Sobolev Spaces and Scattering by Fractal Screens

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With: **Dave Hewett** (Oxford), **Andrea Moiola** (Reading)

Conference en l'honneur de Martin Costabel

JSA 2013, Rennes

This is a talk on

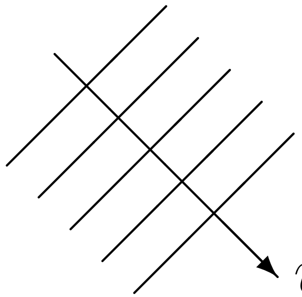
integral equations for acoustic scattering by planar screens

e.g.

$$V_\Gamma \left[\frac{\partial u}{\partial n} \right] = f$$

where V_Γ is the **single-layer potential operator** on the screen Γ.

Scattering by Planar Screens



u^i, incident wave

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D

$u = 0$

Γ

$u - u^i$ satisfies Sommerfeld radiation condition

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New angles:

1. **How do coercivity and continuity constants depend on the wavenumber k?**
2. **How does well-posedness, values of these constants, and the PDE solution, depend on Γ, e.g. its smoothness, its Lebesgue measure, its fractal dimension?**

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A Classical Planar Screen

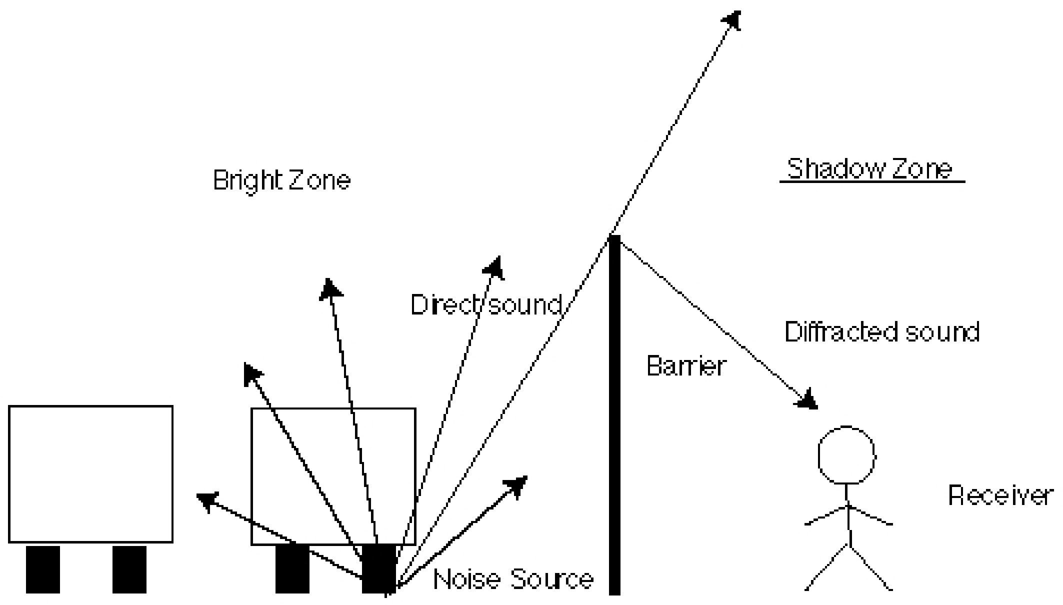
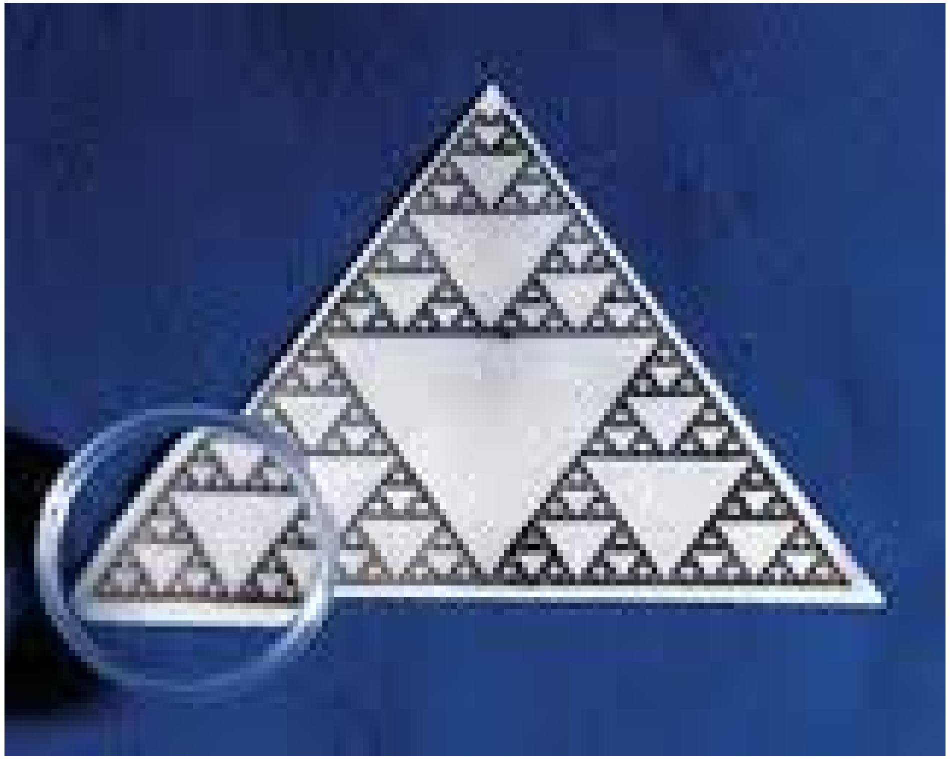


