\sigma = \delta$ for $k \to \infty$. 

Roland Potthast
**Convergence for \( f \in R(H) \)**

**Theorem (Convergence for \( f \in R(H) \))**

Cycled 3dVar for a constant dynamics and identical measurements \( f^{(true)} + \delta \in R(H) \) tends to the true solution \( \varphi^{(true)} + \sigma \) with \( H\sigma = \delta \) for \( k \to \infty \).

*Proof.* We study

\[
\gamma_{n,k} = (1 - q_n^k)\gamma^{(true)}_n + q_n^k \gamma_{n,0} + \frac{(1 - q_n^k)}{\mu_n} \delta_n
\]  

(30)

for \( k \to \infty \).
Convergence Analysis for Cycled Inversion 1: Range Arguments

Convergence for Data $f \in R(H)$ / Divergence for $f \not\in R(H)$

**Theorem (Convergence for $f \in R(H)$)**

*Cycled 3dVar for a constant dynamics and identical measurements* $f^{(true)} + \delta \in R(H)$ *tends to the true solution* $\varphi^{(true)} + \sigma$ *with* $H\sigma = \delta$ *for* $k \to \infty$.

*Proof.* We study

\[
\gamma_{n,k} = (1 - q_n^k)\gamma_n^{(true)} + q_n^k\gamma_{n,0} + \frac{(1 - q_n^k)}{\mu_n}\delta_n
\]

(30)

for $k \to \infty$.

Since $0 < q_n < 1$, we have

\[
q_n^k \to 0, \ k \to \infty, \quad (1 - q_n^k) \to 1, \ k \to \infty.
\]

(31)

Since $\delta = H\sigma$ the element $\sigma$ with spectral coefficients $\delta_n/\mu_n$ is in $X$ and cycled 3dVar converges towards $\varphi^{(true)} + \sigma$.  

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Divergence for \( f \not\in R(H) \)

**Theorem (Divergence for \( f \not\in R(H) \))**

For a constant dynamics and identical measurements \( f^{(\text{true})} + \delta \not\in R(H) \), cycled 3dVar diverges for \( k \to \infty \).
Divergence for $f \notin R(H)$

**Theorem (Divergence for $f \notin R(H)$)**

For a constant dynamics and identical measurements $f^{(\text{true})} + \delta \notin R(H)$ cycled 3dVar **diverges** for $k \to \infty$.

Proof. We study

$$\gamma_{n,k} = (1 - q_n^k)\gamma_n^{(\text{true})} + q_n^k\gamma_{n,0} + \frac{(1 - q_n^k)}{\mu_n}\delta_n$$

for $k \to \infty$. Let $\sigma_k \in X$ denote the element with spectral coefficients

$$\sigma_{n,k} = \frac{(1 - q_n^k)}{\mu_n}\delta_n, \quad k, n \in \mathbb{N}.$$ 

which is well defined since for every fixed $k \in \mathbb{N}$

$$\left| \frac{(1 - q_n^k)}{\mu_n} \right| = \left| \frac{(\alpha + \mu_n^2)^k - \alpha^k}{(\alpha + \mu_n^2)^k \mu_n} \right|$$

is bounded uniformly for $n \in \mathbb{N}$. 

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Since \( \delta \notin R(H) \) we know that

\[
S_L := \sum_{n=1}^{L} \left| \frac{\delta_n}{\mu_n} \right|^2 \rightarrow \infty, \quad L \rightarrow \infty.
\] (34)

Given \( C > 0 \) we can choose \( L \) such that \( S_L > 2C \). Then

\[
||\sigma_k||^2 \geq \sum_{n=1}^{L} \left| \frac{(1 - q_n^k)\delta_n}{\mu_n} \right|^2 > C
\] (35)

for \( k \in \mathbb{N} \) sufficiently large, which proves

\[
||\sigma_k|| \rightarrow \infty, \quad k \rightarrow \infty
\] (36)

and the proof is complete.

\[ \square \]
Convergence Analysis for Cycled Inversion 1: Range Arguments

Convergence for Data \( f \in R(H) \) / Divergence for \( f \notin R(H) \)

