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MELLIN CONVOLUTION OPERATORS IN BESSEL POTENTIAL SPACES WITH ADMISSIBLE MEROMORPHIC KERNELS

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INTRODUCTION

In recent years there is a substantial interest to investigate the following problem: look for a vector-function $u(x) = (u_1(x), u_2(x), u_3(x))^\top$ **in two neighbouring domains** Ω_1 **and** Ω_2

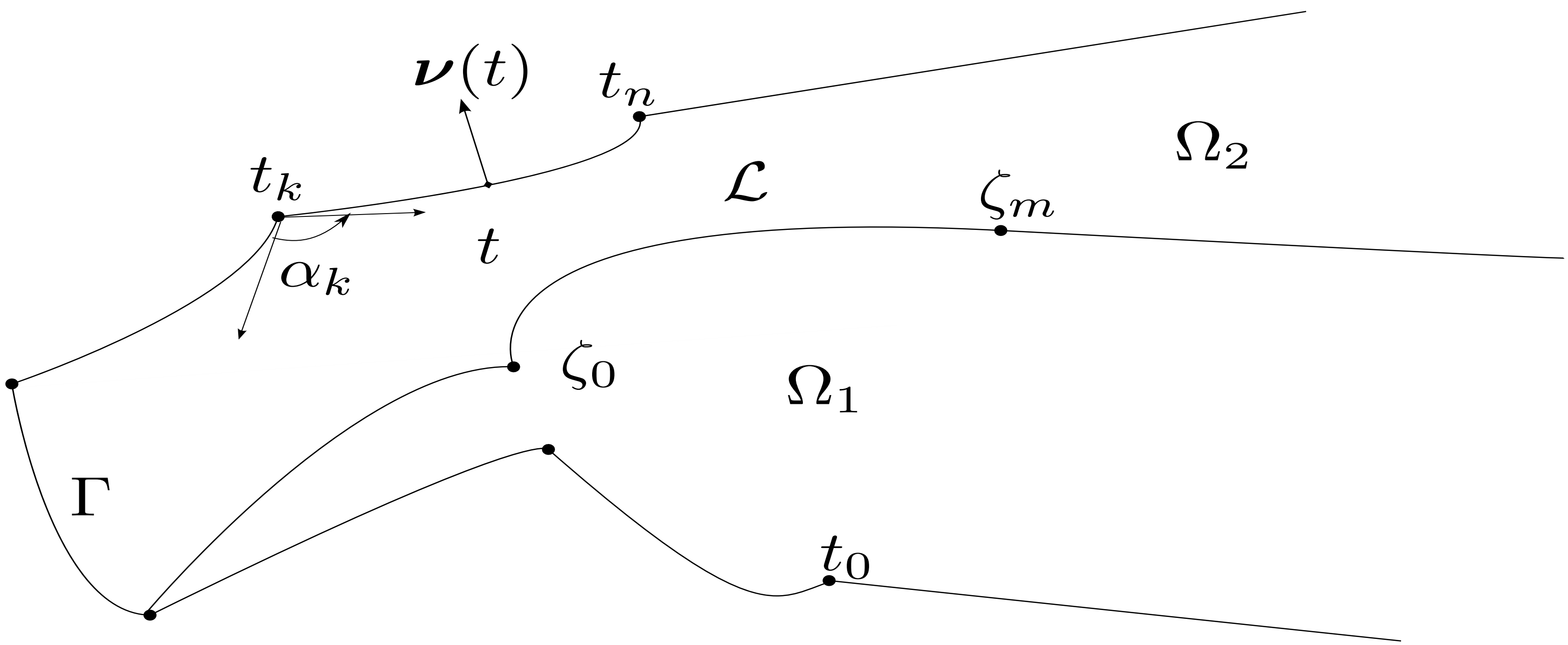

Fig. 1

which solves an "anisotropic" Helholtz system

$$\begin{cases} \operatorname{div}\mathscr{E}_1 \operatorname{grad} u + k_1^2 u = 0 \quad \textbf{in} \quad \Omega_1, \\ \operatorname{div}\mathscr{E}_2 \operatorname{grad} u + k_2^2 u = 0 \quad \textbf{in} \quad \Omega_2, \\ [\partial_{\boldsymbol{\nu}} u]^+ = h \quad \text{or} \quad u^+ = g \quad \textbf{on} \quad \Gamma := \partial(\Omega_1 \cup \Omega_2), \\ u^-(t) = u^+(t), \qquad [\partial_{\boldsymbol{\nu}} u]^-(t) = [\partial_{\boldsymbol{\nu}} u]^+(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad \textbf{on} \quad \mathscr{L} := \partial\Omega_1 \cap \partial\Omega_2, \end{cases} \tag{1}$$