Figure 1: The barrier blocks the direct path from the noise source to the receiver. A shadow zone is created behind the barrier, in which listeners are protected from the noise. However, sound can still reach the receiver by sound diffraction at the top edge of the barrier.

... back to my origins in traffic noise barriers ... reading Costabel & Stephan (1985).

A Classical Planar Screen

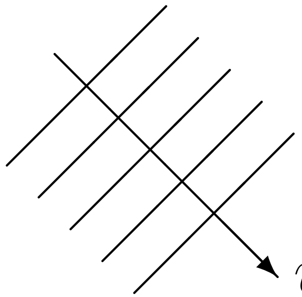
Fractal Planar Screen



... but also, e.g., Puente-Baliarda, Romeu, Pous, Cardama, IEEE Trans. Ant. Prop., **46** (1998), 517–524.

On the Behavior of the **Sierpinski Multiband Fractal Antenna**

Scattering by Planar Screens



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Scattering by Planar Screens

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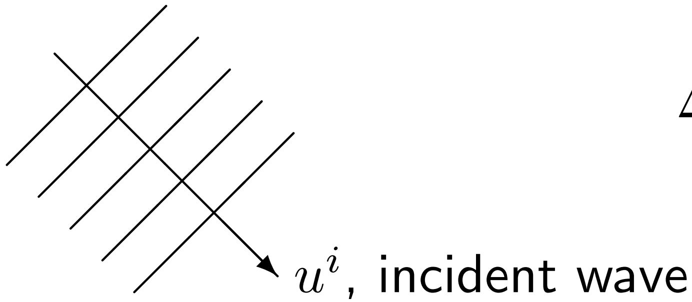
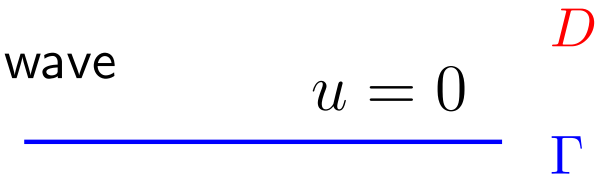




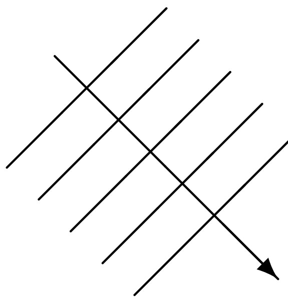
Green's representation theorem:

$$u(x) = u^i(x) - \int_\Gamma G(x-y) \left[\frac{\partial u}{\partial n}\right](y) ds(y), \quad x \in D,$$

where

$$G(x) := \tfrac{\mathrm{i}}{4} H_0^{(1)}(k|x|) \text{ (2D)}, \qquad := \frac{1}{4\pi} \frac{\mathrm{e}^{\mathrm{i}k|x|}}{|x|} \text{ (3D)}.$$

