DPG BEM with optimal test functions

Norbert Heuer

P. Universidad Católica de Chile, Santiago, Chile

in collaboration with

Felipe Pinochet, PUC

Outline

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DPG method with optimal test functions

Model Problem and Ultra-weak Formulation

Test Norms

Test Functions

Numerical Experiments

Petrov-Galerkin method with optimal test functions

$$\text{solve} \qquad u \in U : \quad Bu = L \quad \text{in } V'$$

$$u \in U : \quad b(u, v) = L(v) \quad \forall v \in V$$

Petrov-Galerkin approximation:

$$u_{hp} \in U_{hp} \subset U : \qquad b(u_{hp}, v) = L(v) \quad \forall v \in T(U_{hp})$$

with trial-to-test operator T:

$$T : \; U \to V : \quad (Tu, v)_V = b(u, v) \quad \forall v \in V$$

$$Tu = R_V^{-1} Bu \quad \text{with Riesz operator} \quad R_V : \; V \to V'$$

Petrov-Galerkin method with optimal test functions

Important properties [Demkowicz, Gopalakrishnan *et al.*]

- Discrete system is SPD:

$$u_{hp} \in U_{hp} : \quad (Tu_{hp}, Tw)_V = b(u_{hp}, Tw) = L(Tw) \quad \forall w \in U_{hp}.$$

$$(R_V^{-1} Bu_{hp}, R_V^{-1} Bw)_V = (R_V^{-1} L, Tw)_V = (R_V^{-1} Bu, R_V^{-1} Bw)_V$$

$$u_{hp} \in U_{hp} : \quad \|B(u - u_{hp})\|_{V'} \to \min \quad \text{least squares}$$

- Method provides best approximation:

$$\|u - u_{hp}\|_E = \inf_{w_{hp} \in U_{hp}} \|u - w_{hp}\|_E,$$

$$\|u\|_E := \sup_{\|v\|_V = 1} b(u, v) \qquad \text{(energy norm)}$$

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Petrov-Galerkin method with optimal test functions

Important properties [Demkowicz, Gopalakrishnan *et al.*]

- The energy norm of the error can be evaluated by solving

$$\psi \in V: \ (\psi, v)_V = b(u - u_{hp}, v) = L(v) - b(u_{hp}, v) \quad \forall v \in V.$$

Then

$$\|u - u_{hp}\|_E = \sup_{\|v\|_V = 1} b(u - u_{hp}, v) = \|\psi\|_V$$

There is no need for a posteriori error estimation.

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This talk:

- Does this technique apply to boundary integral equations?
- Is there any advantage in using DPG-BEM?
- We analyze the simplest case in two dimensions, cf.

  Demkowicz, Gopalakrishnan: *DPG for Poisson*, SINUM '11,
  Demkowicz, Heuer: *DPG for confusion*, SINUM (to appear)

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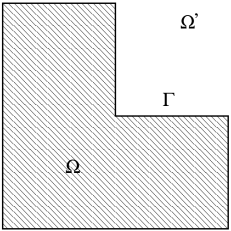
Model Problem



$$\Delta u = 0 \qquad \text{in} \quad \Omega$$

Representation formula:

$$u(y) = \int_\Gamma \frac{\partial u}{\partial n} G(\cdot, y) - u \frac{\partial G}{\partial n}(\cdot, y)\, ds, \quad y \in \Omega$$

Fundamental solution:

$$G(x, y) = -\frac{1}{2\pi} \log |x - y|$$

$\frac{\partial}{\partial n} \cdot |_\Gamma$ and jump relations yield (on Γ):

$$-\frac{\partial}{\partial n} \int_\Gamma u \frac{\partial G}{\partial n}\, ds = \frac{\partial}{\partial n} \int_\Gamma \frac{\partial u}{\partial n} G\, ds + \frac{1}{2}\frac{\partial u}{\partial n} \qquad \boxed{\mathcal{W} u = (\mathcal{K}' + \frac{1}{2} I) \frac{\partial u}{\partial n}}$$

$$\int_\Gamma \frac{\partial u}{\partial n} G : \qquad \mathcal{V} \frac{\partial u}{\partial n} \quad \text{on} \quad \Gamma$$