when material constants filling neighboring domains Ω_1 and Ω_2 "change sign": the 3×3 matrix $\mathscr{E}_1$ is negative definite, while $\mathscr{E}_2$ is positive definite; $\Gamma = \partial(\Omega_1 \cup \Omega_2)$ is the boundary of the unified domain $\Omega := \Omega_1 \cup \Omega_2$, while $\mathscr{L} := \partial\Omega_1 \cap \partial\Omega_2$ is the interface.

If $\mathscr{E}_j = \text{const} = c_j$, then

$$\operatorname{div}\mathscr{E}_j \operatorname{grad} u(x) + k_j^2 u(x) = c_j \Delta\, u(x) + k_j^2 u(x) \quad \textbf{in} \quad \Omega_j,$$

and we have usual "isotropic" vector Helmholtz equation.

Similar BVPs can be formulated for any elliptic equation which we encounter in Mathematical Physics, e.g., for Láme system.

A strong interest to BVPs (1) is motivated by the rapid expansion of research into nanophotonics: Ω_2 is occupied by metamaterial and Ω_1 is occupied by dielectric material. The goal is to design highly integrated photonic signal-processing systems, nanoresolution optical imaging techniques and sensors.

In the recent papers by A. Bonnet-Ben Dhia, L. Chesnel, P. Ciarlet, Jr, X. Claeys, M. Dauge and others spectral properties of BVPs type (1) were investigated in cases, when $\mathscr{E}_1$, $\mathscr{E}_2$ are scalars, but variable functions and boundary conditions are zero. The investigations are carried out with the help of Lax-Milgram Lemma and its modification for T-coercive operators. We had some such reports on the present conference.

The purpose of our research is to find a criterion of unique solvability of BVPs (1) with the classical potential method reducing the BVPs to equivalent boundary integral equations and investigating them in Bessel

potential spaces $\mathbb{H}_p^s$ for arbitrary $s \in \mathbb{R}$ and $1 < p < \infty$.

If we consider the BVP (1) in the classical finite energy setting, i.e., look for a solution in the Sobolev space $\mathbb{W}_2^1(\Omega_1 \cap \Omega_2)$, then the trace of a solution $u^+(t)$ on the boundary $\mathscr{L} := \partial\Omega_1 \cap \partial\Omega_2$ belongs to the Bessel potential space $\mathbb{H}_2^{1/2}(\mathscr{L})$ and we are forced to treat the corresponding boundary integral equations in the Bessel potential spaces.

Moreover, if we consider BVP's in the non-classical setting, namely in the Sobolev space $\mathbb{W}_p^1(\Omega_1 \cap \Omega_2)$ setting, the corresponding boundary integral equations should be treated in the Besov (Sobolev-Slobodeckij) spaces $\mathbb{W}_p^{1-1/p}(\mathscr{L}) = \mathbb{B}_{p,p}^{1-1/p}(\mathscr{L})$.

1. MELLIN AND FOURIER CONVOLUTION OPERATORS AND BESSEL POTENTIALS

The participants of this conference need't to explain what is a Sobolev $\mathbb{W}^m_p(\mathbb{R}^n)$, $m = 1, 2, \ldots$, $1 < p < \infty$, Bessel potential $\mathbb{H}^s_p(\mathbb{R}^n)$, $s \in \mathbb{R}$, Besov (Sobolev-Slobodeckij) $\mathbb{W}^s_p(\mathbb{R}^n) = \mathbb{B}^s_{p,p}(\mathbb{R}^n)$ spaces, also on domains with boundary $\widetilde{\mathbb{H}}^s_p(\Omega)$, $\mathbb{H}^s_p(\Omega)$ and $\widetilde{\mathbb{W}}^s_p(\Omega)$, $\mathbb{W}^s_p(\Omega)$ etc. and we skip the definitions.

