On spectrum of an elastic solid with cusps

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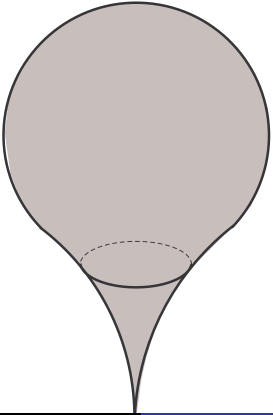
Elastic solid with a cusp

$$\Pi_d = \{x = (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R} : z = x_n \in (0, d),\ z^{-1-\gamma} y \in \omega\},$$

where d is a positive number, ω is a bounded, two-dimensional domain with Lipschitz boundary, $\gamma > 0$ (the sharpness exponent)



General second order system in n-dimensional domain

Consider the following spectral problem in variational formulation

$$a(u,v) = (A\mathcal{D}(\nabla_x)u, \mathcal{D}(\nabla_x)v)_\Omega = \lambda(u,v)_\Omega \qquad (1)$$

which must be satisfied for all $v \in C_c^\infty(\Omega \setminus \mathcal{O})^k$. Here u and v are k-vector functions, $\mathcal{D}(\nabla_x)$ is the first order matrix-operator of the size N and A is a positive definite matrix of the size $N \times N$. It is assumed that the matrix $\mathcal{D}(\xi)$ has rank k for every $\xi \in \mathbb{R}^n$. This implies that the form a degenerates only on a finite-dimensional space of polynomials

$$\mathcal{P} = \{p = p(x) : \mathcal{D}(\nabla_x)p(x) = 0,\ x \in \mathbb{R}^n\}$$

Denote by $\mathcal{E}$ the closure of $C_c^\infty(\Omega \setminus \mathcal{O})^k$ in the norm

$$a(u,u) + (u,u)_\Omega.$$

Denote by $\mathfrak{A}$ the Friedrichs extension corresponding to the form a on $C_c^\infty(\Omega \setminus \mathcal{O})^k$.

Second order system in n-dimensional domain

The following result is due S.Nazarov: Funkt. Anal. i Prilozhen. 2009. V. 43, N 1. P. 55–67.

Let J be the maximal power of z in polynomial from $\mathcal{P}$. Then

- if $J = 0$ then the spectrum of problem (1) is discrete;
- if $J > 0$ then the inclusion $\mathcal{E} \in L^2(\Omega)^k$ is not compact when $\gamma \geq 1/J$ and operator $\mathfrak{A}$ corresponding to the form a has continuous spectrum; moreover if $\gamma > 1/J$ then 0 belongs to the continuous spectrum.

These assertions contain no information about the discrete spectrum and on the structure of the continuous spectrum.

Elasticity problem

In the elasticity case $k = 3$ (or 2), $N = 6$ and

$$\mathcal{D}(\nabla_x)^T = \begin{pmatrix} \partial_1 & 0 & 2^{-1/2}\partial_2 & 2^{-1/2}\partial_3 & 0 & 0 \\ 0 & \partial_2 & 2^{-1/2}\partial_1 & 0 & 2^{-1/2}\partial_3 & 0 \\ 0 & 0 & 0 & 2^{-1/2}\partial_1 & 2^{-1/2}\partial_2 & \partial_3 \end{pmatrix}, \quad \partial_j = \frac{\partial}{\partial x_j}.$$

Indeed, the strain tensor is written as a column vector

$$\varepsilon(u)=(\varepsilon_{11}(u),\varepsilon_{22}(u),\sqrt{2}\varepsilon_{12}(u),\sqrt{2}\varepsilon_{13}(u),\sqrt{2}\varepsilon_{23}(u),\varepsilon_{33}(u))^T \quad (2)$$

where

$$\varepsilon_{jk}(u) = \frac{1}{2}\Big(\frac{\partial u_j}{\partial x_k} + \frac{\partial u_k}{\partial x_j}\Big), \quad j, k = 1, 2, 3. \quad (3)$$

