Boundary Integral Equations on Complex Screens

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Journés Singulières Augmentées 2013
Conférence en l'honneur de Martin Costabel
26-30 Aug 2013 Rennes (France)

Modern View of Boundary Integral Operators

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Our source of inspiration:

SIAM J. MATH. ANAL.
Vol. 19, No. 3, May 1988

BOUNDARY INTEGRAL OPERATORS ON LIPSCHITZ DOMAINS: ELEMENTARY RESULTS*

MARTIN COSTABEL†

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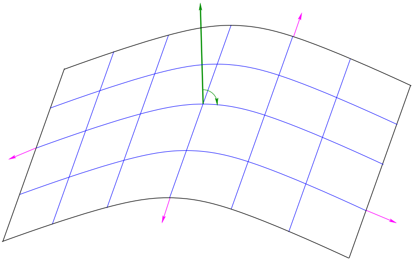
- boundary $\partial\Omega$ $\xrightarrow{\text{Green's formula}}$ PDE on domain Ω
- framework: energy (Sobolev & trace) spaces
- strictly distributional/variational perspective

Screens

Simple Screens

Simple Screens



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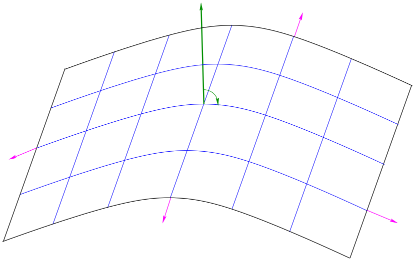
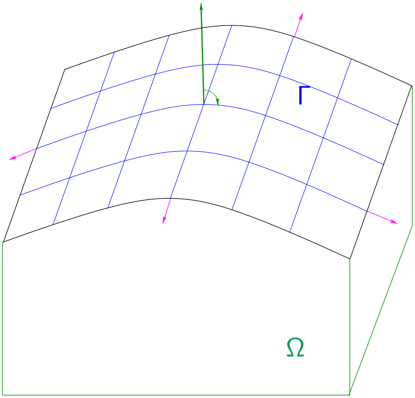


◁ Orientable Lipschitz screen

Simple Screens



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= part of the boundary of a domain Ω

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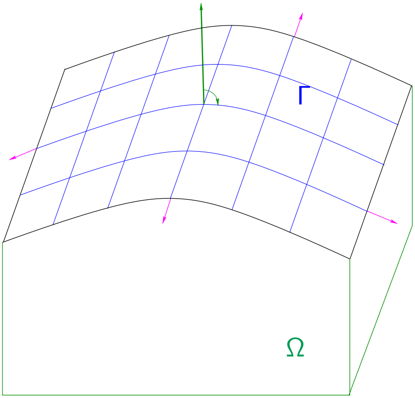

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Jump trace spaces (with "zero boundary conditions on" $\partial\Gamma$)

$$\widetilde{H}^{\frac{1}{2}}(\Gamma) \subset H^{\frac{1}{2}}(\partial\widetilde{\Omega}) ,$$
$$\widetilde{H}^{-\frac{1}{2}}(\Gamma) \subset H^{-\frac{1}{2}}(\partial\widetilde{\Omega})$$

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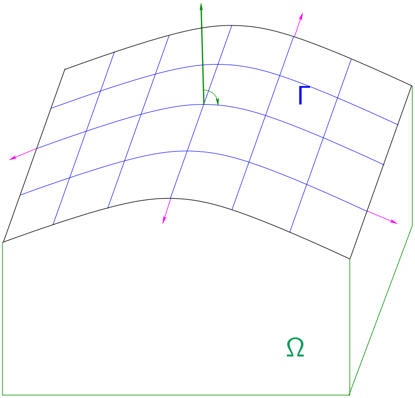