Scattering by Planar Screens



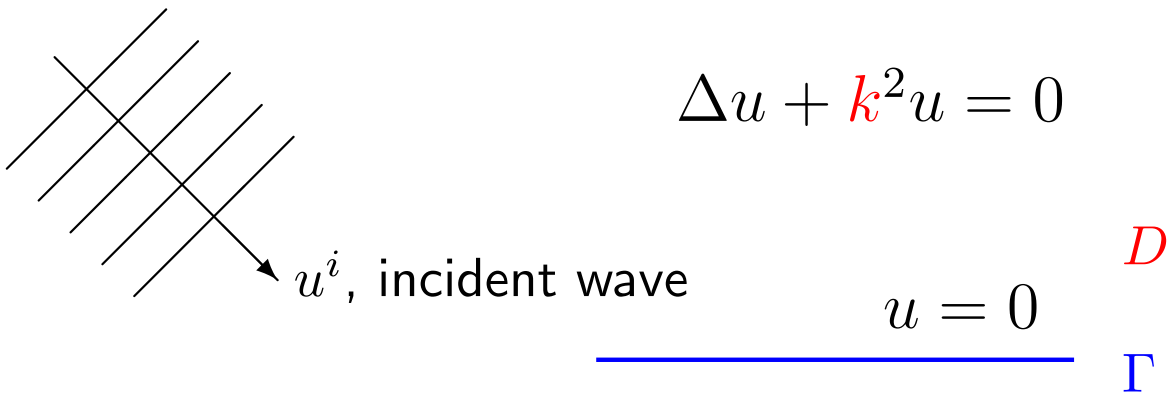
$$\Delta u + k^2 u = 0$$

... giving the boundary integral equation

$$0 = u^i(x) - \int_\Gamma G(x-y) \left[\frac{\partial u}{\partial n}\right](y) ds(y), \quad x \in \Gamma.$$

$$V_\Gamma \left[\frac{\partial u}{\partial n}\right] = u^i|_\Gamma$$

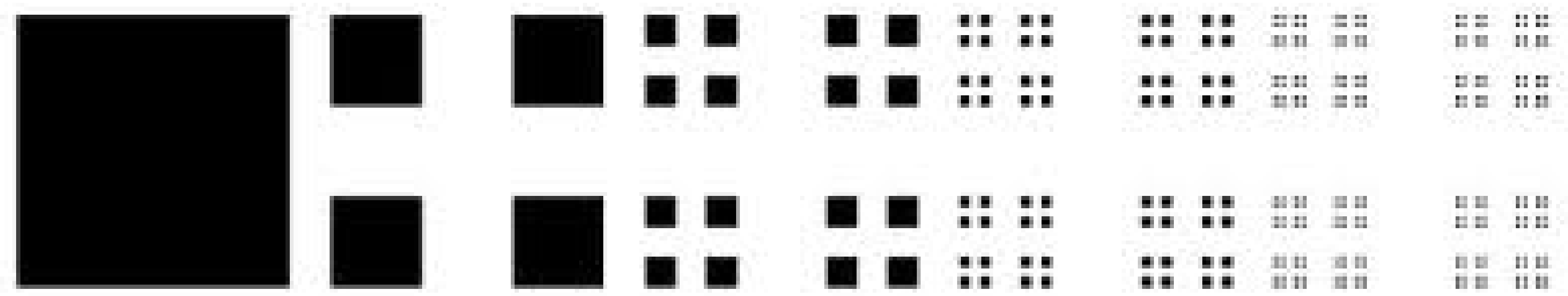
Scattering by Planar Screens



1. Do SP and the integral equation make sense for arbitrary (possibly fractal) screens? ... and do fractal screens make sense? (In all integral equation literature Γ is **Lipschitz** or smoother, even C^∞.)

2. Well-posedness: for which Γ and k is $V_\Gamma : \tilde{H}^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ well-defined/bounded/invertible/coercive, and dependence of constants on Γ and k.

Fractal Screen Example: Cantor Dust



Consider a sequence of screens $\Gamma_0 \supset \Gamma_1 \supset ...$, with Γ_n consisting of 4^n closed squares of side length α^n: here $0 < \alpha < 1/2$ and shown is the classic "middle-third" case $\alpha = 1/3$.

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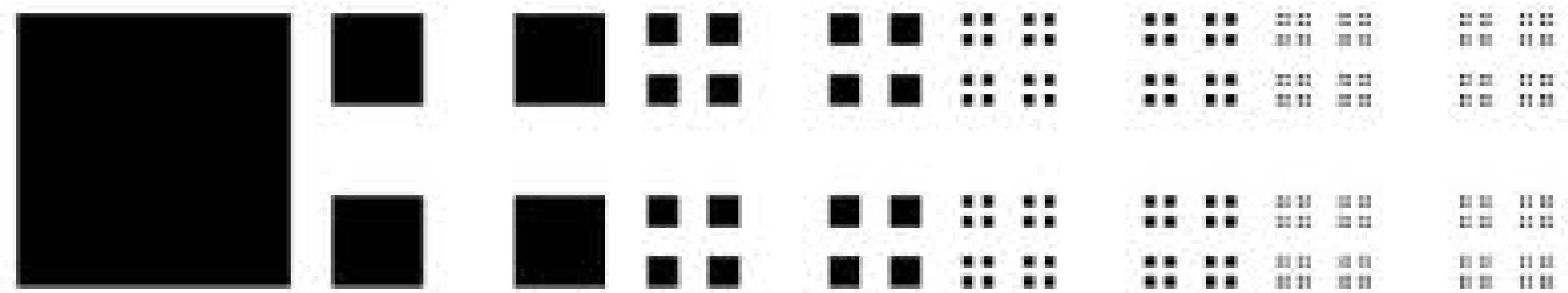

