Boundary Integral Equations for the Transmission Eigenvalue Problem for Maxwell's Equations

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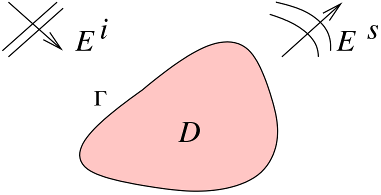
Scattering by an Inhomogeneous Media



$$\nabla \times \nabla \times E^s - k^2 E^s = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{D}$$
$$\nabla \times \nabla \times E - k^2 N(x) E = 0 \quad \text{in } D$$
$$\nu \times E = \nu \times (E^s + E^i) \quad \text{on } \Gamma$$
$$\nu \times \nabla \times E = \nu \times \nabla \times (E^s + E^i) \quad \text{on } \Gamma$$
$$\lim_{|x| \to \infty} (\nabla \times E^s \times x - ik|x|E^s) = 0$$

$N = \dfrac{\epsilon(x)}{\epsilon_0}$ is relative electric permittivity, k the wave number, and E^i incident electric field.

Question: Is there an incident wave E^i that does not scatter?

The answer to this question leads to the transmission eigenvalue problem.

Transmission Eigenvalues

If there exists a nontrivial solution to the transmission eigenvalue problem

$$
\begin{aligned}
\nabla \times \nabla \times E - k^2 N(x) E = 0 \quad & \text{in} \quad D \\
\nabla \times \nabla \times E_0 - k^2 E_0 = 0 \quad & \text{in} \quad D \\
\nu \times E = \nu \times E_0 \quad & \text{on} \quad \Gamma \\
\nu \times (\nabla \times E) = \nu \times (\nabla \times E_0) \quad & \text{on} \quad \Gamma
\end{aligned}
$$

such that E_0 can be extended outside D as a solution to $\nabla \times \nabla \times E_0 - k^2 E_0 = 0$, then the scattered field due to this extended field as incident wave is identically zero.

Values of k for which this problem has a non trivial solution are referred to as transmission eigenvalues.

Transmission Eigenvalues

In general such an extension of E_0 does not exits!

Since superposition of plane waves, so-called electric Herglotz wave functions

$$E_g(x) := ik \int_{\Omega} e^{ikx\cdot d} g(d)ds(d)), \qquad g \in L^2_t(\Omega), \qquad \Omega := \{d : |d| = 1\}$$

or superposition of point sources

$$S_\varphi(x) := \nabla \times \nabla \times \int_{\Lambda} \varphi(y)\Phi(x,y)ds_y, \quad \varphi \in L^2_t(\Lambda),$$

where Λ is a surface in $\mathbb{R}^3 \setminus \overline{D}$ and $\Phi(x,y) = \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}$, are dense in

$$\left\{W \in L^2(\Omega) : \ \nabla \times \nabla \times W - k^2 W = 0\right\},$$

at a transmission eigenvalue there is an incident field that produces an arbitrarily small scattered field

Motivation

The transmission eigenvalue problem is non-selfadjoint and non-linear.

Why study the transmission eigenvalue problem?

- Fredholm property of the interior transmission problem. It arises in important questions such as uniqueness of inverse problems for inhomogenous media.
- Discreteness of transmission eigenvalues. Methods for solving the inverse problem for inhomogeneous media such as linear sampling method and factorization method fail at a transmission eigenvalue.
- Existence of transmission eigenvalues
  - Real transmission eigenvalues can be determined from the scattering data.
  - Transmission eigenvalues carry information about material properties.

Historical Overview

- The transmission eigenvalue problem in scattering theory was introduced by *Kirsch (1986)* and *Colton-Monk (1988)*
- Research was focused on the discreteness of transmission eigenvalues for variety of scattering problems: *Colton-Kirsch-Päivärinta (1989)* and *Rynne-Sleeman (1991)*.
- The first proof of existence of at least one transmission eigenvalues for large contrast *Päivärinta-Sylvester (2009)*.
- The existence of an infinite set of transmission eigenvalues is proven by *Cakoni-Gintides-Haddar (2010)*.
- The determination of real transmission eigenvalues from scattering data by *Cakoni-Colton-Haddar (2010)* improved for simple scattering problems by *Kirsch-Lechleiter (2013)*.
- Since the appearance of these papers there has been an explosion of interest in the transmission eigenvalue problem.