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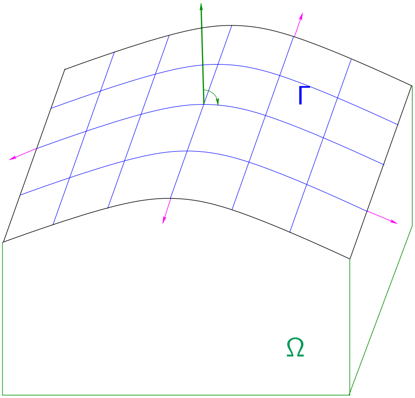

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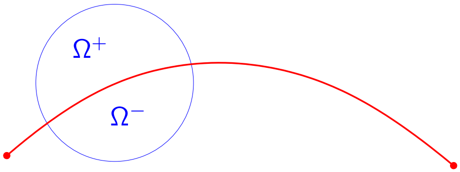

Simple Screens: BVP & BIE

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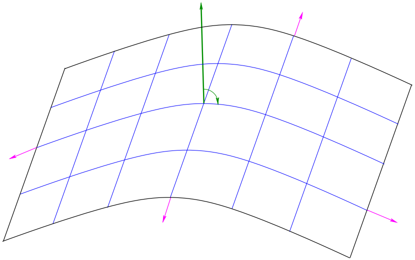

Exterior Dirichlet problem:

$$-\Delta u + u = 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma\,,$$
$$u = g \quad \text{on } \Gamma\,,$$
+ decay conditions at ∞ .

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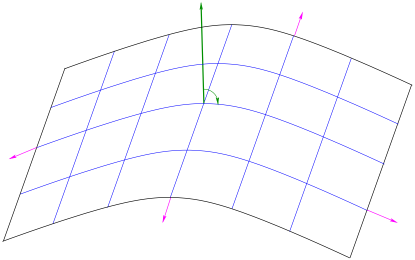

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▶ 1st-kind boundary integral equation:

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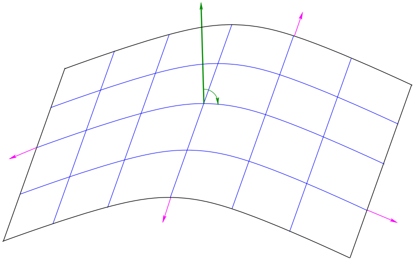

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$$\langle \mathsf{V}_k \varphi, \varphi' \rangle := \int_\Gamma \int_\Gamma G_k(\boldsymbol{x}, \boldsymbol{y}) \varphi(\boldsymbol{y}) \varphi'(\boldsymbol{x}) \, \mathrm{d}S(\boldsymbol{y}) \mathrm{d}S(\boldsymbol{x}) = - \int_\Gamma g(\boldsymbol{x}) \varphi'(\boldsymbol{x}) \, \mathrm{d}S(\boldsymbol{x})$$

for all $\varphi' \in \widetilde{H}^{-\frac{1}{2}}(\Gamma)$.

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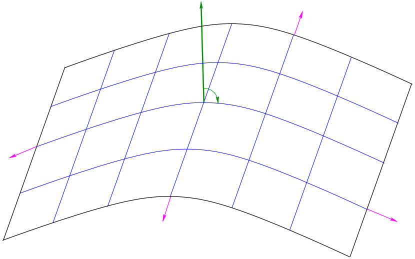

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Neumann jump: $\frac{\partial u}{\partial n}_+ - \frac{\partial u}{\partial n}_-$

Complex Screens

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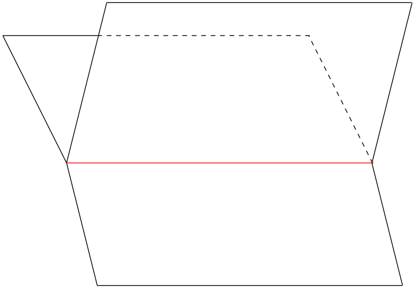