The factor $\sqrt{2}$ is put into (2) in order to make the Euclidean norm of the strain vector (2) be equal to the corresponding Euclidean norm of the strain tensor. The stress column $\sigma(u)$ is defined similarly to (2) and is related with the strain column by Hooke's law

$$\sigma(u) = A\varepsilon(u),$$

where A is a symmetric positive definite matrix composed from elastic moduli. In the sequel we assume that A is constant, i.e. the elastic body Ω is homogeneous but anisotropic.
In new notation, (3) takes the form

$$\varepsilon(u) = D(\nabla_x)u,$$

where $D(\nabla_x)$ is the 6×3-matrix of first order differential operators defined above.
The form a degenerates on the vectors

$$(1,0,0), (0,1,0), (0,0,1),$$
$$(x_2, -x_1, 0), (x_3, 0, -x_1), (0, x_3, -x_2)$$

and hence $J = 1$. In this case one can say more about the spectrum of the operator $\mathfrak{A}$.

On spectrum of $\mathfrak{A}$

the following results are proved in
Nazarov S.A., Sibirsk. Mat. Zh. 2008. V. 49.
and
Bakharev F. L., Nazarov S.A., Sibirsk. Mat. Zh. 2009. V. 50.

- (N) The embedding $\mathcal{E} \subset H^1(\Omega)^3$ fails for $\gamma > 0$.
- (N) The embedding $\mathcal{E} \subset L^2(\Omega)^3$ fails for $\gamma > 1$ and is not compact for $\gamma = 1$.
- (N) If $\gamma \in (0, 1)$, then the spectrum is discrete.
- (N) If $\gamma = 1$, then there exist two positive numbers $\lambda_\bullet < \lambda_\dagger$ such that the half-open interval $[0, \lambda_\bullet)$ contains the unique eigenvalue $\lambda = 0$ with multiplicity six and eigenspace (**??**), and the ray $[\lambda_\dagger, \infty)$ is included in the continuous spectrum.
- (BN) If $\gamma > 1$, then the continuous spectrum of $\mathfrak{A}$ coincides with $[0, \infty)$.
- (N) Another interesting property is the presence of eigenvalues on the continuous part of the spectrum when ω posses some symmetries and $\gamma > 1$.

Black Hole

Let us mention an important application of the above phenomenon, i.e. appearance of the continuous spectrum for $\gamma \geq 1$. So called Vibration Black Holes are used in the modern engineering as dampers and filters for elastic and acoustic waves (see M.A. Mironov, Propagation of a exural wave in a plate whose thickness decreases smoothly to zero in a finite interval. Soviet Physics: Acoustics 34 nr. 3 (1988), pp. 318-319. Krylov V.V., New type of vibration dampers utilising the effect of acoustic "black holes", Acta Acustica united with Acustica 90 nr. 5 (2004), pp. 830-837). These applications are based on a discovered in M.A.M. and verified experimentally effect of absorbing of an elastic wave by sharp cusps at the boundary ($\gamma = 1$). Namely, after the contact with this sharpening, the propagation velocity of the elastic wave decreases logarithmically when approaching the top of the cusp and hence it requires infinite time for reflection by the cusp top and return of the wave to the massive part of the body.

Black Hole

In other words, elastic cusps play a role of a storage of the elastic energy. Corresponding engineering justification were carried out on the basis of an engineering theory of converging beam. A first mathematical explanation of the above effect was presented in S.Nazarov, Sibirsk. Mat. Zh. 2008.

