Contraction estimates for the double layer boundary integral operator and applications

Olaf Steinbach

Institut für Numerische Mathematik, TU Graz, Austria

IABEM Symposium Graz, 2006

M. Costabel: Some historical remarks on the positivity of boundary integral operators. Lecture Notes in Applied and Computational Mechanics, vol. 29, Springer, Heidelberg, pp. 1–27, 2007.

Interior Dirichlet boundary value problem

$$-\Delta u_i(x) = 0 \quad \text{for } x \in \Omega \subset \mathbb{R}^n, \quad u_i(x) = g(x) \quad \text{for } x \in \Gamma = \partial\Omega$$

Double layer potential representation

$$u_i(x) = -\int_\Gamma \frac{\partial}{\partial n_y} U^*(x,y)v(y)ds_y \quad \text{for } x \in \Omega$$

Second kind boundary integral equation to find $v \in H^{1/2}(\Gamma)$ such that

$$\frac{1}{2}v(x) - \int_\Gamma \frac{\partial}{\partial n_y} U^*(x,y)v(y)ds_y = g(x) \quad \text{for } x \in \Gamma$$

Interior Dirichlet boundary value problem

$$-\Delta u_i(x) = 0 \quad \text{for } x \in \Omega \subset \mathbb{R}^n, \quad u_i(x) = g(x) \quad \text{for } x \in \Gamma = \partial\Omega$$

Double layer potential representation

$$u_i(x) = -\int_\Gamma \frac{\partial}{\partial n_y} U^*(x,y) v(y) ds_y \quad \text{for } x \in \Omega$$

Second kind boundary integral equation to find $v \in H^{1/2}(\Gamma)$ such that

$$\frac{1}{2} v(x) - \int_\Gamma \frac{\partial}{\partial n_y} U^*(x,y) v(y) ds_y = g(x) \quad \text{for } x \in \Gamma$$

Neumann series [Neumann 1877]

$$v = \sum_{\ell=0}^\infty (\frac{1}{2} I + K)^\ell g \quad \text{in } H^{1/2}(\Gamma)$$

Interior Dirichlet boundary value problem

$$-\Delta u_i(x) = 0 \quad \text{for } x \in \Omega \subset \mathbb{R}^n, \quad u_i(x) = g(x) \quad \text{for } x \in \Gamma = \partial\Omega$$

Double layer potential representation

$$u_i(x) = -\int_\Gamma \frac{\partial}{\partial n_y} U^*(x,y) v(y) ds_y \quad \text{for } x \in \Omega$$

Second kind boundary integral equation to find $v \in H^{1/2}(\Gamma)$ such that

$$\frac{1}{2} v(x) - \int_\Gamma \frac{\partial}{\partial n_y} U^*(x,y) v(y) ds_y = g(x) \quad \text{for } x \in \Gamma$$

Neumann series [Neumann 1877]

$$v = \sum_{\ell=0}^{\infty} (\frac{1}{2} I + K)^\ell g \quad \text{in } H^{1/2}(\Gamma)$$

Contraction estimate [OS, Wendland 2001]

$$\|(\frac{1}{2}I+K)v\|_{V^{-1}} \leq c_K \|v\|_{V^{-1}}, \; c_K = \frac{1}{2}+\sqrt{\frac{1}{4} - c_0} < 1, \; c_0 = \inf_{v \in H^{1/2}_{\mathcal{R}}(\Gamma)} \frac{\langle Dv, v\rangle_\Gamma}{\langle V^{-1} v, v\rangle_\Gamma}$$

Proof [OS, Wendland 2001]: For $v \in H^{1/2}_{\mathcal{R}}(\Gamma)$ we have

$$\|(\frac{1}{2}I + K)v\|^2_{V^{-1}} \quad = \quad \langle(\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)v, v\rangle_\Gamma$$

Proof [OS, Wendland 2001]: For $v \in H^{1/2}_{\mathcal{R}}(\Gamma)$ we have

$$\begin{aligned} \|(\frac{1}{2}I + K)v\|^2_{V^{-1}} &= \langle(\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)v, v\rangle_\Gamma \\ &= \langle Sv, v\rangle_\Gamma - \langle Dv, v\rangle_\Gamma \end{aligned}$$