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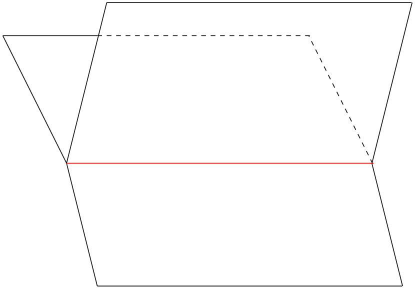
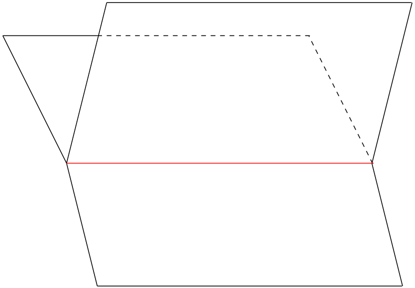


◁ Non-Lipschitz, non-orientable complex screen

Complex Screens



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(Lipschitz/orientable only locally away from “junction sets”)

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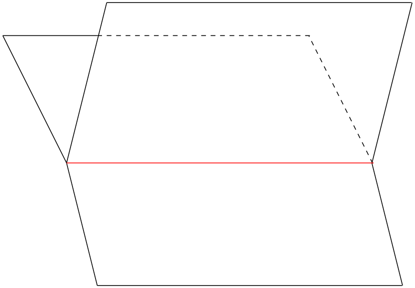

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Definition. Complex screen $\Gamma \subset \mathbb{R}^d$: $\exists$ mutually disjoint Lipschitz domains $\{\Omega_j\}_{j=1}^n$, such that

$$\Gamma \cap \partial\Omega_j \quad \text{is an orientable Lipschitz screen} \quad \forall j = 1, \ldots, n \,.$$

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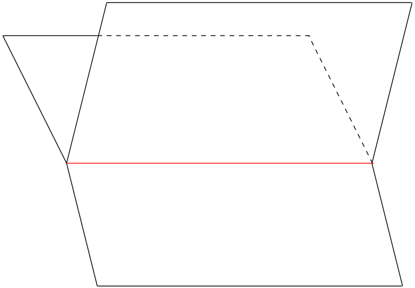

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Meaning of jumps **?**

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Coming up next

Function Spaces off the Screen

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Recall: Ω bounded, Lipschitz ➥ trace spaces as quotient spaces:

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$\Gamma \;\hat{=}$ complex screen

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$\Gamma \triangleq$ complex screen

- Functions which jump across Γ : $H^1(\mathbb{R}^d \setminus \Gamma) \quad , \quad \boldsymbol{H}(\mathrm{div},\mathbb{R}^d \setminus \Gamma)$

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- Functions which jump across Γ : $H^1(\mathbb{R}^d \setminus \Gamma)$, $\boldsymbol{H}(\mathrm{div}, \mathbb{R}^d \setminus \Gamma)$
- Functions "continuous" across Γ : $H^1(\mathbb{R}^d)$, $\boldsymbol{H}(\mathrm{div}, \mathbb{R}^d)$

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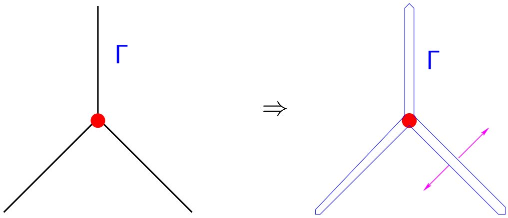
Closed subspaces:

$$H^1_{0,\Gamma}(\mathbb{R}^d) \subset H^1(\mathbb{R}^d) \subset H^1(\mathbb{R}^d \setminus \Gamma) \,,$$
$$\boldsymbol{H}_{0,\Gamma}(\mathrm{div},\mathbb{R}^d) \subset \boldsymbol{H}(\mathrm{div},\mathbb{R}^d) \subset \boldsymbol{H}(\mathrm{div},\mathbb{R}^d \setminus \Gamma) \,.$$

Multi-Trace Spaces

Mental picture:

thick screen



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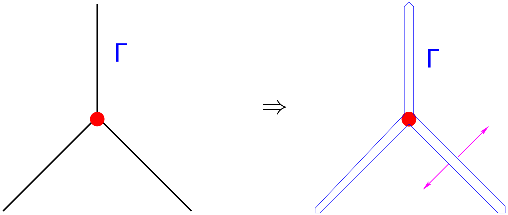