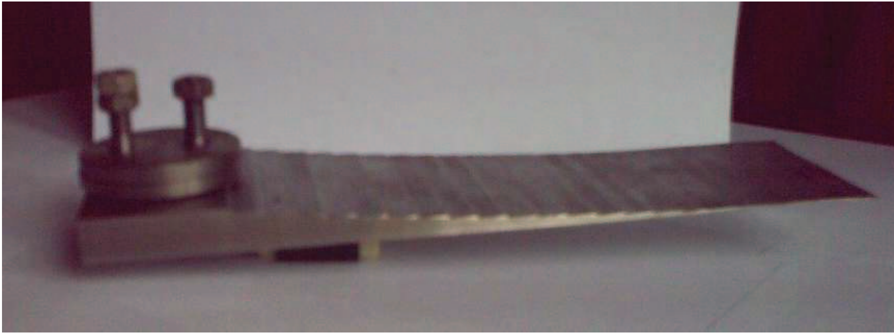


Figure:

Main results

Consider the case $\gamma = 1$. The main result is the existence of a threshold $\lambda_\dagger > 0$ such that

- the half-line $[\lambda_\dagger, \infty)$ consists of continuous spectrum of $\mathfrak{A}$;
- the interval $[0, \lambda_\dagger)$ contains only discrete spectrum of the operator $\mathfrak{A}$.

Here

$$\lambda_\dagger = \frac{5^2 7^2 \mu_1}{16|\omega|},$$

where μ_1 is the least positive eigenvalue of the matrix $\mathcal{M}$, defined later.

The study of spectrum in the interval $[0, \lambda_\dagger)$ is based on the following assertion

Let $\lambda < \lambda_\dagger$ and let $u \in \mathcal{E}(\Omega)$ solve the problem

$$a(u, v) - \lambda(u, v)_\Omega = (f, v)_\Omega, \quad v \in \mathcal{E}(\Omega),$$

where $f \in L^2(\Omega)^3$. Then there exist a positive number $\sigma < d$ such that

$$||u; \mathcal{E}(\Omega)||^2 \leq C\big(||f; L^2(\Omega)^3||^2 + ||u; L^2(\Omega_\sigma)^3||^2\big),$$

where $\Omega_\sigma = \Omega \setminus \Pi_\sigma$. The constant C here may depend on λ.

The stiffness matrix $\mathcal{A}$

It appears that an important role in describing of the behavior of solutions to the spectral problem near the cusp is played by a 4×4 matrix $\mathcal{A}$. Let us describe this matrix. We denote by Y_k, $k = 1, 2, 3, 4$, the kth column in the matrix

$$\mathcal{Y} = \begin{pmatrix} 0 & 0 & 0 & -2^{-1/2}\eta_2 \\ 0 & 0 & 0 & 2^{-1/2}\eta_1 \\ \eta_1 & \eta_2 & 1 & 0 \end{pmatrix}.$$

Let also $X_k \in H^1(\omega)^3$ be a solution to the following two-dimensional elasticity problem

$$(AD(\nabla_\eta, 0)X, D(\nabla_\eta, 0)w)_\omega = (AD(0,1)Y_k, D(\nabla_\eta, 0)w)_\omega, \quad (4)$$

where the test function w belongs to $H^1(\omega)^3$ and $\eta = (\eta_1, \eta_2) \in \omega$. This problem is solvable since the right-hand side is zero when $w = \mathbf{e}_{(k)}$, $k = 1, 2, 3, 4$, where

$$\mathbf{e}_{(1)} = (1, 0, 0),\ \mathbf{e}_{(2)} = (0, 1, 0),\ \mathbf{e}_{(3)} = (0, 0, 1),$$

$$\mathbf{e}_{(4)} = \theta := 2^{-1/2}(\eta_1 \mathbf{e}_{(2)} - \eta_2 \mathbf{e}_{(1)}).$$