The integral operator

$$A\varphi(t) := c_0\varphi(t) + \frac{c_1}{\pi i}\int_0^\infty \frac{\varphi(\tau)\,dt}{\tau - t} + \int_0^\infty \mathscr{K}\left(\frac{t}{\tau}\right)\varphi(\tau)\frac{d\tau}{\tau} = f(t), \quad (2)$$

with the kernel $\mathscr{K}$ satisfying the condition

$$\int_0^\infty t^{\beta-1}|\mathscr{K}(t)\,dt < \infty, \quad 0 < \beta < 1, \quad (3)$$

is a classical Mellin convolution. More generally, if $a \in \mathbb{L}_\infty(\mathbb{R})$ is an essentially bounded measurable $N \times N$ matrix function, the Mellin con-

volution operator $\mathfrak{M}_a^0$ is defined by

$$\mathfrak{M}_a^0\varphi(t) := \mathscr{M}_\beta^{-1} a \mathscr{M}_\beta \varphi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} a(\xi) \int_0^\infty \left(\frac{t}{\tau}\right)^{i\xi-\beta} \varphi(\tau)\frac{d\tau}{\tau}\, d\xi, \qquad \varphi \in \mathbb{S}(\mathbb{R}^+),$$

where $\mathbb{S}(\mathbb{R}^+)$ is the Schwartz space of fast decaying functions on $\mathbb{R}^+$, whereas $\mathscr{M}_\beta$ and $\mathscr{M}_\beta^{-1}$ are the Mellin transform and its inverse, i.e.

$$\mathscr{M}_\beta\psi(\xi) := \int_0^\infty t^{\beta-i\xi}\psi(t)\frac{dt}{t}, \quad \xi \in \mathbb{R},$$

$$\mathscr{M}_\beta^{-1}\varphi(t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} t^{i\xi-\beta}\varphi(\xi)d\xi, \qquad t \in \mathbb{R}^+.$$

The function $a(\xi)$ is usually referred to as symbol of the Mellin operator $\mathfrak{M}_a^0$. Further, if the corresponding Mellin convolution operator $\mathfrak{M}_a^0$ is bounded on the weighted Lebesgue space $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ of N-vector

functions endowed with the norm

$$\|\varphi|\mathbb{L}_p(\mathbb{R}^+, t^\gamma)\| := \left[\int_0^\infty t^\gamma\, |\varphi(t)|^p\, dt\right]^{1/p},$$

then the symbol $a(\xi)$ is called an $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$Mellin multiplier. The transformations

$$\mathbf{Z}_\beta \ : \mathbb{L}_p(\mathbb{R}^+, t^\gamma) \to \mathbb{L}_p(\mathbb{R}), \qquad \mathbf{Z}_\beta\varphi(\xi) := e^{-\beta\xi}\varphi(e^{-\xi}), \quad \xi \in \mathbb{R},$$
$$\mathbf{Z}_\beta^{-1} : \mathbb{L}_p(\mathbb{R}) \to \mathbb{L}_p(\mathbb{R}^+, t^\gamma), \qquad \mathbf{Z}_\beta^{-1}\psi(t) := t^{-\beta}\psi(-\ln\, t), \quad t \in \mathbb{R}^+$$

generate an isometrical isomorphism between the corresponding $\mathbb{L}_p$-spaces. Moreover, the relations

$$\mathscr{M}_\beta = \mathscr{F}\mathbf{Z}_\beta, \qquad \mathscr{M}_\beta^{-1} = \mathbf{Z}_\beta^{-1}\mathscr{F}^{-1},$$
$$\mathfrak{M}_a^0 = \mathscr{M}_\beta^{-1} a \mathscr{M}_\beta = \mathbf{Z}_\beta^{-1}\mathscr{F}^{-1} a \mathscr{F}\mathbf{Z}_\beta = \mathbf{Z}_\beta^{-1} W_a^0 \mathbf{Z}_\beta,$$

where $\mathscr{F}$ and $\mathscr{F}^{-1}$ are the Fourier transform and its inverse, show a close connection between Mellin $\mathfrak{M}_a^0$ and Fourier convolution operators

$$W_a^0\varphi := \mathscr{F}^{-1} a \mathscr{F}\varphi, \qquad \varphi \in \mathbb{S}(\mathbb{R}).$$

where $\mathbb{S}(\mathbb{R})$ denotes the Schwartz class of smooth fast decaying functions.