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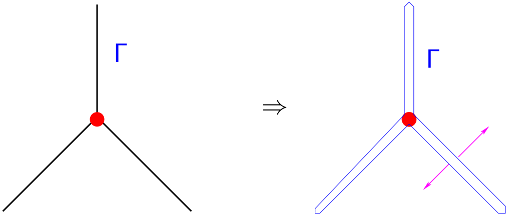

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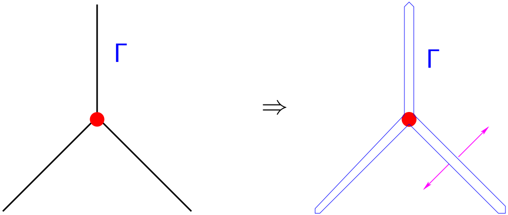

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Duality pairing: $\ll\,,\,\gg : \mathbb{H}^{+\frac{1}{2}}(\Gamma) \times \mathbb{H}^{-\frac{1}{2}}(\Gamma) \to \mathbb{C}$

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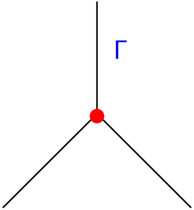
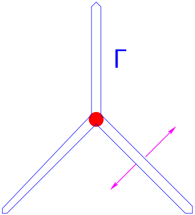


$\Rightarrow$



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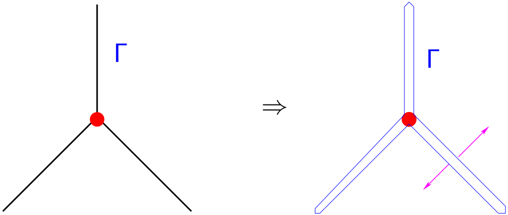

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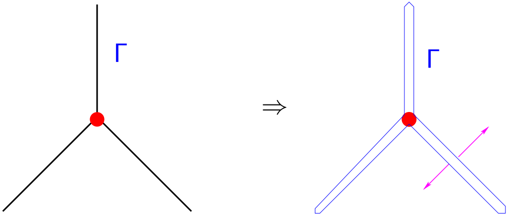

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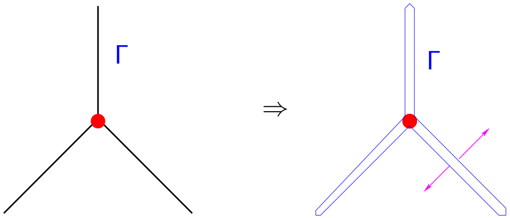

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$\ll\,,\,\gg$ induces isometric isomorphism: $(\mathbb{H}^{+\frac{1}{2}}(\Gamma))' \cong \mathbb{H}^{-\frac{1}{2}}(\Gamma)$

Duality

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Multi-Trace Spaces: Examples

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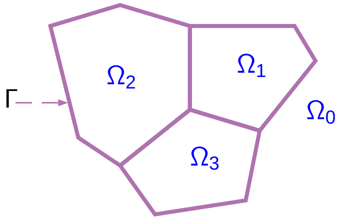

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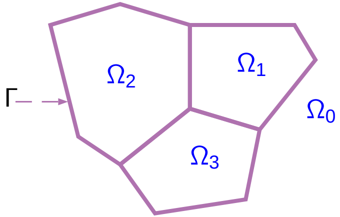

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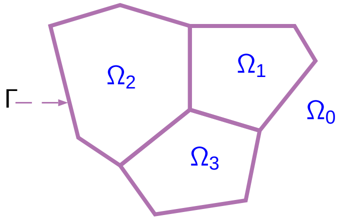

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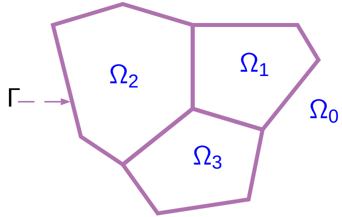