Proof [OS, Wendland 2001]: For $v \in H^{1/2}_{\mathcal{R}}(\Gamma)$ we have

$$\begin{aligned} \|(\frac{1}{2}I + K)v\|^2_{V^{-1}} &= \langle(\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)v, v\rangle_\Gamma \\ &= \langle Sv, v\rangle_\Gamma - \langle Dv, v\rangle_\Gamma \\ &\leq \langle V^{-1}(\frac{1}{2}I + K)v, v\rangle_\Gamma - c_0\langle V^{-1}v, v\rangle \end{aligned}$$

Proof [OS, Wendland 2001]: For $v \in H^{1/2}_{\mathcal{R}}(\Gamma)$ we have

$$\begin{aligned}
\|(\frac{1}{2}I+K)v\|^2_{V^{-1}} &= \langle(\frac{1}{2}I+K')V^{-1}(\frac{1}{2}I+K)v,v\rangle_\Gamma \\
&= \langle Sv,v\rangle_\Gamma - \langle Dv,v\rangle_\Gamma \\
&\leq \langle V^{-1}(\frac{1}{2}I+K)v,v\rangle_\Gamma - c_0\langle V^{-1}v,v\rangle \\
&\leq \|(\frac{1}{2}I+K)v\|_{V^{-1}}\|v\|_{V^{-1}} - c_0\|v\|^2_{V^{-1}}
\end{aligned}$$

Proof [OS, Wendland 2001]: For $v \in H^{1/2}_{\mathcal{R}}(\Gamma)$ we have

$$\begin{aligned} \|(\frac{1}{2}I + K)v\|^2_{V^{-1}} &= \langle (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)v, v\rangle_\Gamma \\ &= \langle Sv, v\rangle_\Gamma - \langle Dv, v\rangle_\Gamma \\ &\leq \langle V^{-1}(\frac{1}{2}I + K)v, v\rangle_\Gamma - c_0\langle V^{-1}v, v\rangle \\ &\leq \|(\frac{1}{2}I + K)v\|_{V^{-1}}\|v\|_{V^{-1}} - c_0\|v\|^2_{V^{-1}} \end{aligned}$$

i.e.

$$\|(\frac{1}{2}I + K)v\|^2_{V^{-1}} - \|(\frac{1}{2}I + K)v\|_{V^{-1}}\|v\|_{V^{-1}} + c_0\|v\|^2_{V^{-1}} \leq 0$$

Proof [OS, Wendland 2001]: For $v \in H^{1/2}_{\mathcal{R}}(\Gamma)$ we have

$$\begin{aligned}
\|(\frac{1}{2}I + K)v\|^2_{V^{-1}} &= \langle(\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)v, v\rangle_\Gamma \\
&= \langle Sv, v\rangle_\Gamma - \langle Dv, v\rangle_\Gamma \\
&\leq \langle V^{-1}(\frac{1}{2}I + K)v, v\rangle_\Gamma - c_0\langle V^{-1}v, v\rangle \\
&\leq \|(\frac{1}{2}I + K)v\|_{V^{-1}}\|v\|_{V^{-1}} - c_0\|v\|^2_{V^{-1}}
\end{aligned}$$

i.e.

$$\|(\frac{1}{2}I + K)v\|^2_{V^{-1}} - \|(\frac{1}{2}I + K)v\|_{V^{-1}}\|v\|_{V^{-1}} + c_0\|v\|^2_{V^{-1}} \leq 0$$

Hence,

$$\|(\frac{1}{2}I + K)v\|_{V^{-1}} \leq \left(\frac{1}{2} + \sqrt{\frac{1}{4} - c_0}\right)\|v\|_{V^{-1}}$$

Questions

- characterisation/computation of c_K (circle: $c_K = \frac{1}{2}$)

Questions

- characterisation/computation of c_K (circle: $c_K = \frac{1}{2}$)
- equivalence with constants/eigenvalues of partial differential equations

Questions

- characterisation/computation of c_K (circle: $c_K = \frac{1}{2}$)
- equivalence with constants/eigenvalues of partial differential equations
- computation of eigenfunctions