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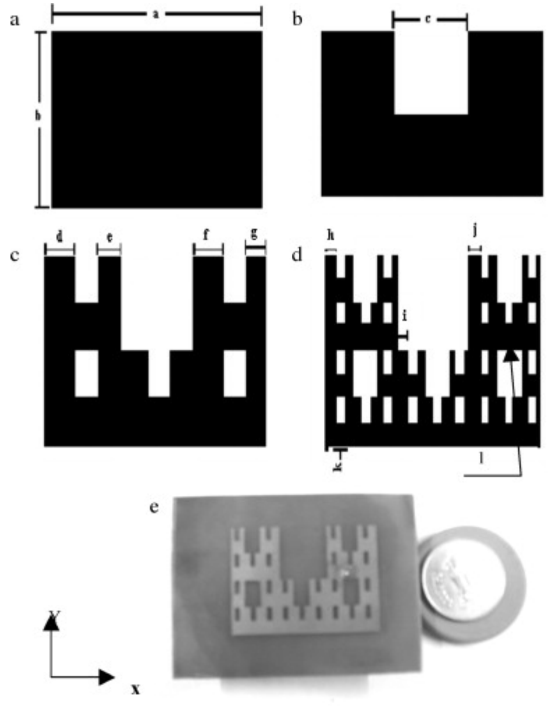
Cantor dust is the closed set $\Gamma = \bigcap_{n=0}^{\infty} \Gamma_n$ which has Lebesgue measure zero and Haussdorf dimension $2 \log 2 / \log(1/\alpha) \in (0, 2)$.

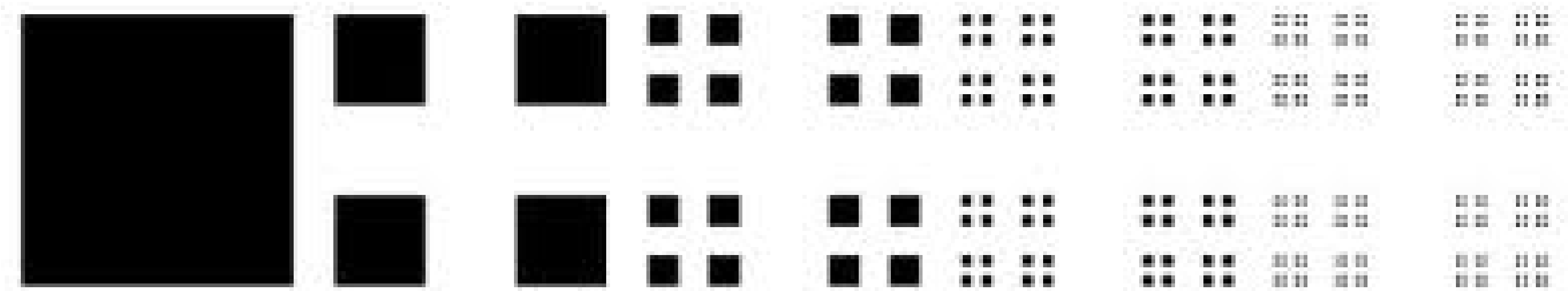
Planar Screens



Srivatsun, Rani, Krishnan, Wireless Eng. Tech. **2** (2011), 107–111.
A Self-Similar Fractal Cantor Antenna for MICS Band Wireless Applications

My new patent ...

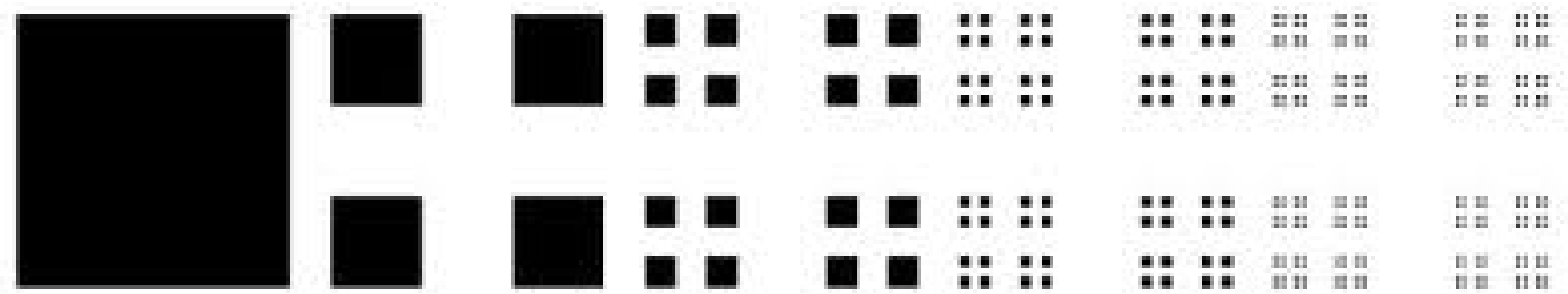
THE CANTOR DUST NOISE SCREEN®



The weightless, invisible solution to your noise control problems!