These functions constitute the kernel of the form in the left hand side of (4). The matrix $\mathcal{A}$ consists of elements

$$\mathcal{A}_{kj} = (AD(\nabla_\eta, 0)X_j, D(0,1)\eta_k \mathbf{e}_{(3)})_\omega - (AD(0,1)X_j, D(0,1)\eta_k \mathbf{e}_{(3)})_\omega$$

for $k = 1, 2$, and

$$\mathcal{A}_{kj} = (AD(\nabla_\eta, 0)X_j, D(0,1)\mathbf{e}_{(k)})_\omega - (AD(0,1)X_j, D(0,1)\mathbf{e}_{(k)})_\omega$$

for $k = 3, 4$. Here $j = 1, 2, 3, 4$ is the number of the column and $k = 1, 2, 3, 4$ is the number of the row. This matrix is positive definite. We represent the matrix $\mathcal{A}$ block-wise

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}^{\dagger\dagger} & \mathcal{A}^{\dagger\sharp} \\ \mathcal{A}^{\sharp\dagger} & \mathcal{A}^{\sharp\sharp} \end{pmatrix}, \qquad (5)$$

where block entries are 2×2-matrix. Then

$$\mathcal{M} = \mathcal{A}^{\dagger\dagger} - \mathcal{A}^{\dagger\sharp}(\mathcal{A}^{\sharp\sharp})^{-1}\mathcal{A}^{\sharp\dagger} \qquad (6)$$

is a positive definite matrix also with eigenvalues $\mu_1 \leq \mu_2$.

Asymptotic representation of solutions to the spectral problem near the vertex of the cusp.

In order to describe this result let us consider the following system of ordinary differential equations

$$(\partial_t - 2t^{-1})(-\partial_t + 4t^{-1})\big(\mathcal{A}^{\dagger\dagger}(\partial_t + 2t^{-1})\partial_t(w_1, w_2)^T + \mathcal{A}^{\dagger\sharp}(\partial_t w_j, (\partial$$
$$+\lambda t^{-4}|\omega|(w_1, w_2)^T = 0$$

and

$$(\partial_t - 2t^{-1}, -\partial_t + 4t^{-1})\Big(\mathcal{A}^{\sharp\dagger}(\partial_t + 2t^{-1})\partial_t(w_1, w_2)^T$$
$$+\mathcal{A}^{\sharp\sharp}(\partial_t w_3, (\partial_t + 2t^{-1})w_4)^T\Big) = 0.$$

This system has twelve linear independent solutions:
$\mathbf{w}^s = (w_1^s, w_2^s, w_3^s, w_4^s)$, $s = 1, \dots, 12$.

Then the asymptotic representation of solution of the spectral problem subject to

$$\int_{\Pi_\varsigma} (|u|^2 + |\sigma(u)|^2) z^N \, dzdy < \infty$$

for a certain N, can be written as

$$u(y,z) = \sum_{s=1}^{12} C_s \Big(\sum_{k=1}^{4} w_k^s(z) \mathbf{e}_{(k)} + (y_1 \partial_z w_1^s(z) + y_2 \partial_z w_1^s(z)) \mathbf{e}_{(3)} \Big) + \cdots$$

Some ideas of the proof

1. **Change of variables.** Applying the change of variables in a neighborhood of the cusp:

$$\eta_k = z^{-2} y_k, \quad k = 1,2, \quad t = z^{-1}, \quad U(\eta, t) = u(y(\eta, t), z(\eta, t)),$$

we obtain

$$\int_{\mathcal{C}_\sigma} (A(D(\nabla_\eta, -\partial_t - 2t^{-1}\varrho(\eta, \nabla_\eta))U, (D(\nabla_\eta, -\partial_t - 2t^{-1}\varrho(\eta, \nabla_\eta))v)t^{-2} d\eta dt$$
$$= \lambda \int_{\mathcal{C}_\sigma} (U, v)\, t^{-6} d\eta\, dt, \tag{7}$$

where $\mathcal{C}_\sigma = \omega \times (\sigma, \infty)$ and $\varrho(\eta, \nabla_\eta) = \eta_1 \partial_{\eta_1} + \eta_2 \partial_{\eta_2}$.