Questions

- characterisation/computation of c_K (circle: $c_K = \frac{1}{2}$)
- equivalence with constants/eigenvalues of partial differential equations
- computation of eigenfunctions

Observation for harmonic functions v:

$$\langle Sv, v\rangle_\Gamma = \int_\Gamma \frac{\partial}{\partial n_x} v \, v \, ds_x = \int_\Omega |\nabla v|^2 dx$$

Questions

- characterisation/computation of c_K (circle: $c_K = \frac{1}{2}$)
- equivalence with constants/eigenvalues of partial differential equations
- computation of eigenfunctions

Observation for harmonic functions v:

$$\langle Sv, v\rangle_\Gamma = \int_\Gamma \frac{\partial}{\partial n_x} v\, v\, ds_x = \int_\Omega |\nabla v|^2 dx$$

$Vw = v$ on Γ:

$$\langle V^{-1} v, v\rangle_\Gamma = \langle Vw, w\rangle_\Gamma = \int_\Omega |\nabla v|^2 dx + \int_{\Omega^c} |\nabla v|^2 dx$$

Interior Dirichlet boundary value problem

$$-\Delta u_i(x) = 0 \quad \text{for } x \in \Omega \subset \mathbb{R}^n, \quad u_i(x) = g(x) \quad \text{for } x \in \Gamma$$

Exterior Dirichlet boundary value problem

$$-\Delta u_e(x) = 0 \text{ for } x \in \Omega^c,\; u_e(x) = g(x) \text{ for } x \in \Gamma,\; u_e(x) = \mathcal{O}(\frac{1}{|x|}) \text{ as } |x| \to \infty$$

Dirichlet integrals

$$J_i = \int_\Omega |\nabla u_i(x)|^2 dx, \quad J_e = \int_{\Omega^c} |\nabla u_e(x)|^2 dx$$

Interior Dirichlet boundary value problem

$$-\Delta u_i(x) = 0 \quad \text{for } x \in \Omega \subset \mathbb{R}^n, \quad u_i(x) = g(x) \quad \text{for } x \in \Gamma$$

Exterior Dirichlet boundary value problem

$$-\Delta u_e(x) = 0 \text{ for } x \in \Omega^c, \; u_e(x) = g(x) \text{ for } x \in \Gamma, \; u_e(x) = \mathcal{O}(\frac{1}{|x|}) \text{ as } |x| \to \infty$$

Dirichlet integrals

$$J_i = \int_\Omega |\nabla u_i(x)|^2 dx, \quad J_e = \int_{\Omega^c} |\nabla u_e(x)|^2 dx$$

Theorem [Poincaré 1896; Costabel 2007]

$$J_i \leq \mu \, J_e, \qquad \mu = \mu(\Omega) > 0$$

Single layer potential representation

$$u(x) = \int_\Gamma U^*(x,y)w(y)ds_y \quad \text{for } x \in \mathbb{R}^n$$

Green's formulae and jump relations

$$J_i = \int_\Omega |\nabla u_i(x)|^2 dx = \int_\Gamma \frac{\partial}{\partial n_x} u_i(x) u_i(x) ds_x = \langle (\frac{1}{2} I + K')w, Vw\rangle_\Gamma = \langle Bw, w\rangle_\Gamma$$

$$J_e = \int_{\Omega^c} |\nabla u_e(x)|^2 dx = -\int_{\Gamma^c} \frac{\partial}{\partial n_x} u_e(x) u_e(x) ds_x = \langle (\frac{1}{2} I - K')w, Vw\rangle_\Gamma = \langle Aw, w\rangle_\Gamma$$

Single layer potential representation

$$u(x) = \int_\Gamma U^*(x,y)w(y)ds_y \quad \text{for } x \in \mathbb{R}^n$$

Green's formulae and jump relations

$$J_i = \int_\Omega |\nabla u_i(x)|^2 dx = \int_\Gamma \frac{\partial}{\partial n_x} u_i(x) u_i(x) ds_x = \langle (\frac{1}{2} I + K')w, Vw\rangle_\Gamma = \langle Bw, w\rangle_\Gamma$$

$$J_e = \int_{\Omega^c} |\nabla u_e(x)|^2 dx = -\int_{\Gamma^c} \frac{\partial}{\partial n_x} u_e(x) u_e(x) ds_x = \langle (\frac{1}{2} I - K')w, Vw\rangle_\Gamma = \langle Aw, w\rangle_\Gamma$$