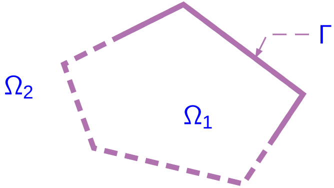

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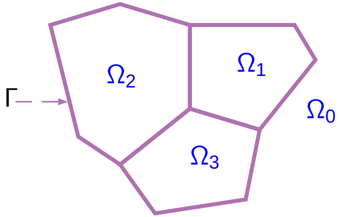

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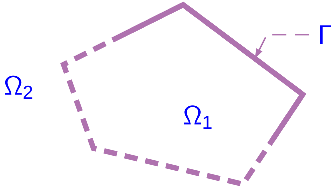

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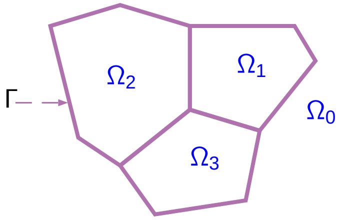

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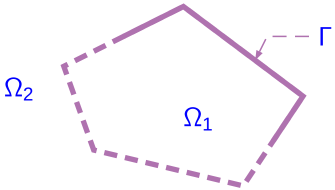

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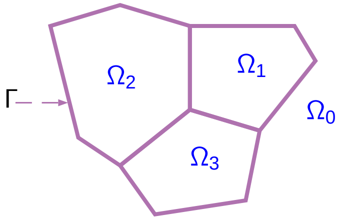

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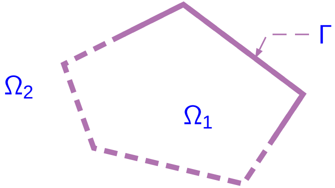

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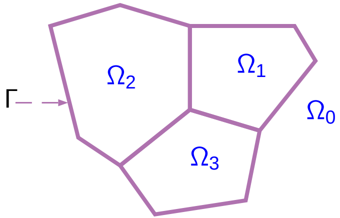

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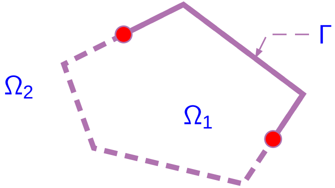

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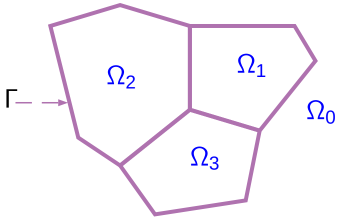

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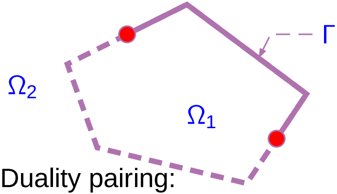

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Polarity by Green’s formula:

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no duality!

$$\dot{u} \in H^{+\frac{1}{2}}([\Gamma]) \quad \Leftrightarrow \quad \ll \dot{u},\dot{q} \gg = 0 \quad \forall \dot{q} \in H^{-\frac{1}{2}}([\Gamma])\,,$$
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Jump Spaces

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Definition. Spaces of Dirichlet- and Neumann jumps

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Example: Jump Spaces on Simple Screens

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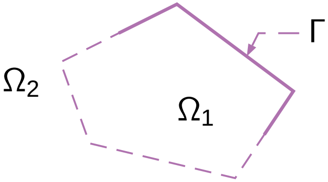

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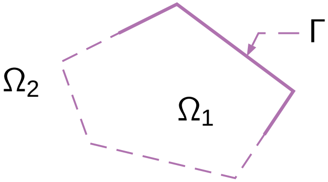

Recall: If $\Gamma \subset \partial\Omega$

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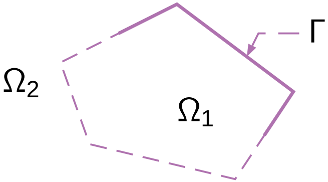