Special issue of Inverse Problems on Transmission Eigenvalues October 2013.

Transmission Eigenvalues

In a "natural" variational form this problem reads

$$\int_D (\nabla \times E) \cdot (\nabla \times \overline{E}') \, dx - \int_D (\nabla \times E_0) \cdot (\nabla \times \overline{E}_0') \, dx$$
$$-k^2 \int_D N E \cdot \overline{E}' \, dx + k^2 \int_D E_0 \cdot \overline{E}_0' \, dx = 0$$

for all $E', E_0' \in X(D)$, where

$$X(D) := \{(w, v) \in H(curl, D) \times H(curl, D) \,|\, \nu \times w = \nu \times v \text{ on } \Gamma\}.$$

Chesnel, Inverse Problems, 2012 – uses $\top$- coercivity to prove discreteness of TE for media $\nabla \times A \nabla \times E - k^2 N E = 0$, provided $A - I$ and $N - I$ are bounded away from zero in a neighborhood of Γ. Existence of TE in this case is proven under stronger assumptions on A and N (*Cakoni - Kirsch, (2010)).*

Transmission Eigenvalues

It is possible to write

$$\begin{array}{lcl} \nabla \times \nabla \times E - k^2 N E = 0 & \text{in} & D \\ \nabla \times \nabla \times E_0 - k^2 E_0 = 0 & \text{in} & D \\ \nu \times E = \nu \times E_0 & \text{on} & \Gamma \\ \nu \times (\nabla \times E) = \nu \times (\nabla \times E_0) & \text{on} & \Gamma \end{array}$$

$E, E_0 \in L^2(D)$, for the difference $W = E - E_0 \in H_0(curl^2, D)$ as

$$(\nabla \times \nabla \times -k^2)(N - I)^{-1}(\nabla \times \nabla \times -k^2 N)W = 0$$

i.e. in the variational form

$$\int_D (N - I)^{-1}(\nabla\times\nabla\times W - k^2 N W)(\nabla\times\nabla\times\overline{W}' - k^2\overline{W}')\, dx = 0, \qquad W' \in \mathcal{U}_0(D)$$

where $H_0(curl^2, D) := \{U \in H_0(curl, D) : \text{ such that } \nabla \times U \in H_0(curl, D)\}$

Transmission Eigenvalues

Definition

Transmission eigenvalues are values of $k \in \mathbb{C}$ for which the transmission eigenvalue problem has non-zero solutions $E \in L^2(D)$, $E_0 \in L^2(D)$ such that $(E - E_0) \in H_0(curl, D)$ and $\nabla \times (E - E_0) \in H_0(curl, D)$.

Theorem (Cakoni-Gintides-Haddar, SIAM J. Math. Anal. (2010))

Assume that either $N - I$ or $I - N$ is positive definite uniformly in D. Then:

- *the set of all transmission eigenvalues is at most discrete.*
- *there exists an infinite set of real transmission eigenvalues accumulating at $+\infty$.*

Note: The interior transmission problem with nonhomgeneous boundary data satisfies the Fredholm alternative.

Integral Equation Formulation

We use an alternative approach to study the transmission eigenvalue problem based on integral equations. The goal is:

- to relax the assumption on the sign of the contrast $N - I$.
- to provide an alternative approach suitable for computation of transmission eigenvalues.

The integral equation formulation of the transmission eigenvalue problem was first introduced for the scalar case in

Cossonnière-Haddar, Surface integral formulation of the interior transmission problem, *J. Int. Eqn. Appl. (to appear)*

Integral Equation Formulation

Throughout here we assume that Γ is smooth enough!