Ultra-weak Formulation

$$\mathcal{W}\phi = f, \qquad \langle \mathcal{W}\phi, w\rangle = \langle \mathcal{V}\phi', w'\rangle, \qquad \mathcal{W}\phi = -(\mathcal{V}\phi')' = f$$

Consider the system

$$\sigma = \mathcal{V}\phi', \qquad -\sigma' = f$$

First equation:

$$\langle \sigma, \tau\rangle = \langle \mathcal{V}\phi', \tau\rangle = -\langle \phi, (\mathcal{V}\tau)'\rangle \qquad \forall \tau \in L^2(\Gamma)$$

Second equation: for $v \in H^1(\mathcal{T})$

$$-\langle \sigma', v\rangle = \sum_{T\in\mathcal{T}} \langle \sigma, v'\rangle_T + \sum_{j=1}^{N} \sigma(x_j)[v]_j = \langle f, v\rangle$$

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$\sigma \in L^2(\Gamma)$, $\phi \in L^2(\Gamma)$, $\hat\sigma \in \mathbb{R}^N$:

$$\begin{aligned} \langle \sigma, \tau \rangle + \langle \phi, (\mathcal{V}\tau)' \rangle &= 0 && \forall \tau \in L^2(\Gamma) \\ \langle \sigma, v' \rangle_{\mathcal{T}} + \hat\sigma \cdot [v] + \langle \phi, 1 \rangle \langle v, 1 \rangle &= \langle f, v \rangle && \forall v \in H^1(\mathcal{T}) \end{aligned}$$

Theorem

The ultra-weak formulation is uniquely solvable with

$$\|\phi\|_{L^2(\Gamma)} + \|\sigma\|_{L^2(\Gamma)} + N^{-1/2}|\hat\sigma| \lesssim \|f\|_{L^2(\Gamma)}.$$

Equivalence:

$$\mathcal{W}\phi_{\mathbb{R}} = f \quad \Rightarrow \quad \sigma := \mathcal{V}\phi'_{\mathbb{R}},\ \phi := \phi_{\mathbb{R}} - |\Gamma|^{-1}\langle \phi_{\mathbb{R}}, 1 \rangle,$$
$$\hat\sigma := (\mathcal{V}\phi'_{\mathbb{R}}(x_j))_{j=1}^N \quad \textit{solve ultra-weak form}$$

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Ultra-weak Formulation

$(\phi, \sigma, \hat{\sigma}) \in U := L^2(\Gamma) \times L^2(\Gamma) \times \mathbb{R}^N$

$$b(\phi, \sigma, \hat{\sigma}; \tau, v) = \langle f, v \rangle \qquad \forall (\tau, v) \in V := L^2(\Gamma) \times H^1(\mathcal{T}).$$

Proof of theorem.

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Proof of theorem.

$$\|(\tau, v)\|_{V,\mathrm{opt},\alpha} := \sup_{(\phi,\sigma,\hat{\sigma}) \in U \setminus \{0\}} \frac{b(\phi, \sigma, \hat{\sigma}; \tau, v)}{\|(\phi, \sigma, \hat{\sigma})\|_{U,\alpha}}$$

$$\|(\phi, \sigma, \hat{\sigma})\|_{E,\alpha} := \sup_{(\tau, v) \in V \setminus \{0\}} \frac{b(\phi, \sigma, \hat{\sigma}; \tau, v)}{\|(\tau, v)\|_{V,\mathrm{opt},\alpha}} = \|(\phi, \sigma, \hat{\sigma})\|_{U,\alpha}$$