(This is a joke!!)

Your First Quiz Questions



1. Does $V_\Gamma \left[\frac{\partial u}{\partial n}\right] = u^i|_\Gamma$ have a solution? In what sense does this equation make sense?

2. Is this solution just $\left[\frac{\partial u}{\partial n}\right] = 0$, i.e., the incident wave does not see the screen, i.e. my noise screen is completely useless?

Sobolev Spaces for Planar Screens: What are $H^s(\Gamma)$ and $\widetilde{H}^s(\Gamma)$?

Opinion 1: Costabel, Penzel, Schneider (1993)

By $H^s(\mathbb{R}^2)$ ($s \in \mathbb{R}$) we denote the usual Sobolev spaces [15]. For a bounded domain $\mathcal{O}$,

$$p_{\mathcal{O}} : u \mapsto u|_{\mathcal{O}} \tag{2.12}$$

denotes the operator of restriction to $\mathcal{O}$. Then

$$H^s(\mathcal{O}) := \{u|_{\mathcal{O}} \mid u \in H^s(\mathbb{R}^2)\} = p_{\mathcal{O}} H^s(\mathbb{R}^2) \tag{2.13}$$

and

$$\tilde{H}^s(\mathcal{O}) := \{u \in H^s(\mathbb{R}^2) \mid \operatorname{supp} u \subset \overline{\mathcal{O}}\}\,. \tag{2.14}$$

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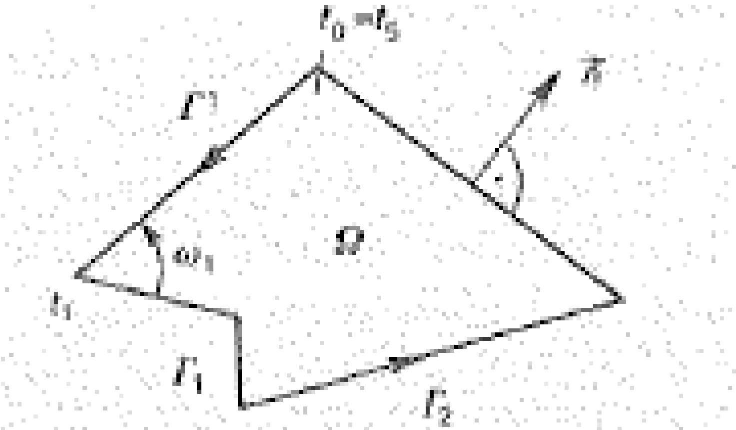
Opinion 2: Costabel & Stephan (1985)



$$\tilde{H}^s(\Gamma_j) = \{u \in H^s(\Gamma_j) | \ \tilde{u} \in H^s(\Gamma)\} (s \geqslant 0)\} .$$

Here $\tilde{u} = \begin{cases} u \text{ on } \Gamma_j \\ 0 \text{ on } \Gamma \setminus \Gamma_j \end{cases}$ means the continuation of u by 0 outside Γ_j.

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Opinion 3: McLean (2000)

For a bounded open set $\Omega \subset \mathbb{R}^d$

$H^s(\Omega) = \{u|_\Omega : u \in H^s(\mathbb{R}^d)\}$, as Opinion 1, Costabel et al (1993),

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but

$\tilde{H}^s(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $H^s(\mathbb{R}^d)$

rather than

$\{u \in H^s(\mathbb{R}^d) : \mathrm{supp}(u) \subset \overline{\Omega}\}$ (Opinion 1)

or (for $s \geq 0$)

$\{u \in H^s(\Omega) : \tilde{u} \in H^s(\mathbb{R}^d)\}$ (Opinion 2).

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Sometimes everyone is right: these definitions coincide if Ω is a C^0 open set, in particular if Lipschitz.

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In general these definitions **do not coincide**: we prefer McLean for a general open set, for then

$$\tilde{H}^s(\Omega) = (H^{-s}(\Omega))^*. \quad \text{New for general } \Omega??$$

First Well-Posedness Theorem

In the 3D case, when Γ is an **open** subset of $\mathbb{R}^2$, and using natural wavenumber dependent norms on $H^s(\mathbb{R}^2)$,

$$V_\Gamma : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$$

is bounded with $\|V_\Gamma\| \leq C(1 + kD)^{1/2}$, where $D = \mathrm{diam}(\Gamma)$.