**Numerical Example: Dynamic Magnetic Tomography**

![Graph showing convergence and divergence in time index k](image)

- **Error** values for different time indices
- **3dVar** line indicating convergence for data within the range of the operator \( R(H) \)
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5 Reading ↔ Rennes
\[
\varphi^{(a)}_{k+1} - \varphi^{(\text{true})}_{k+1} = M(\varphi^{(a)}_k - \varphi^{(\text{true})}_k) + R_\alpha H M (\varphi^{(\text{true})}_k - \varphi^{(a)}_k) + R_\alpha f^{(\delta)}_{k+1}
\]

\[
= (I - R_\alpha H) M (\varphi^{(a)}_k - \varphi^{(\text{true})}_k) + R_\alpha f^{(\delta)}_{k+1}.
\] (37)

We abbreviate the analysis error by

\[
e_k := \varphi^{(a)}_k - \varphi^{(\text{true})}_k, \quad k = 1, 2, \ldots
\] (38)

and define

\[
\Lambda := (I - R_\alpha H) M
\] (39)

to obtain the iteration formula

\[
e_{k+1} = \Lambda e_k + R_\alpha f^{(\delta)}_{k+1}, \quad k = 0, 1, 2, \ldots
\] (40)
System Evolution

Theorem

Assume that the error $f^{(\delta)}_k$ does not depend on $k$, i.e. that we feed some constant error into the data assimilation scheme. Then, the error terms $e_k$ with initial error $e_0$ described by the update formula (40) evolve according to

$$e_k = \Lambda^k e_0 + \left( \sum_{\ell=0}^{k-1} \Lambda^\ell \right) f^{(\delta)}$$  \hspace{1cm} (41)

with $f^{(\delta)} := R^{(\delta)}_\alpha$. If $(I - \Lambda)^{-1}$ exists, it can be written as

$$e_k = \Lambda^k e_0 + (I - \Lambda)^{-1} (I - \Lambda^k) f^{(\delta)}$$  \hspace{1cm} (42)

- The update formula has been already known in the 1970s.
- It was derived for a finite dimensional system with some well-posed observation operator in a stochastic framework for the Kalman filter.
Outline

1. Motivation and Introduction.

2. Cycled Inverse Scattering Problems
   - Cycled Tikhonov and 3dVar
   - Spectral Representation

3. Convergence Analysis for Cycled Inversion 1: Range Arguments
   - Setup: Constant System
   - Convergence for Data $f \in R(H)$ / Divergence for $f \notin R(H)$

4. Convergence Analysis 2: Regularization of Cycled Inversion
   - Setup: High-Frequency Damping Systems
     - Analysis Error Bounds

5. Reading ↔ Rennes
Space Decomposition

Let the orthonormal system \( \{\psi_\ell : \ell \in \mathbb{N}\} \) in \( X \) be given by the singular system of the observation operator \( H : X \to Y \). In this case we define an orthogonal decomposition of the space \( X \) by

\[
X_1^{(n)} := \text{span}\{\psi_1, \ldots, \psi_n\}, \quad X_2^{(n)} := \text{span}\{\psi_{n+1}, \psi_{n+2}, \ldots\}. \tag{43}
\]

Using the orthogonal projection operators \( P_1 \) of \( X \) onto \( X_1 \) and \( P_2 \) of \( X \) onto \( X_2 \), we decompose \( M \) into

\[
M_1 := P_1 M, \quad M_2 := P_2 M. \tag{44}
\]

Using \( N = (I - R_{\alpha}H) \) and \( \Lambda = NM \) we obtain

\[
\Lambda = N|_{X_1} M_1 + N|_{X_2} M_2. \tag{45}
\]
The operator $N$ maps $X_j$, $j = 1, 2$, into itself. This leads to the norm estimate

$$
\| \Lambda \varphi \|^2 = \| (N|_{X_1} M_1 + N|_{X_2} M_2) \varphi \|^2
\]

$$

$$
= \| N|_{X_1} M_1 \varphi \|^2 + \| N|_{X_2} M_2 \varphi \|^2,
$$

leading to

$$
\| \Lambda \|^2 \leq \| N|_{X_1} \|^2 \| M_1 \|^2 + \| N|_{X_2} \|^2 \| M_2 \|^2.
$$