An $N \times N$ matrix function $a(\xi)$, $\xi \in \mathbb{R}$ is called $\mathbb{L}_p$-multiplier if the operator $W_a^0 \ : \ \mathbb{L}_p(\mathbb{R}) \longrightarrow \mathbb{L}_p(\mathbb{R})$ (or, equivalently, the operator $\mathfrak{M}_a^0$: $\mathbb{L}_p(\mathbb{R}^+, t^\gamma) \longrightarrow \mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ is bounded). The set of all $\mathbb{L}_p$-multipliers is denoted by $\mathfrak{M}_p(\mathbb{R})$. It is known, that $\mathfrak{M}_p(\mathbb{R})$ is a Banach algebra which contains the algebra $V_1(\mathbb{R})$ of all functions with finite variation provided that

$$\beta := \frac{1+\gamma}{p}, \qquad 1 < p < \infty, \qquad -1 < \gamma < p-1. \tag{4}$$

As was already mentioned, the primary aim of the present paper is to study Mellin convolution operators $\mathfrak{M}_a^0$ acting in Bessel potential spaces,

$$\mathfrak{M}_a^0 \ : \ \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^s(\mathbb{R}^+). \tag{5}$$

The symbols of these operators are $N \times N$ matrix functions $a \in C\mathfrak{M}_p^0(\overline{\mathbb{R}})$ continuous on the real axis $\mathbb{R}$ with the only possible jump at infinity.

Let $a \in \mathbb{L}_{\infty,loc}(\mathbb{R})$ be a locally bounded $m \times m$ matrix function. The **Fourier convolution operator (FCO)** with the **symbol** a is defined by

$$W_a^0 := \mathscr{F}^{-1} a \mathscr{F}. \tag{6}$$

If the operator

$$W_a^0 \ : \ \mathbb{H}_p^s(\mathbb{R}) \longrightarrow \mathbb{H}_p^{s-r}(\mathbb{R}) \tag{7}$$

is bounded, we say that a is an **$\mathbb{L}_p$-multiplier (of order 0)**. The set of all $\mathbb{L}_p$-multipliers is denoted by $\mathfrak{M}_p(\mathbb{R})$.

The Fourier convolution operator (FCO) on the semi-axis $\mathbb{R}^+$ with the symbol a is defined by

$$W_a := r_+ W_a^0,$$

where $r_+ := r_{\mathbb{R}^+} \ : \ \mathbb{H}_p^s(\mathbb{R}) \to \mathbb{H}_p^s(\mathbb{R}^+)$ is the restriction operator.

Consider a Fourier convolution

$$W_a = r_+ W_a^0 \ : \ \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^{s-r}(\mathbb{R}^+), \tag{8}$$

and Hankel

$$H_a = r_+ \boldsymbol{V} W_a^0 \ : \ \widetilde{\mathbb{H}}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^{s-r}(\mathbb{R}^+), \qquad (9)$$
$$\boldsymbol{V}\psi(t) := \psi(-t)$$

operators, where r_+ is the restriction operator to the semi-axes $\mathbb{R}^+$. Note, that the generalized Hoermander's kernel of a FCO W_a depends on the difference of arguments $\mathscr{K}_a(t-\tau)$, while the Hoermander's kernel of Hankel operator $r_+\boldsymbol{V}W_a^0$ depends of the sum of the arguments $\mathscr{K}_a(t+\tau)$

If W_a in (9) is bounded, we say that W_a has order r and a is an $\mathbb{L}_p$ multiplier of order r. The set of all $\mathbb{L}_p$ multipliers of order r is denoted by $\mathfrak{M}_p^r(\mathbb{R})$.