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(i) $$\|(\phi, \sigma, \hat{\sigma})\|_{E,\alpha} = \sup_{(\tau,v)\in V\setminus\{0\}} \frac{b(\phi, \sigma, \hat{\sigma}; \tau, v)}{\|(\tau, v)\|_{V,\mathrm{opt},\alpha}}$$

$$\left(\|(\phi, \sigma, \hat{\sigma})\|_{E,\alpha} = 0 \quad \Rightarrow \quad (\phi, \sigma, \hat{\sigma}) = 0\right)$$

(ii) $$\sup_{(\tau,v)\in V\setminus\{0\}} \frac{b(\phi, \sigma, \hat{\sigma}; \tau, v)}{\|(\tau, v)\|_{V,\mathrm{opt},\alpha}} \gtrsim \|(\phi, \sigma, \hat{\sigma})\|_{E,\alpha} \qquad \forall (\phi, \sigma, \hat{\sigma}) \in U,$$

(iii) $$\sup_{(\phi,\sigma,\hat{\sigma})\in U\setminus\{0\}} \frac{b(\phi, \sigma, \hat{\sigma}; \tau, v)}{\|(\phi, \sigma, \hat{\sigma})\|_{E,\alpha}} > 0 \qquad \forall (\tau, v) \in V \setminus \{0\},$$

(iv) boundedness of b and functional $L = \langle f, \cdot \rangle$

Outline

Test Norms

$$b(\phi,\sigma,\hat{\sigma};\tau,v) = \langle \phi, (\mathcal{V}\tau)' \rangle + \langle \sigma, \tau \rangle + \langle \sigma, v' \rangle_{\mathcal{T}} + \hat{\sigma}\cdot[v] + \langle \phi, 1 \rangle \langle v, 1 \rangle$$

$$\|(\phi,\sigma,\hat{\sigma})\|_{U,\alpha} = \left(\|\phi\|^2_{L^2(\Gamma)} + \|\sigma\|^2_{L^2(\Gamma)} + \alpha^2 |\hat{\sigma}|^2 \right)^{1/2}$$

Using optimal test norm

$$\|(\tau,v)\|_{V,\mathrm{opt},\alpha} := \sup_{(\phi,\sigma,\hat{\sigma})\in U\setminus\{0\}} \frac{b(\phi,\sigma,\hat{\sigma};\tau,v)}{\|(\phi,\sigma,\hat{\sigma})\|_{U,\alpha}}$$

$$\simeq \|(\mathcal{V}\tau)'\|_{L^2(\Gamma)} + \|\tau + v'\|_{L^2(\mathcal{T})} + \alpha^{-1}|[v]| + |\langle v, 1\rangle|$$

yields optimal convergence

$$\|(\phi-\phi_{hp},\sigma-\sigma_{hp},\hat{\sigma}-\hat{\sigma}_{hp})\|^2_{U,\alpha} = \inf_{(\varphi,\rho,\hat{\rho})\in U_{hp}} \|(\phi-\varphi,\sigma-\rho,\hat{\sigma}-\hat{\rho})\|^2_{U,\alpha}$$

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Test Norms

Optimal test norm is impractical: global, coupled variables

$$\|(\tau, v)\|_{V,\mathrm{opt},\alpha} \simeq \|(\mathcal{V}\tau)'\|_{L^2(\Gamma)} + \|\tau + v'\|_{L^2(\mathcal{T})} + \alpha^{-1}|[v]| + |\langle v, 1\rangle|$$

Instead, we use

$$\|(\tau, v)\|_V^2 := \|\tau\|_{L^2(\Gamma)}^2 + \|v'\|_{L^2(\mathcal{T})}^2 + |v|_h^2, \qquad |v|_h^2 := \sum_{T\in\mathcal{T}} |T| \, (v|_T(x_T))^2$$

Norm equivalence

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$$\|(\tau, v)\|_{V} \simeq \|\tau\|_{L^2(\Gamma)} + \|v'\|_{L^2(\mathcal{T})} + |v|_h$$