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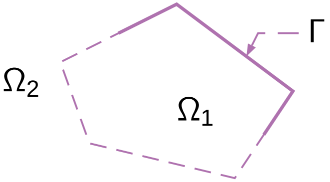

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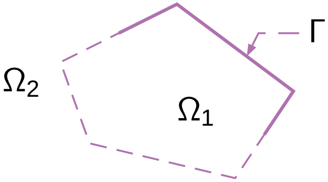

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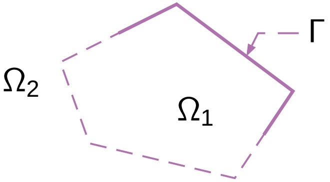

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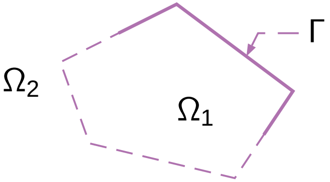

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➢ “Customary jump spaces” recovered

Coming up next

Representation Formula

[PDE: $-\Delta u + u = 0$ in $\mathbb{R}^d \setminus \Gamma$]

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Dirichlet trace: $\gamma_D : H^1(\mathbb{R}^d \setminus \Gamma) \to \mathbb{H}^{+\frac{1}{2}}(\Gamma)$,

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Potential Operators

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Single layer potential: $\quad \mathsf{SL} := \mathsf{N} \circ \gamma_D' \qquad : \mathbb{H}^{-\frac{1}{2}}(\Gamma) \rightarrow H^1(\mathbb{R}^d) \cap H^1(\Delta, \mathbb{R}^d \setminus \Gamma)$,

Potential Operators

Representation formula:

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Potential Operators

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"Integral" representation ($\boldsymbol{x} \notin \Gamma$)

$$(\mathsf{SL}(\dot{q}))(\boldsymbol{x}) := \ll \gamma_D G(\boldsymbol{x} - \cdot), \dot{q} \gg \; , \quad \dot{q} \in \mathbb{H}^{-\frac{1}{2}}(\Gamma) \; ,$$

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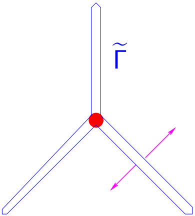

"Integral" representation $\quad (\boldsymbol{x} \notin \Gamma)$

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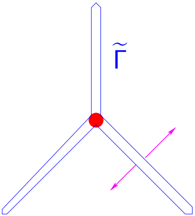

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e.g. "$\mathsf{SL}(\dot{q}) = \int_{\widetilde{\Gamma}} G(\boldsymbol{x}-\boldsymbol{y})\dot{q}(\boldsymbol{y})\,\mathrm{d}\sigma$" .

Potential Operators: Jump Relations

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Potential Operators: Jump Relations

Jumps here!

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Proof. Recall $\mathsf{L} \circ \mathsf{DL} = -\mathsf{L} \circ \mathsf{N} \circ \gamma_N' = -\gamma_N'$ $(\mathsf{L} := -\Delta + \mathsf{Id})$.

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Known: $(\gamma_D, \gamma_N) : \mathcal{D}(\mathbb{R}^d) \to H^{+\frac{1}{2}}([\Gamma]) \times H^{-\frac{1}{2}}([\Gamma])$ has dense range

Boundary Integral Operators

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Single layer operator:

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Kernels.

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$$\mathrm{Kern}(\mathsf{V}) = H^{-\frac{1}{2}}([\Gamma]) \quad , \quad \mathrm{Kern}(\mathsf{W}) = H^{+\frac{1}{2}}([\Gamma]) \, .$$

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Ellipticity.