For the moment we consider $N = nI$ where $n > 1$ or $0 < n < 1$ is constant. Denote by $k_1 := \sqrt{n}k$ and

$$\Phi_k(x,y) = \frac{1}{4\pi}\frac{e^{ik|x-y|}}{|x-y|}.$$

From Stratton-Chu formula we have

$$E(x) := curl \int_\Gamma (E \times \nu)\Phi_{k_1}(\cdot, x)\, ds$$
$$+\frac{1}{{k_1}^2}\nabla \int_\Gamma div_\Gamma \cdot (curl\, E \times \nu)\Phi_{k_1}(\cdot, x)\, ds + \int_\Gamma (curl\, E \times \nu)\Phi_{k_1}(\cdot, x)\, ds$$

with similar expression for E_0 where we replace k_1 by k.

Integral Equation Formulation

Define the boundary integral operators:

$\mathbf{T}_k : H^{-1/2}(div, \Gamma) \to H^{-1/2}(curl, \Gamma)$

$$\mathbf{T}_k(\psi) := \gamma_\Gamma \left(k \int_\Gamma \psi(y)\Phi_k(\cdot, y)\, ds + \frac{1}{k}\, grad_\Gamma \int_\Gamma div_\Gamma \psi(y)\, \Phi_k(\cdot, y)\, ds \right)$$

and

$\mathbf{K}_k : H^{-1/2}(div, \Gamma) \to H^{-1/2}(curl, \Gamma)$

$$\mathbf{K}_k(\psi) := \gamma_\Gamma \left(curl \int_\Gamma \psi(y)\Phi_k(\cdot, y)\, ds \right)$$

where $\gamma_\Gamma u := \nu \times (u \times \nu)$

Integral Equation Formulation

Taking the tangential traces we have

$$\begin{aligned}\gamma_\Gamma E &= \mathbf{K}_{k_1}(E\times\nu)+\frac{1}{k_1}\mathbf{T}_{k_1}(curl\,E\times\nu)=\frac{1}{2}\gamma_\Gamma E\\ \gamma_\Gamma(curl\,E) &= \mathbf{K}_{k_1}(curl\,E\times\nu)+k_1\mathbf{T}_{k_1}(E\times\nu)=\frac{1}{2}\gamma_\Gamma(curl\,E)\end{aligned}$$

with similar expression for E_0 where we replace k_1 by k.

Recall that $k_1 := \sqrt{n}k$

Next consider the difference $E - E_0$ and the fact that the Cauchy data coincide on Γ.

Integral Equation Formulation

We formally obtain the following system of integral equations for

$$M := E \times \nu = E_0 \times \nu \qquad \text{and} \qquad J := (\nabla \times E) \times \nu = (\nabla \times E_0) \times \nu$$

$$\mathcal{L}(k) \begin{bmatrix} M \\ J \end{bmatrix} = \begin{bmatrix} k_1 \mathbf{T}_{k_1} - k\mathbf{T}_k & \mathbf{K}_{k_1} - \mathbf{K}_k \\ \mathbf{K}_{k_1} - \mathbf{K}_k & \frac{1}{k_1}\mathbf{T}_{k_1} - \frac{1}{k}\mathbf{T}_k \end{bmatrix} \begin{bmatrix} M \\ J \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Recall the Helmholtz orthogonal decomposition of tangential fields

$$U = curl_\Gamma p + \nabla_\Gamma q$$

Integral Equation Formulation

Note that $E \in L^2(D)$, $E_0 \in L^2(D)$ and hence $\nabla \times \nabla \times E \in L^2(D)$, $\nabla \times \nabla \times E_0 \in L^2(D)$. Hence $M \in H_t^{-1/2}(\Gamma)$ and

$$J \in \mathcal{H}(\Gamma) := \left\{ u \in H_t^{-3/2}(\Gamma) \text{ such that } \; div_\Gamma\, u \in H^{-1/2}(\Gamma) \right\}.$$