Corollary [Costabel 2007]

$$J_i \le \mu J_e, \quad \langle Bw, w\rangle_\Gamma \le \mu \langle Aw, w\rangle_\Gamma, \quad \langle Vw, w\rangle_\Gamma \le (1+\mu)\langle Aw, w\rangle_\Gamma$$

Single layer potential representation

$$u(x) = \int_\Gamma U^*(x,y)w(y)ds_y \quad \text{for } x \in \mathbb{R}^n$$

Green's formulae and jump relations

$$J_i = \int_\Omega |\nabla u_i(x)|^2 dx = \int_\Gamma \frac{\partial}{\partial n_x} u_i(x) u_i(x) ds_x = \langle (\frac{1}{2}I + K')w, Vw\rangle_\Gamma = \langle Bw, w\rangle_\Gamma$$

$$J_e = \int_{\Omega^c} |\nabla u_e(x)|^2 dx = -\int_{\Gamma^c} \frac{\partial}{\partial n_x} u_e(x) u_e(x) ds_x = \langle (\frac{1}{2}I - K')w, Vw\rangle_\Gamma = \langle Aw, w\rangle_\Gamma$$

Corollary [Costabel 2007]

$$J_i \le \mu J_e, \quad \langle Bw, w\rangle_\Gamma \le \mu \langle Aw, w\rangle_\Gamma, \quad \langle Vw, w\rangle_\Gamma \le (1+\mu)\langle Aw, w\rangle_\Gamma$$

$$\langle Bw, w\rangle_\Gamma = \langle Vw, w\rangle_\Gamma - \langle Aw, w\rangle_\Gamma \le \frac{\mu}{1+\mu}\langle Vw, w\rangle_\Gamma$$

Single layer potential representation

$$u(x) = \int_\Gamma U^*(x,y)w(y)ds_y \quad \text{for } x \in \mathbb{R}^n$$

Green's formulae and jump relations

$$J_i = \int_\Omega |\nabla u_i(x)|^2 dx = \int_\Gamma \frac{\partial}{\partial n_x} u_i(x) u_i(x) ds_x = \langle (\frac{1}{2} I + K')w, Vw\rangle_\Gamma = \langle Bw, w\rangle_\Gamma$$

$$J_e = \int_{\Omega^c} |\nabla u_e(x)|^2 dx = -\int_{\Gamma^c} \frac{\partial}{\partial n_x} u_e(x) u_e(x) ds_x = \langle (\frac{1}{2} I - K')w, Vw\rangle_\Gamma = \langle Aw, w\rangle_\Gamma$$

Corollary [Costabel 2007]

$$J_i \leq \mu J_e, \quad \langle Bw, w\rangle_\Gamma \leq \mu \langle Aw, w\rangle_\Gamma, \quad \langle Vw, w\rangle_\Gamma \leq (1+\mu)\langle Aw, w\rangle_\Gamma$$

$$\langle Bw, w\rangle_\Gamma = \langle Vw, w\rangle_\Gamma - \langle Aw, w\rangle_\Gamma \leq \frac{\mu}{1+\mu}\langle Vw, w\rangle_\Gamma$$

Contraction $\frac{1}{2} I + K' : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ with $c_K = \dfrac{\mu}{1+\mu}$

Steklov eigenvalue problem

$$-\Delta u(x) = 0 \quad \text{for } x \in \Omega, \quad \frac{\partial}{\partial n_x} u(x) = \lambda u(x) \quad \text{for } x \in \Gamma$$

Steklov–Poincaré operator eigenvalue problem

$$(Su)(x) = \lambda u(x) \quad \text{for } x \in \Gamma, \quad S : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

Steklov eigenvalue problem

$$-\Delta u(x) = 0 \quad \text{for } x \in \Omega, \qquad \frac{\partial}{\partial n_x} u(x) = \lambda u(x) \quad \text{for } x \in \Gamma$$