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Ellipticity. There is $C > 0$: for all $\dot{q} \in \widetilde{H}^{-\frac{1}{2}}([\Gamma])$, $\dot{v} \in \widetilde{H}^{+\frac{1}{2}}([\Gamma])$

$$\ll \mathsf{V}\dot{q}, \dot{q} \gg \geq C \, \|\dot{q}\|^2_{\widetilde{H}^{-\frac{1}{2}}([\Gamma])} \quad , \quad \ll \mathsf{W}\dot{v}, \dot{v} \gg \geq C \, \|\dot{v}\|^2_{\widetilde{H}^{+\frac{1}{2}}([\Gamma])} \; .$$

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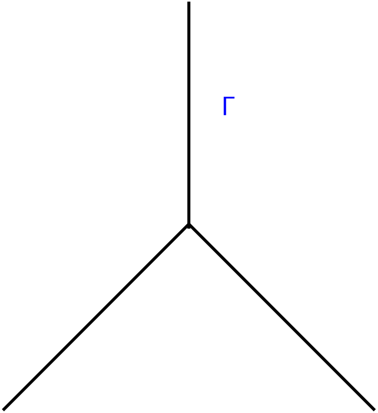
► Isomorphisms: $\mathsf{V} : \widetilde{H}^{-\frac{1}{2}}([\Gamma]) \to H^{+\frac{1}{2}}([\Gamma])$, $\mathsf{W} : \widetilde{H}^{+\frac{1}{2}}([\Gamma]) \to H^{-\frac{1}{2}}([\Gamma])$

Outlook: Galerkin BEM

Work in progress!

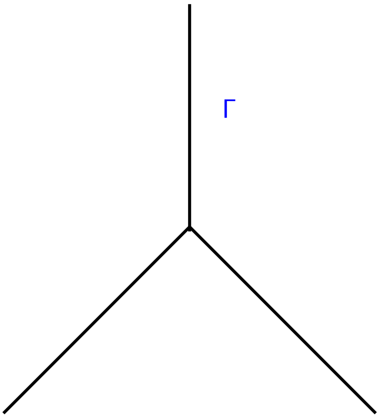
Outlook: Galerkin BEM



Exterior Neumann problem:

$$-\Delta u + u = 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma ,$$
$$\gamma_N u = h \in H^{-\frac{1}{2}}([\Gamma]) \quad \text{on } \Gamma ,$$
+ decay conditions at ∞ .

Outlook: Galerkin BEM



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$$\Downarrow$$

Seek $\dot{u} \in \widetilde{H}^{+\frac{1}{2}}([\Gamma])$ such that

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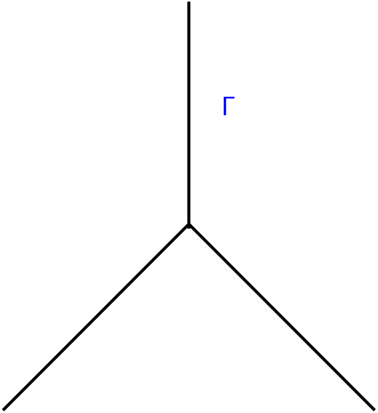

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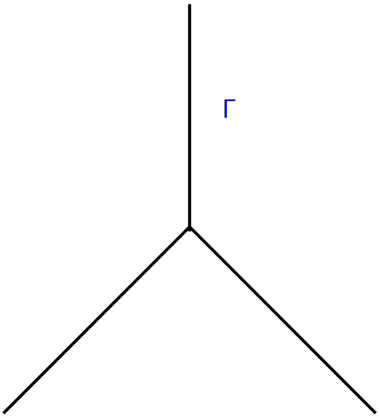
? BEM space $V_h \subset \widetilde{H}^{+\frac{1}{2}}([\Gamma])$ (piecewise linear)

Outlook: Galerkin BEM



Exterior Neumann problem:

$$-\Delta u + u = 0 \quad \text{in } \mathbb{R}^d \setminus \Gamma \,,$$

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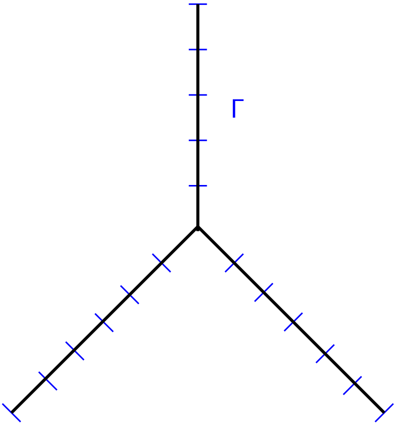