We derive estimates for all the above terms.
Norm Estimates 1

\[ ||\Lambda||^2 \leq ||N|_{X_1}||^2 ||M_1||^2 + ||N|_{X_2}||^2 ||M_2||^2. \]  \hspace{1cm} (48)

**Lemma**

The norm of the operator \( N|_{X_2} \) is given by

\[ ||N|_{X_2}|| = 1 \]  \hspace{1cm} (49)

for all \( n \in \mathbb{N} \) and \( \alpha > 0 \).

**Lemma**

Assume that \( M \) is self-adjoint with bounded Frobenius norm. Then, given \( \rho > 0 \) there is \( n \in \mathbb{N} \) such that for \( M_2 = M_2^{(n)} \) we have

\[ ||M_2|| < \rho. \]  \hspace{1cm} (50)
Norm Estimates 2

\[ \| \Lambda \|^2 \leq \| N_{X_1} \|^2 \| M_1 \|^2 + \| N_{X_2} \|^2 \| M_2 \|^2. \]  \hspace{1cm} (51)

We need the freedom

\[ \| M_1 \| = c. \]  \hspace{1cm} (52)

**Lemma**

On \( X_1 \) for \( N = I - R_\alpha H \) we have the norm estimate

\[ \| N_{X_1} \| = \sup_{\ell=1,..,n} \left| \frac{\alpha}{\alpha + \mu_n^2} \right| \]  \hspace{1cm} (53)

where \( \mu_n \) are the singular values of the operator \( H \) ordered according to their size and multiplicity. In particular, given \( \epsilon > 0 \) and \( n \in \mathbb{N} \) we can choose \( \alpha > 0 \) sufficiently small such that

\[ \| N_{X_1} \| < \epsilon. \]  \hspace{1cm} (54)
Stabilization of data assimilation

Recall

\[ e_k = \Lambda^k e_0 + (I - \Lambda)^{-1} (I - \Lambda^k) e(\delta) \]  \hspace{1cm} (55)

**Theorem**

Assume that the system M is self-adjoint and its Frobenius norm is bounded and let \( \alpha \) denote the regularization parameter for a cycled data assimilation scheme. Then, for \( \alpha > 0 \) sufficiently small, we have \( \|\Lambda\| < 1 \). Assume that the observation error \( f(\delta) \) is bounded in norm by \( \delta > 0 \). In this case, the analysis error is bounded over time with

\[
\limsup_{k \to \infty} \| e_k \| \leq \frac{\| R_\alpha \| \delta}{1 - \|\Lambda\|} .
\]  \hspace{1cm} (56)
Numerical Example

\( \alpha = 0.27 \)

\( \alpha = 0.26 \)
Summary on Cycled Inversion Instability I

- 3dVar (and 4dVar for linear systems) can be analysed as a **cycled Tikhonov regularization**.
- We have shown that even for very stable dynamics where $M = \text{const}$ cycled Tikhonov with ill-posed observation operators **diverges** if we have data $f \not\in R(H)$.
- The divergence carries over to 3dVar and 4dVar for simple systems.
Summary on Cycled Inversion Instability I

- 3dVar (and 4dVar for linear systems) can be analysed as a cycled Tikhonov regularization.
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Summary on Cycled Inversion Instability II

- For systems damping high frequencies we have shown that stability depends on the choice of the regularization parameter $\alpha$.
- For self-adjoint systems $M$ with bounded Frobenius norm we can always achieve a stable assimilation by choosing $\alpha$ appropriately, even if $\|M_2\| > 1$. 
Rennes Research Group

1. Martin Costabel, Eric Darrigrand, Monique Dauge - Visiting Position 2007-2009 (3 years)
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2. Fréchet differentiability by boundary integral equations, Inversion Studio Software
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5. Being part of a well-developed maturing community, focus on mathematics ...