Lemma ($\|\cdot\|_{V,\mathrm{opt},\alpha} \lesssim \|\cdot\|_V$)

$$\|(\mathcal{V}\tau)'\|_{L^2(\Gamma)} + \|\tau + v'\|_{L^2(\mathcal{T})} + h_{\min}^{1/2}|[v]| + |\langle v, 1\rangle| \lesssim \|\tau\|_{L^2(\Gamma)} + \|v'\|_{L^2(\mathcal{T})} + |v|_h$$

Proof.
Triangle inequality, continuity of $\mathcal{V}$, Sobolev embedding theorem, Poincaré-Friedrichs inequality. □

Norm Equivalence

$$\|(\tau, v)\|_{V,\mathrm{opt},\alpha} \simeq \|(\mathcal{V}\tau)'\|_{L^2(\Gamma)} + \|\tau + v'\|_{L^2(\mathcal{T})} + \alpha^{-1}|[v]| + |\langle v, 1\rangle|$$

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Lemma ($\|\cdot\|_{V,\mathrm{opt},\alpha} \lesssim \|\cdot\|_V$)

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Stability analysis of adjoint problem

$$\tau + v' = g_1 \quad \text{in } L^2(\mathcal{T}), \qquad (\mathcal{V}\tau)' = g_2 \quad \text{in } L^2(\Gamma)$$

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Adjoint Problem

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$g_1 := \tau + v', \; g_2 := (\mathcal{V}\tau)'$

$\tau_0 \in L^2(\Gamma), \quad v_0 \in H^1(\mathcal{T}):$

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Lemma

For $g_1, g_2 \in L^2(\Gamma)$ the adjoint problem has a unique solution $(\tau, v) \in L^2(\Gamma) \times H^1(\Gamma)/\mathbb{R}$ with

$$\|\tau\|_{L^2(\Gamma)} + \|v'\|_{L^2(\Gamma)} \lesssim \|g_1\|_{L^2(\Gamma)} + \|g_2\|_{L^2(\Gamma)}.$$

Proof.

Regularity theory for

$$\mathcal{W}v = -(\mathcal{V}v')' = g_2 - (\mathcal{V}g_1)' \in L^2(\Gamma).$$

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$$\tau + v' = 0 \quad \text{in } L^2(\mathcal{T}), \qquad (\mathcal{V}\tau)' = 0 \quad \text{in } L^2(\Gamma)$$

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Any solution $(\tau, v) \in L^2(\Gamma) \times H^1(\mathcal{T})$ of the homogeneous adjoint problem satisfies

$$\|\tau\|_{L^2(\Gamma)} + \|v'\|_{L^2(\mathcal{T})} \lesssim \sqrt{N}|[v]|.$$

Proof.

$$a := \mathcal{V}\tau \in \mathbb{R}, \qquad \langle \tau, 1\rangle = -\langle v', 1\rangle_{\mathcal{T}} = [v]\cdot \mathbf{1}$$

$$\tau \in H^{-1/2}(\Gamma),\ a \in \mathbb{R}: \quad \mathcal{V}\tau - a = 0, \quad \langle \tau, 1\rangle = [v]\cdot \mathbf{1}.$$

Use ellipticity regularity. □

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Test Norms: Conclusion

energy norm $\qquad \|(\phi,\sigma,\hat{\sigma})\|_E := \sup_{(\tau,v)\in V\setminus\{0\}} \frac{b(\phi,\sigma,\hat{\sigma};\tau,v)}{\|\tau\|_{L^2(\Gamma)} + \|v'\|_{L^2(\mathcal{T})} + |v|_h}$

Norm estimates:

$$\|(\phi,\sigma,\hat{\sigma})\|_E \lesssim \|\phi\|_{L^2(\Gamma)} + \|\sigma\|_{L^2(\Gamma)} + h_{\min}^{-1/2}|\hat{\sigma}|$$

$$\|\phi\|_{L^2(\Gamma)} + \|\sigma\|_{L^2(\Gamma)} + N^{-1/2}|\hat{\sigma}| \lesssim \|(\phi,\sigma,\hat{\sigma})\|_E$$