Further (cf. Costabel (2004)) V_Γ is coercive, with

$$|(V\phi, \phi)| \geq \frac{1}{\sqrt{2}} \|\phi\|^2, \quad \phi \in \tilde{H}^{-1/2}(\Gamma).$$

[The above assumes we use the natural norm $\left(\int_{\mathbb{R}^2} |\hat{\phi}(\xi)|^2 (k^2 + \xi^2)^s \, d\xi \right)^{1/2}$ on $H^s(\mathbb{R}^2)$, and these results are sharp.]

Another Quiz Question

Is the following question dumb: **"Does it matter whether the screen Γ is open or closed"**?

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No!

1. If Ω is open there is a well-defined solution in the case that $\Gamma = \Omega$, and in the case that $\Gamma = \overline{\Omega}$. These are the same if Ω is C^0 but not in general! E.g. Ω and $\overline{\Omega}$ may have different Lebesgue measure.

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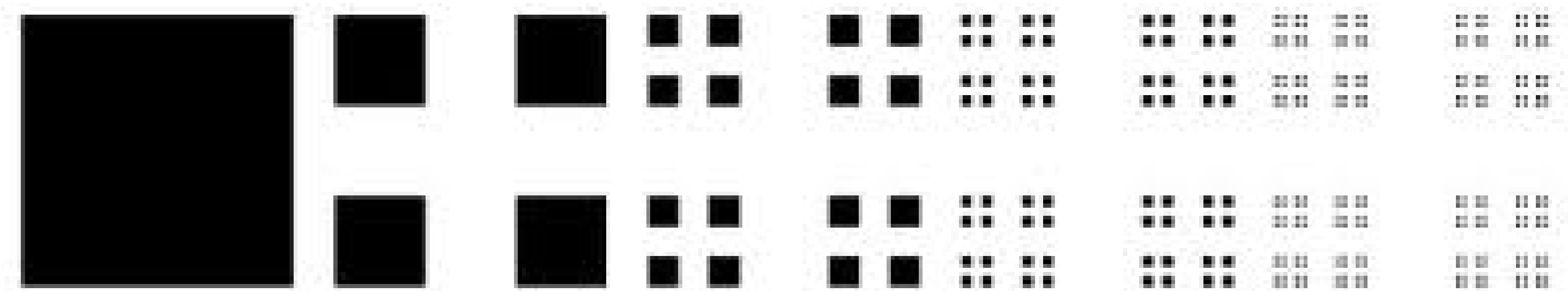
2. Γ may be closed with empty interior and still scatter waves!

Your Next Quiz Question



Is there a sensible integral equation formulation in this case, when Γ is the Cantor dust, a closed set with zero Lebesgue measure?

Theorem. If u satisfies the 3D scattering problem then

$$u(x) = u^i(x) - \int_\Gamma G(x-y)\left[\frac{\partial u}{\partial n}\right](y)ds(y), \quad x \in D,$$

with $[\partial u/\partial n] \in H_\Gamma^{-1/2} := \{\phi \in H^2(\mathbb{R}^2) : \mathrm{supp}(\phi) \subset \Gamma\}$ and

$$\gamma u = \gamma u^i - V\left[\frac{\partial u}{\partial n}\right]$$

where $V : H^{-1/2}_{\mathrm{comp}}(\mathbb{R}^2) \to H^{1/2}_{\mathrm{loc}}(\mathbb{R}^2)$ is defined by

$$V\phi(x) = \int_{\mathbb{R}^2} G(x-y)\phi(y)ds(y), \quad x \in \mathbb{R}^2,$$

for $\phi \in C_0^\infty(\mathbb{R}^2)$, and extended to $\phi \in H_\Gamma^{-1/2}$ by density.

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with $[\partial u/\partial n] \in H_\Gamma^{-1/2} := \{\phi \in H^2(\mathbb{R}^2) : \mathrm{supp}(\phi) \subset \Gamma\}$ and

$$0 = (\gamma u, \phi) = (\gamma u^i, \phi) - \left(V\left[\frac{\partial u}{\partial n}\right], \phi\right), \quad \phi \in H_\Gamma^{-1/2}.$$

N.B. We've interpreted "$u = 0$ on Γ" as meaning $\gamma u \in \tilde{H}^{1/2}(\mathbb{R}^2 \setminus \Gamma)$.

.

Theorem. For $\phi, \psi \in H_\Gamma^{-1/2}$ let

$$a(\phi, \psi) := (V\phi, \psi).$$

Then a is bounded and coercive:

$$|a(\phi, \psi)| \leq C(1 + kD)^{1/2} \|\phi\| \; \|\psi\|$$

and

$$|a(\phi, \phi)| \geq \frac{1}{\sqrt{2}} \|\phi\|^2.$$

Corollary (by Lax-Milgram). The problem: find $[\partial u/\partial n] \in H_\Gamma^{-1/2}$ such that

$$a([\partial u/\partial n], \phi) = g(\phi), \quad \forall \, \phi \in H_\Gamma^{-1/2}$$

has exactly one solution.