Steklov–Poincaré operator eigenvalue problem

$$(Su)(x) = \lambda u(x) \quad \text{for } x \in \Gamma, \qquad S : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

Generalized Steklov eigenvalue problem

$$-\Delta u(x) = 0 \quad \text{for } x \in \Omega, \qquad \frac{\partial}{\partial n_x} u(x) = \lambda (V^{-1} u)(x) \quad \text{for } x \in \Gamma$$

Steklov eigenvalue problem

$$-\Delta u(x) = 0 \quad \text{for } x \in \Omega, \qquad \frac{\partial}{\partial n_x} u(x) = \lambda u(x) \quad \text{for } x \in \Gamma$$

Steklov–Poincaré operator eigenvalue problem

$$(Su)(x) = \lambda u(x) \quad \text{for } x \in \Gamma, \qquad S : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

Generalized Steklov eigenvalue problem

$$-\Delta u(x) = 0 \quad \text{for } x \in \Omega, \qquad \frac{\partial}{\partial n_x} u(x) = \lambda (V^{-1}u)(x) \quad \text{for } x \in \Gamma$$

Eigenvalue problem in variational form to find $(u, \lambda) \in H^1(\Omega) \times \mathbb{R}_+$:

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx = \lambda \langle V^{-1} u_{|\Gamma}, v_{|\Gamma} \rangle_\Gamma \quad \text{for all } v \in H^1(\Omega)$$

Steklov eigenvalue problem

$$-\Delta u(x) = 0 \quad \text{for } x \in \Omega, \qquad \frac{\partial}{\partial n_x} u(x) = \lambda u(x) \quad \text{for } x \in \Gamma$$

Steklov–Poincaré operator eigenvalue problem

$$(Su)(x) = \lambda u(x) \quad \text{for } x \in \Gamma, \qquad S : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$$

Generalized Steklov eigenvalue problem

$$-\Delta u(x) = 0 \quad \text{for } x \in \Omega, \qquad \frac{\partial}{\partial n_x} u(x) = \lambda (V^{-1} u)(x) \quad \text{for } x \in \Gamma$$

Eigenvalue problem in variational form to find $(u, \lambda) \in H^1(\Omega) \times \mathbb{R}_+$:

$$\int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx = \lambda \langle V^{-1} u_{|\Gamma}, v_{|\Gamma} \rangle_\Gamma \quad \text{for all } v \in H^1(\Omega)$$

Lemma

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots, \qquad \langle \nabla u_k, \nabla u_\ell \rangle_{L_2(\Omega)} = \langle u_k, u_\ell \rangle_{V^{-1}} = 0 \quad \text{for } k \neq \ell$$

$(u, \lambda) \in H^1(\Omega) \times \mathbb{R}_+$, $Vw = u$ on Γ:

$$\int_\Omega |\nabla u(x)|^2 dx = \lambda \langle V^{-1} u, u \rangle_\Gamma = \lambda \langle Vw, w \rangle_\Gamma$$

$(u,\lambda) \in H^1(\Omega) \times \mathbb{R}_+$, $Vw = u$ on Γ:

$$\int_\Omega |\nabla u(x)|^2 dx = \lambda \langle V^{-1}u, u\rangle_\Gamma = \lambda \langle Vw, w\rangle_\Gamma$$

i.e.

$$J_i = \lambda(J_i + J_e),$$

$(u, \lambda) \in H^1(\Omega) \times \mathbb{R}_+$, $Vw = u$ on Γ:

$$\int_\Omega |\nabla u(x)|^2 dx = \lambda \langle V^{-1} u, u \rangle_\Gamma = \lambda \langle Vw, w \rangle_\Gamma$$

i.e.

$$J_i = \lambda(J_i + J_e), \quad \lambda = \frac{J_i}{J_1 + J_e} < 1, \quad J_i = \frac{\lambda}{1 - \lambda} J_e$$