Main Result

Theorem

Calculating test functions with inner product

$$\langle (\tau, v), (\delta_\tau, \delta_v) \rangle_V := \langle \tau, \delta_\tau \rangle + \langle v', \delta_v' \rangle_{\mathcal{T}} + \sum_{T \in \mathcal{T}} |T| \, v(x_T) \, \delta_v(x_T)$$

the DPG boundary element method converges quasi-optimally:

$$\|\phi - \phi_{hp}\|_{L^2(\Gamma)} + \|\sigma - \sigma_{hp}\|_{L^2(\Gamma)} + N^{-1/2} |\hat{\sigma} - \hat{\sigma}_{hp}| \lesssim \inf_{\varphi \in U^0_{hp}, \rho \in U^0_{hp}} \Big(\|\phi - \varphi\|_{L^2(\Gamma)} + \|\sigma - \rho\|_{L^2(\Gamma)} \Big).$$

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Outline

DPG method with optimal test functions

Model Problem and Ultra-weak Formulation

Test Norms

Test Functions

Numerical Experiments

Optimal Test Functions

Recall trial-to-test operator

$T(\phi, \sigma, \hat{\sigma}) = (\tau, v) \in V = L^2(\Gamma) \times H^1(\mathcal{T})$:

$$\langle (\tau, v), (\delta_\tau, \delta_v) \rangle_V = b((\phi, \sigma, \hat{\sigma}), (\delta_\tau, \delta_v)) \qquad \forall (\delta_\tau, \delta_v) \in V$$

Three types of basis functions in U_{hp}:

- Type 1: $(\phi, \sigma, \hat{\sigma}) = (0, 0, e_j)$, scalar 1 at node x_j
- Type 2: $(\phi, \sigma, \hat{\sigma}) = (0, \sigma_j, 0)$, Legendre polynomial on element T_j
- Type 3: $(\phi, \sigma, \hat{\sigma}) = (\phi_j, 0, 0)$, Legendre polynomial on element T_j

Optimal Test Functions Type 1

$$(\phi, \sigma, \hat{\sigma}) = (0, 0, e_j)$$

$$\begin{aligned} b(\phi, \sigma, \hat{\sigma}; \tau, v) &= \langle \phi, (\mathcal{V}\tau)' \rangle + \langle \phi, 1 \rangle \langle v, 1 \rangle + \langle \sigma, \tau + v' \rangle_{\mathcal{T}} + \hat{\sigma} \cdot [v] \\ &= [v](x_j) \end{aligned}$$

$$(\tau, v) \in V : \quad \langle \tau, \delta_\tau \rangle + \langle v', \delta_v' \rangle_{\mathcal{T}} + \langle v, \delta_v \rangle_h = [\delta_v](x_j)$$

$$\tau = 0, \qquad v(s) = \begin{cases} -s - |T_{j-1}|^{-1} & \text{on } T_{j-1} \simeq s \in (0, |T_{j-1}|), \\ |T_j|^{-1} & \text{on } T_j, \\ 0 & \text{otherwise.} \end{cases}$$

Optimal Test Functions Type 2

$$(\phi, \sigma, \hat{\sigma}) = (0, \sigma_j, 0)$$

$$\begin{aligned} b(\phi, \sigma, \hat{\sigma}; \tau, v) &= \langle \phi, (\mathcal{V}\tau)' \rangle + \langle \phi, 1 \rangle \langle v, 1 \rangle + \langle \sigma, \tau + v' \rangle_{\mathcal{T}} + \hat{\sigma} \cdot [v] \\ &= \langle \sigma_j, \tau + v' \rangle_{T_j} \end{aligned}$$

$$(\tau, v) \in V : \quad \langle \tau, \delta_\tau \rangle + \langle v', \delta_v' \rangle_{\mathcal{T}} + \langle v, \delta_v \rangle_h = \langle \sigma_j, \delta_\tau + \delta_v' \rangle_{T_j}$$

$$\tau = \sigma_j, \qquad v(s) = \int_0^s \sigma_j(x(t))\, dt \quad \text{on} \quad T_j \simeq s \in (0, |T_j|)$$