2. **ODE with operator coefficients.** We put

$$H_0 = (L^2(\omega))^3, \quad H_1 = (H^1(\omega))^3 \ \text{ and } \ H_{-1} = (H^1(\omega)^*)^3$$

Then (7) transforms into the abstract equation

$$\partial_t((\mathcal{A}_0(\partial_t) - \mathcal{N}_0(t))u(t)) + (\mathcal{A}_1(\partial_t) - \mathcal{N}_1(t, \partial_t))u(t) = t^{-4}\lambda u(t).$$

Here,

$$\mathcal{A}_0(\partial_t) = -\mathbf{A}_{00}\partial_t + \mathbf{A}_{01}, \;\; \mathcal{A}_1(\partial_t) = -\mathbf{A}_{10}\partial_t + \mathbf{A}_{11}$$

and

$$\mathcal{N}_0(t) = \frac{1}{t}\mathbf{M}_1, \;\; \mathcal{N}_1(t, \partial_t) = \frac{1}{t}\Big(\mathbf{M}_0' + \mathbf{M}_1'\Big) - \frac{1}{t}\mathbf{M}_0\partial_t - \frac{1}{t^2}\mathbf{M}.$$

3. **Reduction to a first order system** We introduce

$$\mathcal{U} = (\mathcal{U}_1, \mathcal{U}_2), \quad \mathcal{U}_1 = u \quad \text{and} \quad \mathcal{U}_2 = \mathcal{A}_0(\partial_t)u - \mathcal{N}_0(t)u.$$

From the last relation, one finds

$$\partial_t \mathcal{U}_1 = \mathbf{A}_{00}^{-1}\big(\mathbf{A}_{01}\mathcal{U}_1 - \mathcal{U}_2 - \mathcal{N}_0\mathcal{U}_1\big).$$

The above abstract equation can be written in new notation as

$$\partial_t \mathcal{U} + \mathfrak{A}\mathcal{U} = t^{-1}\mathfrak{N}\mathcal{U} + \lambda t^{-4}(0, \mathcal{U}_1)^T + +t^{-4}(0, f)^T,$$

where

$$\mathfrak{A} = \begin{pmatrix} -\mathbf{A}_{00}^{-1}\mathbf{A}_{01} & \mathbf{A}_{00}^{-1} \\ -\mathbf{A}_{10}\mathbf{A}_{00}^{-1}\mathbf{A}_{01} + \mathbf{A}_{11} & \mathbf{A}_{10}\mathbf{A}_{00}^{-1} \end{pmatrix}$$

and

$$\mathfrak{N} = \begin{pmatrix} -\mathbf{A}_{00}^{-1}\mathbf{M}_1 & 0 \\ 0 & \mathbf{M}_0\mathbf{A}_{00}^{-1} \end{pmatrix}.$$

4. **Spectral splitting.** Consider the equation

$$\mathcal{A}(\partial_t)U(t) = F(t) \quad \text{for } t \in \mathbb{R}, \tag{8}$$

where

$$\mathcal{A}(\partial_t) = \partial_t \mathcal{A}_0(\partial_t) + \mathcal{A}_1(\partial_t).$$

We associate with the problem (8) the operator pencil

$$\mathcal{A}(\lambda) : H_1 \to H_{-1}.$$

The spectrum of the pencil $\mathcal{A}(\lambda)$ consists of isolated eigenvalues of finite algebraic multiplicity and every strip $-K < \Re\lambda < K$ contains only finite number of such eigenvalues. Moreover there is only one eigenvalue $\lambda = 0$ on the line $\Re\lambda = 0$ and its geometric multiplicity is 4, partial multiplicities are 4, 4, 2 and 2 and hence the algebraic multiplicity is 12. This eigenvalue generates the 12 -dimensional space $\mathcal{Z}$ of polynomial solutions to the homogeneous equation

$$\mathcal{A}(\partial_t)U(t) = 0.$$

Spectral splitting

This implies that the operator $\mathfrak{A}$ has the eigenvalue 0 of geometric multiplicity 4 and algebraic multiplicity 12. Let $\mathcal{P}$ be the projector on the eigenvalue subspace of the operator $\mathfrak{A}$ corresponding to the eigenvalue 0.