The dual $\mathcal{H}^*(\Gamma)$ (with respect to L^2-inner product) is

$$\mathcal{H}^*(\Gamma) := \left\{ u \in H_t^{-1/2}(\Gamma) \text{ such that } \; curl_\Gamma\, u \in H^{1/2}(\Gamma) \right\}.$$

Lemma

For a fixed k, the linear operator $\mathcal{L}(k) : H_t^{-1/2}(\Gamma) \times \mathcal{H}(\Gamma) \to H_t^{1/2}(\Gamma) \times \mathcal{H}^*(\Gamma)$ is bounded. The family of operators $\mathcal{L}(k)$ depends analytically on $k \in \mathbb{C} \setminus \mathbb{R}_-$.

Integral Equation Formulation

The following statements are equivalent:

1. There exists $E, E_0 \in L^2(D)$, $E - E_0 \in H(curl^2, D)$ a non trivial solution of TEP.
2. There exists $M \in H_t^{-1/2}(\Gamma)$ and $J \in \mathcal{H}(\Gamma)$ nonzero such that

$$\mathcal{L}(k) \begin{bmatrix} M \\ \\ J \end{bmatrix} = 0, \text{ and either } E_0^\infty(M, J) = 0 \text{ or } E_1^\infty(M, J) = 0$$

where

$$E_1^\infty(M, J)(\hat{x}) = \hat{x} \times \left(\frac{1}{4\pi} curl \int_\Gamma M\, e^{-ik_1 \hat{x}\cdot y}\, ds_y \right.$$
$$\left. + \frac{1}{4\pi {k_1}^2} \nabla \int_\Gamma div_\Gamma J\, e^{-ik_1 \hat{x}\cdot y}\, ds + \int_\Gamma J\, e^{-ik_1 \hat{x}\cdot y}\, ds_y \right) \times \hat{x}$$

with same expression for $E_0^\infty(M, J)$ where we replace k_1 by k.

Integral Equation Formulation

- $\mathbf{S}_{k_1} - \mathbf{S}_k$ is smoothing of order 3 where

$$\mathbf{S}_k \psi := \int_\Gamma \psi(y) \Phi_k(\cdot, y)\, ds_y$$

- $\mathbf{K}_{k_1} - \mathbf{K}_k$ is a smoothing operator of order 2.

  (*see Cossonnière-Haddar, Hsiao-Wendland*)

- $k_1 \mathbf{T}_{k_1} - k\mathbf{T}_k = ({k_1}^2 \mathbf{S}_{k_1} - k^2 \mathbf{S}_k) + grad_\Gamma \circ (S_{k_1} - S_k) \circ div_\Gamma$
- $\frac{1}{k_1} \mathbf{T}_{k_1} - \frac{1}{k} \mathbf{T}_k = (\mathbf{S}_{k_1} - \mathbf{S}_k) + grad_\Gamma \circ \left(\frac{1}{{k_1}^2} S_{k_1} - \frac{1}{k^2} S_k \right) \circ div_\Gamma$

Integral Equation Formulation

Define $\mathcal{H}_0(\Gamma)$ the space of $u \in \mathcal{H}(\Gamma)$ such that $div_\Gamma u = 0$

Lemma

$\mathcal{L}(i|k|)$ is strictly coercive in $H_t^{-1/2}(\Gamma) \times \mathcal{H}_0(\Gamma)$.

$U \in L^2(\mathbb{R}^3)$, $\nabla \times U \in L^2(\mathbb{R}^3)$, $\nabla \times \nabla \times U \in L^2(\mathbb{R}^3)$ satisfying

$$\begin{aligned}
\left(\nabla \times \nabla \times + |k_1|^2\right)\left(\nabla \times \nabla \times + |k|^2\right) U &= 0 && \text{in } \mathbb{R}^3 \setminus \Gamma \\
[\nu \times \nabla \times \nabla \times U] &= M({k_1}^2 - k^2) && \text{on } \Gamma \\
[\nu \times \nabla \times \nabla \times \nabla \times U] &= J({k_1}^2 - k^2) && \text{on } \Gamma
\end{aligned}$$

for $M \in H_t^{-1/2}(\Gamma)$, $J \in \mathcal{H}_0(\Gamma)$.