Optimal Test Functions Type 3

$$(\phi, \sigma, \hat{\sigma}) = (\phi_j, 0, 0)$$

$$\begin{aligned} b(\phi, \sigma, \hat{\sigma}; \tau, v) &= \langle \phi, (\mathcal{V}\tau)' \rangle + \langle \phi, 1 \rangle \langle v, 1 \rangle + \langle \sigma, \tau + v' \rangle_{\mathcal{T}} + \hat{\sigma} \cdot [v] \\ &= \langle \phi_j, (\mathcal{V}\tau)' \rangle + \langle \phi_j, 1 \rangle \langle v, 1 \rangle \end{aligned}$$

$$(\tau, v) \in V : \quad \langle \tau, \delta_\tau \rangle + \langle v', \delta_v' \rangle_{\mathcal{T}} + \langle v, \delta_v \rangle_h = \langle \phi_j, (\mathcal{V}\delta_\tau)' \rangle + \langle \phi_j, 1 \rangle \langle \delta_v, 1 \rangle$$

$$\begin{aligned} &\tau \in L^2(\Gamma) : \quad \langle \tau, \delta_\tau \rangle = \langle \phi_j, (V\delta_\tau)' \rangle \quad \forall \delta_\tau \in L^2(\Gamma) \\ &v(s) = \langle \phi_j, 1 \rangle \big((|T_j| - s/2)s + 1 \big) \quad \text{on } T_j \simeq s \in (0, |T_j|) \end{aligned}$$

Outline

Numerical Experiment

$\Gamma = (-1, 1) \times \{0\}$ open slit

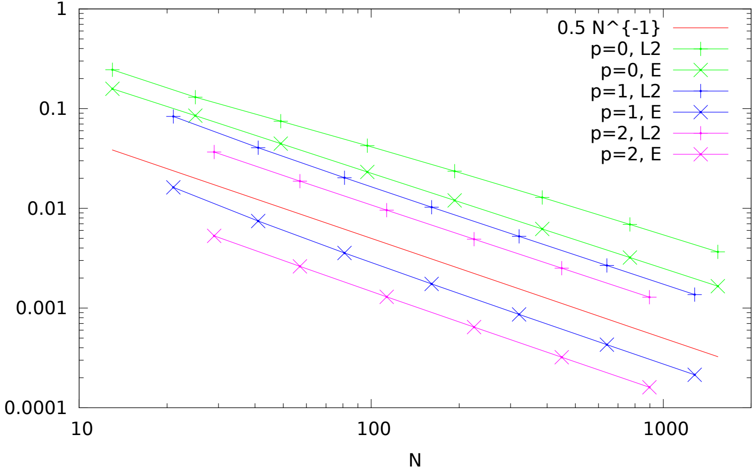
$\mathcal{W}\phi = 1/2$ with exact solution $\phi(x_1, x_2) = \sqrt{1 - x_1^2}$

Ultra-weak formulation:

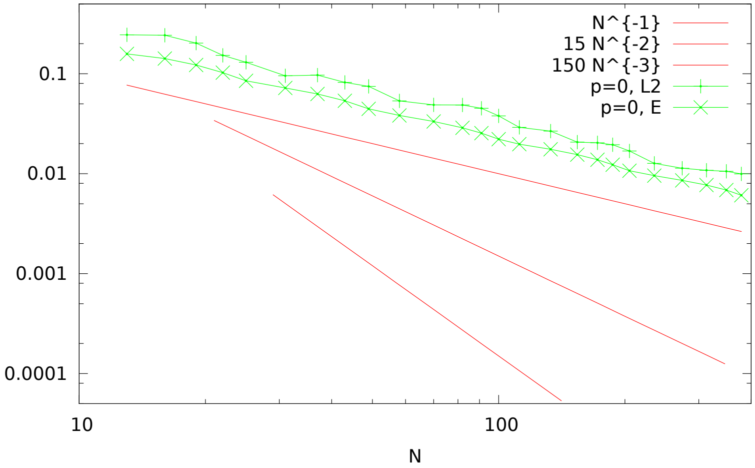
$\phi \in L^2(\Gamma)$, $\sigma \in L^2(\Gamma)$, $\hat{\sigma} \in \mathbb{R}^N$ s.th.