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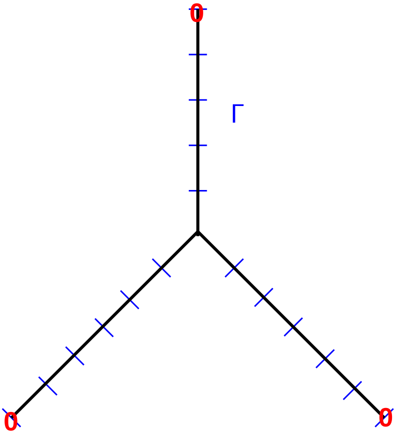

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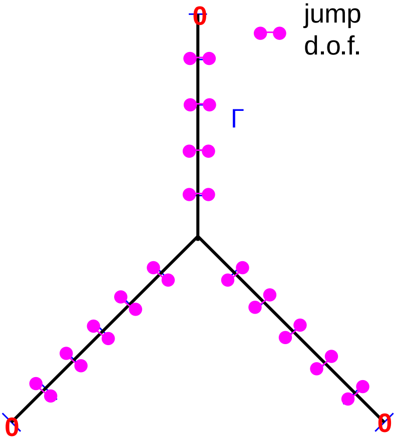

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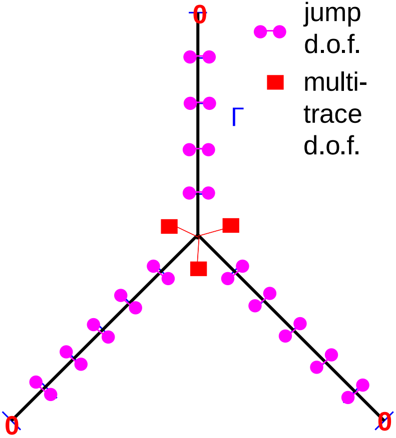

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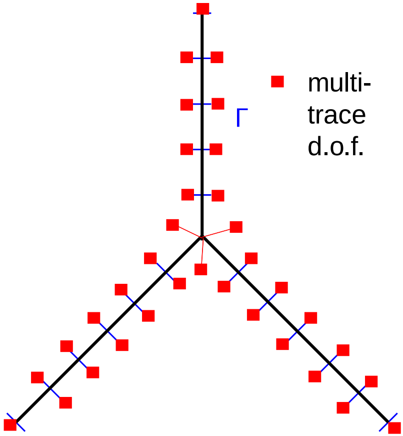

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Is all this aimless abstraction ?

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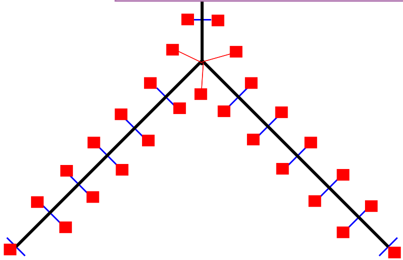

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X. Claeys and R. Hiptmair, *Integral equations on multi-screens*, Integral Equations and Operator Theory, (2013), pp. 1–31.

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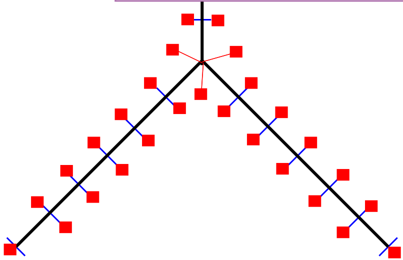

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Outlook: Galerkin BEM

LIEBER MARTIN,

AUCH IN ZUKUNFT

VIEL FREUDE

AM LEBEN UND AN

DER MATHEMATIK

Γ])

Outlook: Galerkin BEM

DEAR MARTIN,

CONTINUE ENJOYING

A GOOD LIFE

&

MATHEMATICS

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