Theorem. For $\phi, \psi \in H_\Gamma^{-1/2}$ let $a(\phi,\psi) := (V\phi,\psi)$. Then a is bounded and coercive, specifically

$$|a(\phi,\psi)| \leq C(1+kD)^{1/2}\|\phi\|\,\|\psi\| \quad \text{and} \quad |a(\phi,\phi)| \geq \frac{1}{\sqrt{2}}\|\phi\|^2.$$

Proof. It is enough to show that

$$|a(\phi,\psi)| \leq C(1+kD)^{1/2}\|\phi\|\,\|\psi\| \quad \text{and} \quad |a(\phi,\phi)| \geq \frac{1}{\sqrt{2}}\|\phi\|^2$$

for $\phi, \psi \in C_0^\infty(\mathbb{R}^2)$, with $D = \mathrm{diam}(\mathrm{supp}(\phi) \cup \mathrm{supp}(\psi))$.

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Proof. Further, for $\phi, \psi \in C_0^\infty(\mathbb{R}^2)$,

$$(V\phi,\psi) = (\widehat{G*\phi}, \widehat{\psi}) = (\widehat{G}\widehat{\phi}, \widehat{\psi}),$$

with $\widehat{G}(\xi) = 1/\sqrt{k^2-\xi^2}$. So

$$\begin{aligned} |(V\phi,\phi)| &= \left|\int_{\mathbb{R}^{d-1}} \frac{|\widehat{\phi}(\xi)|^2}{\sqrt{k^2-\xi^2}}\mathrm{d}\xi\right| \geq \frac{1}{\sqrt{2}}\int_{\mathbb{R}^{d-1}} \frac{|\widehat{\phi}(\xi)|^2}{|\sqrt{k^2-\xi^2}|}\mathrm{d}\xi \\ &\geq \frac{1}{\sqrt{2}}\int_{\mathbb{R}^{d-1}} \frac{|\widehat{\phi}(\xi)|^2}{\sqrt{k^2+\xi^2}}\mathrm{d}\xi = \frac{1}{\sqrt{2}}\|\phi\|^2. \end{aligned}$$

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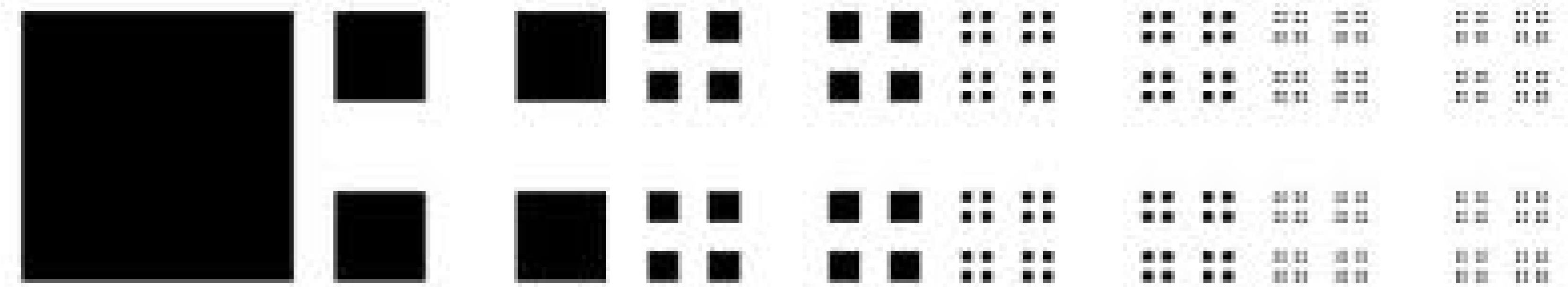

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2. Is this solution just $\left[\frac{\partial u}{\partial n}\right] = 0$, i.e., the incident wave does not see the screen, i.e. my noise screen is completely useless?

YES! if: find $[\partial u/\partial n] \in H_\Gamma^{-1/2}$ such that

$$\langle S[\partial u/\partial n], \phi\rangle = \langle \gamma u^i, \phi\rangle, \quad \forall\, \phi \in H_\Gamma^{-1/2}$$