$$
\begin{aligned}
\langle \sigma, \tau \rangle \quad + \langle \phi, (\mathcal{V}\tau)' \rangle &= 0 \qquad && \forall \tau \in L^2(\Gamma) \\
\langle \sigma, v' \rangle_{\mathcal{T}} + \quad \hat{\sigma} \cdot [v] \quad &= \langle f, v \rangle \qquad && \forall v \in H^1(\mathcal{T})
\end{aligned}
$$

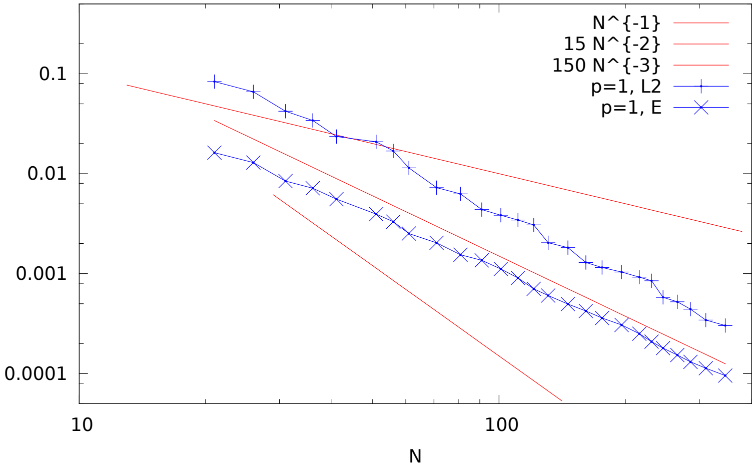
Uniform meshes, $p = 0, 1, 2$, errors in energy and L^2-norms



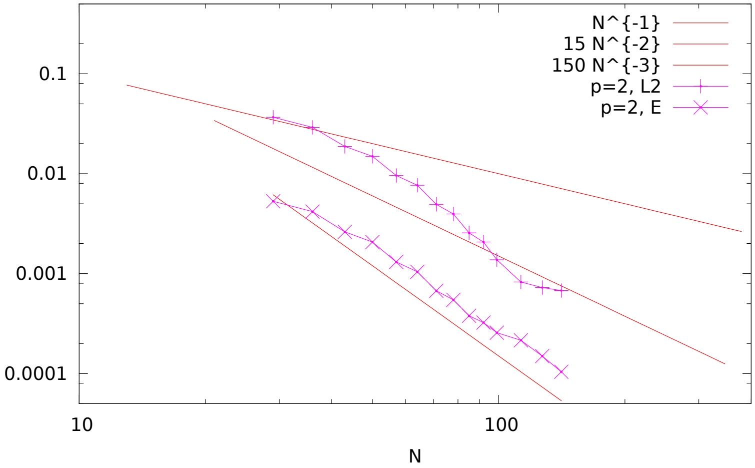
Adaptively refined meshes, $p = 0$, errors in energy and L^2-norms



Adaptively refined meshes, $p = 1$, errors in energy and L^2-norms



Adaptively refined meshes, $p = 2$, errors in energy and L^2-norms



Summary

- Ultra-weak formulation of a hypersingular integral equation

$$H^{1/2}(\Gamma) \times H^{1/2}(\Gamma) \longrightarrow (L^2(\Gamma), L^2(\Gamma), \mathbb{R}^N) \times (L^2(\Gamma), H^1(\mathcal{T}))$$

- Discontinuous approximation improves flexibility.
- Error estimation/calculation is inherent to the method and is local.
- Calculation of test functions and errors is based on sparse matrices.
- Globality of the problem enters through the right-hand side, the trial-to-test operator.
- Happy Birthday, Martin!

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