Let us consider the problem

$$\partial_t \mathcal{U} + \mathfrak{A}\mathcal{U} = t^{-1}\mathfrak{N}\mathcal{U} + \lambda t^{-4}\mathfrak{J}\mathcal{U} + \mathcal{F}, \tag{9}$$

where $\mathcal{F} = t^{-4}(0, f)$ and $\mathfrak{J}\mathcal{U} = (0, \mathcal{U}_1)^T$. We put

$$\mathbf{u}(t) = \mathcal{P}\mathcal{U}(t), \quad \mathbf{v}(t) = (\mathcal{I} - \mathcal{P})\mathcal{U}(t).$$

Using these vector functions we write system (9) as

$$(\partial_t + \mathfrak{A})\mathbf{u}(t) - t^{-1}\mathcal{P}\mathfrak{N}\mathbf{u} - \lambda t^{-4}\mathcal{P}\mathfrak{J}\mathbf{u} = t^{-1}\mathcal{P}\mathfrak{N}\mathbf{v} + \lambda t^{-4}\mathcal{P}\mathfrak{J}\mathbf{v} + \mathcal{P}\mathcal{F}$$

and

$$(\partial_t + \mathfrak{A})\mathbf{v}(t) = t^{-1}(\mathcal{I}-\mathcal{P})\mathfrak{N}(\mathbf{u}+\mathbf{v}) + \lambda t^{-4}(\mathcal{I}-\mathcal{P})\mathfrak{J}(\mathbf{u}+\mathbf{v}) + (\mathcal{I}-\mathcal{P})\mathcal{F}.$$

Scaling of vector functions **u** and **v**

We introduce the operator $\Lambda = \Lambda(t)$ by

$$\Lambda \mathbf{U} = \sum_{k=1}^{2} \sum_{j=0}^{3} t^{j-3} U_{kj} \mathcal{U}_{kj} + \sum_{k=3}^{4} \sum_{j=0}^{1} t^{j-2} U_{kj} \mathcal{U}_{kj},$$

where

$$\mathbf{U} = \sum_{k=1}^{2} \sum_{j=0}^{3} U_{kj} \mathcal{U}_{kj} + \sum_{k=3}^{4} \sum_{j=0}^{1} U_{kj} \mathcal{U}_{kj}.$$

We use the following change of variables

$$\mathbf{u} = \Lambda \mathbf{U}, \quad \mathbf{v} = t^{-3} \mathbf{V}$$

and

$$\mathbf{F} = t \Lambda^{-1} \mathcal{P} \mathcal{F} \ \text{ and } \ \mathbf{G} = t^3 (\mathcal{I} - \mathcal{P}) \mathcal{F}.$$

After these substitutions we arrive at

$$(\mathcal{S}t\partial_t + \mathcal{T})\mathbf{U} = -t^{-1}\mathcal{K}(t^{-1})\mathbf{U} + t^{-2}\mathcal{L}_{12}(t^{-1})\mathbf{V} + \mathbf{F}$$

and

$$(\partial_t+\mathfrak{A})\mathbf{V}(t)=(\mathcal{I}-\mathcal{P})\mathfrak{N}\sum_{k=3}^{4} U_{k0}\mathcal{U}_{k0}+t^{-1}\big(\mathcal{L}_{21}(t^{-1})\mathbf{U}+\mathcal{L}_{22}(t^{-1})\mathbf{V}\big)+\mathbf{G}$$