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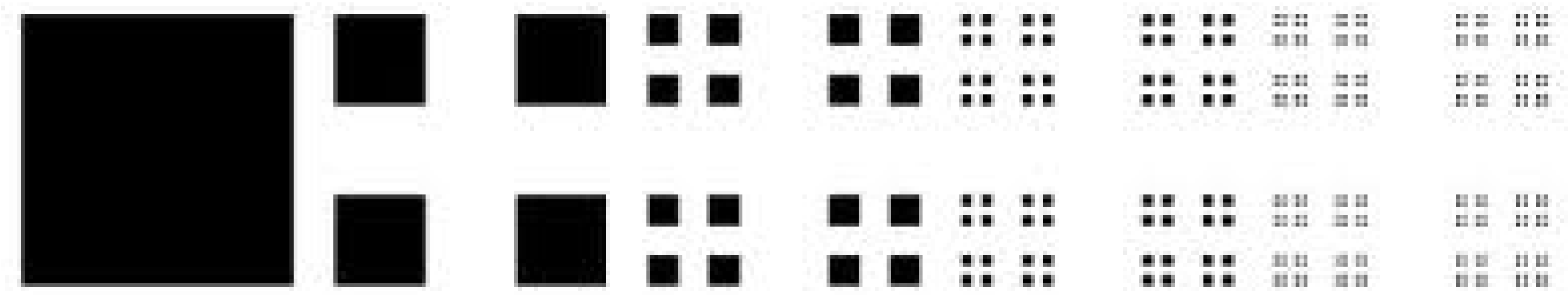

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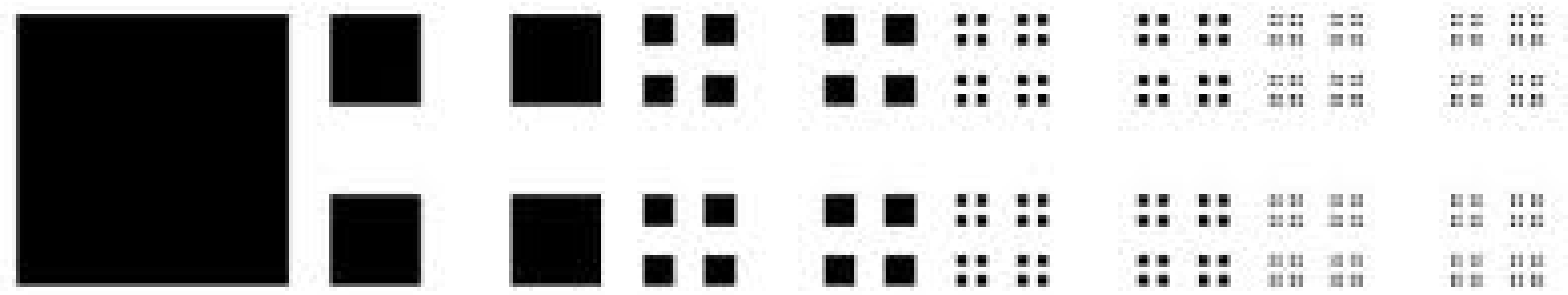

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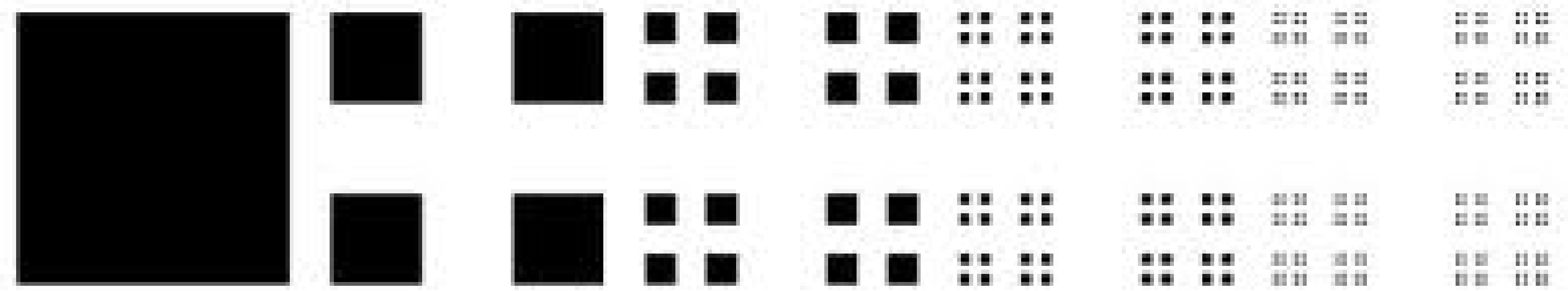

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Theorem. $u^s = 0$ for every incident field iff $H_\Gamma^{-1/2} = \{0\}$. Further,

$$H_\Gamma^{-1/2} = \{0\} \Leftrightarrow \dim_H(\Gamma) \leq 1 \Leftrightarrow \alpha \leq 1/4.$$

Summary

Reviewed recent work on scattering by planar screens that:

- Requires detailed understanding of Sobolev spaces on general open and closed sets
- Establishes existence of solution by integral equation methods for arbitrary (including fractal) screens.
- Clarifies when a screen has an effect (whenever $\dim_H(\Gamma) > 1$ in 3D even if $m(\Gamma) = 0$).
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Missed out today: Neumann screen/hypersingular integral equation.

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Thank you Martin for feedback, and Happy Birthday!