$(u, \lambda) \in H^1(\Omega) \times \mathbb{R}_+$, $Vw = u$ on Γ:

$$\int_\Omega |\nabla u(x)|^2 dx = \lambda \langle V^{-1} u, u \rangle_\Gamma = \lambda \langle Vw, w \rangle_\Gamma$$

i.e.

$$J_i = \lambda(J_i + J_e), \quad \lambda = \frac{J_i}{J_1 + J_e} < 1, \quad J_i = \frac{\lambda}{1-\lambda} J_e$$

Corollary

$$\mu = \sup_\lambda \frac{\lambda}{1-\lambda} = \frac{\sup \lambda}{1 - \sup \lambda}$$

$(u,\lambda) \in H^1(\Omega) \times \mathbb{R}_+$, $Vw = u$ on Γ:

$$\int_\Omega |\nabla u(x)|^2 dx = \lambda \langle V^{-1} u, u\rangle_\Gamma = \lambda \langle Vw, w\rangle_\Gamma$$

i.e.

$$J_i = \lambda(J_i + J_e), \quad \lambda = \frac{J_i}{J_1 + J_e} < 1, \quad J_i = \frac{\lambda}{1-\lambda} J_e$$

Corollary

$$\mu = \sup_\lambda \frac{\lambda}{1-\lambda} = \frac{\sup \lambda}{1 - \sup \lambda}$$

$$c_K = \frac{\mu}{1+\mu} = \sup \lambda$$

generalized Steklov–Poincaré operator eigenvalue problem

$$Su = \lambda V^{-1} u$$

generalized Steklov–Poincaré operator eigenvalue problem

$$Su = \lambda V^{-1} u$$

Double layer boundary integral operator eigenvalue problem

$$(\frac{1}{2} I + K)u = \lambda u \quad \text{in } H^{1/2}(\Gamma)$$

generalized Steklov–Poincaré operator eigenvalue problem

$$Su = \lambda V^{-1} u$$

Double layer boundary integral operator eigenvalue problem

$$(\frac{1}{2} I + K) u = \lambda u \quad \text{in } H^{1/2}(\Gamma)$$

Symmetric representation of the Steklov–Poincaré operator

$$[D + (\frac{1}{2} I + K') V^{-1} (\frac{1}{2} I + K) u = \lambda V^{-1} u$$

generalized Steklov–Poincaré operator eigenvalue problem

$$Su = \lambda V^{-1} u$$

Double layer boundary integral operator eigenvalue problem

$$(\frac{1}{2}I + K)u = \lambda u \quad \text{in } H^{1/2}(\Gamma)$$

Symmetric representation of the Steklov–Poincaré operator

$$[D + (\frac{1}{2}I + K')V^{-1}(\frac{1}{2}I + K)u = \lambda V^{-1} u$$

Boundary element discretization

$$[D_h + (\frac{1}{2}M_h^\top + K_h^\top)V_h^{-1}(\frac{1}{2}M_h + K_h)]\underline{u} = \lambda M_h^\top V_h^{-1} M_h \underline{u}$$

generalized Steklov–Poincaré operator eigenvalue problem

$$Su = \lambda V^{-1} u$$

Double layer boundary integral operator eigenvalue problem

$$(\frac{1}{2} I + K) u = \lambda u \quad \text{in } H^{1/2}(\Gamma)$$

Symmetric representation of the Steklov–Poincaré operator

$$[D + (\frac{1}{2} I + K') V^{-1} (\frac{1}{2} I + K) u = \lambda V^{-1} u$$

Boundary element discretization

$$[D_h + (\frac{1}{2} M_h^\top + K_h^\top) V_h^{-1} (\frac{1}{2} M_h + K_h)] \underline{u} = \lambda M_h^\top V_h^{-1} M_h \underline{u}$$

Finite element discretization

$$[A_{CC} - A_{IC} A_{II}^{-1} A_{CI}] \underline{u}_C = \lambda M_h^\top V_h^{-1} M_h \underline{u}_C$$

Remarks

- stability and error analysis
- solution methods for discrete eigenvalue problems

Remarks

- stability and error analysis
- solution methods for discrete eigenvalue problems

Applications

- solution of second kind boundary integral equations
- preconditioning by operators of opposite order
- one–equation coupling of finite and boundary elements
- ...