Here $[\cdot]$ is the jump across Γ

Integral Equation Formulation

Lemma

Restricted to $H_t^{-1/2}(\Gamma) \times \mathcal{H}_0(\Gamma)$

$$\mathcal{L}(k) + \frac{{k_1}^2 - k^2}{|k_1|^2 - |k|^2} \mathcal{L}(i|k|)$$

is compact.

Using the Helmholtz orthogonal decomposition

$$J = P + Q, \quad P = curl_\Gamma p, \quad Q = \nabla_\Gamma q$$

and writing $\mathcal{L}(k)$ as a 3×3- matrix operator acting on $M \in H_t^{-1/2}(\Gamma)$, $P \in \mathcal{H}_0(\Gamma)$, Q with $div_\Gamma \in H^{-1/2}(\Gamma)$ we can now show:

Theorem

$\mathcal{L}(k) : H_t^{-1/2}(\Gamma) \times \mathcal{H}(\Gamma) \to H_t^{1/2}(\Gamma) \times \mathcal{H}^*(\Gamma)$ *is Fredholm operator of index zero. Its kernel fails to be trivial for at most a discrete set of values* $k \in \mathbb{C} \setminus \mathbb{R}_-$.

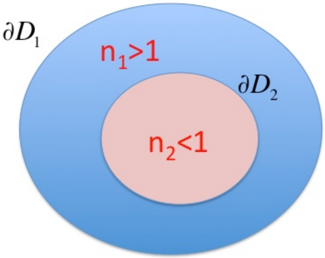
Discreteness of Transmission Eigenvalues

- If $0 < n \neq 1$ is constant, then the set of transmission eigenvalues is discrete.
- This approach does not provide existence of transmission eigenvalues.
- In the context of using transmission eigenvalues to obtain information on material properties, one needs to solve the transmission eigenvalue problem for homogeneous media.

  This approach provides an alternative computational framework to the finite element method for a forth order curl equation.

  (see *Kleefeld, Inverse Problems (2013)* for the scalar case)

Media with Contrast that Changes Sign



- $\mathcal{L}_{n_1}(k)$ is the operator we analyzed replacing n by n_1
- $\mathcal{A}_{12}^{-1}(k)$ is the operator corresponding the the transmission problem with interface Γ_2.
- $\mathcal{Z}^{\Gamma_2 \to \Gamma_1}$ and $\mathcal{Z}^{\Gamma_1 \to \Gamma_2}$ are compact operators.

$$\mathcal{L}(k) = \mathcal{L}_{n_1}(k) + \mathcal{Z}^{\Gamma_2 \to \Gamma_1} \mathcal{A}_{12}^{-1}(k) \mathcal{Z}^{\Gamma_1 \to \Gamma_2}$$

In the general case if N is such that in a neighborhood of Γ is positive constant greater or less than one, than

$$\mathcal{L}(k) = \mathcal{L}_{n_1}(k) + \text{compact perturbation.}$$

Media with Contrast that Changes Sign

- In general to show that there is $k \in \mathbb{C}$ which is not a transmission eigenvalue one needs to work directly on the the transmission eigenvalue problem in the PDE form. It is possible to show there is a k real large enough not transmission eigenvalue

  (*see Sylvester, SIAM J. Math Anal. (2012) for the scalar case*).
- Under the above assumption on N the set of transmission eigenvalues is discrete.
- The existence of transmission eigenvalues is open for electromagnetic scattering problem for inhomogenouos media with contrast changing sign. Progress is recently made in this direction for the scalar problem by Lakshtanov-Vainberg, and Robbiano, both to appear in the special issue of Inverse Problems.