The equation

$$(\mathcal{S}t\partial_t + \mathcal{T})\mathbf{U} = \mathbf{F}$$

in the coordinate form can be written as

Operator $(\mathcal{S}t\partial_t + \mathcal{T})$

$$t\partial_t U_{k3} - U_{k2} = F_{k3},\ (t\partial_t + 1)U_{k2} - U_{k1} = F_{k2},\quad k = 1, 2,$$

$$(t\partial_t - 1)U_{31} - U_{30} - 2\sum_{m=1}^{2} \gamma_m U_{m2} = F_{31},$$
$$(t\partial_t + 1)U_{41} - U_{40} = F_{41}.$$

$$\sum_{j=1}^{2} \mathcal{M}_{kj}(t\partial_t - 5)U_{j0} - \lambda|\omega|U_{k3} = \sum_{j=1}^{2} \mathcal{M}_{kj}F_{j0},\quad k = 1, 2.$$

$$(t\partial_t - 4)U_{30} = F_{30},\ (t\partial_t - 6)U_{40} = F_{40}.$$

$$(t\partial_t - 6)U_{k1} - U_{k0} - 2a_k \sum_{n=3}^{4} U_{n0}\mathcal{A}_{n3} = F_{k1},\quad k = 1, 2.$$

Homogeneous system $(\mathcal{S}t\partial_t + \mathcal{T})\mathbf{U} = 0$

Simple calculations show that this system has solutions of order t^{-1}, t and t^4, t^6. Other eight solutions are found from

$$\sum_{j=1}^{2} \mathcal{M}_{kj} p(t\partial_t) U_{j3} - \lambda|\omega| U_{k3} = 0, \quad k = 1, 2, \tag{10}$$

where

$$p(z) = (z-6)(z-5)(z+1)z = \left(\left(z - \frac{5}{2}\right) - \frac{25}{4}\right)\left(\left(z - \frac{5}{2}\right) - \frac{49}{4}\right).$$

In order to solve (10) we note that the matrix $\mathcal{M}$ is positive definite because the same is valid for the matrix $\mathcal{A}$. We denote by μ_1, μ_2 the eigenvalues of $\mathcal{M}$, we fix them to satisfy $0 < \mu_1 \leq \mu_2$. The corresponding eigenvectors $\mathbf{w}^1$ are $\mathbf{w}^2$ are orthogonal to each other. We introduce representations

$$(w_1(t), w_2(t)) = \alpha_1(t)\mathbf{w}^1 + \alpha_2(t)\mathbf{w}^2.$$

Homogeneous system $(\mathcal{S}t\partial_t + \mathcal{T})\mathbf{U} = 0$

Then equation (10) converts into

$$p(t\partial_t)\alpha_k = \kappa_k\alpha_k, \quad \kappa_k = \frac{\lambda|\omega|}{\mu_k}, \quad k = 1, 2.$$

We need the roots of the equation

$$p(z) = \kappa, \quad \kappa \geq 0. \tag{11}$$

If $\kappa < 5^2 7^2/4^2$ then

$$\Lambda = \frac{5}{2} \pm z_\pm^{1/2}, \tag{12}$$

where

$$z_\pm = z_\pm(\kappa) = \frac{1}{8}\Big(7^2 + 5^2 \pm \sqrt{(7^2 - 5^2)^2 + 64\kappa}\Big).$$

If $\kappa > 5^2 7^2/4^2$ then equation (11) has two roots (12) with z_+ and two roots

$$\Lambda = \frac{5}{2} \pm i(-z_-)^{1/2}. \tag{13}$$

If $\kappa = 5^2 7^2/4^2$ then equation (11) has two roots (12) with z_+ and

Asymptotics of eigenfunctions near the cusp

If $\lambda < \lambda_\dagger$ then the corresponding eigenfunction has the following asymptotics

$$u(y,z) = \sum_{k=1}^{4} \alpha_k \Big(\sum_{j=1}^{2} w_{kj}(z) e_{(j)} - \sum_{j=1}^{2} y_j \partial_z w_{kj}(z) e_{(3)} \Big) + \cdots ,$$

where

$$(w_{k1}(z), w_{k2}(z)) = z^{-\Lambda_{\pm}(\kappa_s)} \mathbf{w}_s$$

THANK YOU