Theorem 1 *Let* $1 < p < \infty$. *Then*

1. *For any* $r, s \in \mathbb{R}$, $\gamma \in \mathbb{C}$, $\operatorname{Im}\gamma > 0$ *the convolution operators (*Ψ*DOs)*

$$\begin{aligned} &\Lambda^r_\gamma = W_{\lambda^r_\gamma} \ : \ \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \widetilde{\mathbb{H}}^{s-r}_p(\mathbb{R}^+) \\ &\Lambda^r_{-\gamma} = W_{\lambda^r_{-\gamma}}\ell \ : \ \mathbb{H}^s_p(\mathbb{R}^+) \to \mathbb{H}^{s-r}_p(\mathbb{R}^+)\,, \\ &\lambda^r_{\pm\gamma}(\xi) := (\xi \pm \gamma)^r, \qquad \xi \in \mathbb{R}^+, \end{aligned} \tag{10}$$

where $\ell \ : \ \mathbb{H}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R})$ *is some extension operator, define an isomorphism between the corresponding spaces. The final result is independent of the choice of an extension* ℓ.

2. *For any operator* $A \ : \ \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^{s-r}_p(\mathbb{R}^+)$ *of the order* r, *the following diagram is commutative*

$$\begin{array}{ccc} \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) & \stackrel{A}{\longrightarrow} & \mathbb{H}^{s-r}_p(\mathbb{R}^+) \\ \uparrow \Lambda^{-s}_+ & & \downarrow \Lambda^{s-r}_- \\ \mathbb{L}_p(\mathbb{R}^+) & \stackrel{\Lambda^{s-r}_- A \Lambda^{-s}_+}{\longrightarrow} & \mathbb{L}_p(\mathbb{R}^+). \end{array} \tag{11}$$

The diagram (10) *provides an equivalent lifting of the operator* A *of order* r *to the operator* $\Lambda_{-}^{s-r} A \Lambda_{+}^{-s} \ : \ \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *of order* 0.

3. *If* $W_a \ : \ \mathbb{H}_p^s(\mathbb{R}^+) \longrightarrow \mathbb{H}_p^{s-r}(\mathbb{R}^+)$ *is a bounded convolution operator of order* r*, then the lifted operator* $W_{a_0} = \Lambda_{-}^{s-r} W_a \Lambda_{+}^{-s} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *is again a convolution operator with the symbol*

$$a_0(\xi) = \lambda_{-\gamma}^{s-r}(\xi) a(\xi) \lambda_{\gamma}^{-s}(\xi) = \left(\frac{\xi - \gamma}{\xi + \gamma}\right)^{s-r} \frac{a(\xi)}{(\xi + i)^r}.$$

For a Mellin convolution operator we have two major problems:

- **The boundeeness result in the Bessel potential spaces;**
- **The lifting.**

Concerning the first problem the following was known.

Proposition 2 *Let* $1 < p < \infty$ *and let* $m = 1, 2, \ldots$ *be an integer. If a function* $\mathscr{K}$ *satisfies the condition*

$$\int_0^1 t^{\frac{1}{p}-m-1} |\mathscr{K}(t)|\, dt + \int_1^\infty t^{\frac{1}{p}-1} |\mathscr{K}(t)|\, dt < \infty,$$

then the Mellin convolution operator (see (1)*)*

$$\boldsymbol{A} = \mathfrak{M}^0_{\mathscr{A}_{1/p}} \;:\; \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \;\longrightarrow\; \mathbb{H}^s_p(\mathbb{R}^+)$$

with the symbol

$$\mathscr{A}_{1/p}(\xi) := c_0 + c_1 \coth \pi \left(\frac{i}{p} + \xi\right) + \int_0^\infty t^{\frac{1}{p}-i\xi} \mathscr{K}(t) \frac{dt}{t}, \quad \xi \in \mathbb{R}$$

is bounded for any $0 \leqslant s \leqslant m$.

The formulated result has rather restricted application and we need better result on the boundedness, nothing to say about lifting.

To overcome this difficulty and involve all operators which we encounter in applications, let us consider kernels which are meromorphic on the complex plane $\mathbb{C}$, vanishing at infinity

$$\mathscr{K}(t) := \sum_{j=0}^{\infty} \frac{d_j}{(t-c_j)^{m_j}}, \qquad c_j \neq 0, \quad j = 0, 1, \ldots, \tag{12}$$
$$0 < \alpha_k := \arg c_k \leqslant 2\pi, \quad k = \ell+1, \ell+2, \ldots$$

having poles at $c_0, c_1, \ldots \in \mathbb{C} \setminus \{0\}$ and complex coefficients $d_j \in \mathbb{C}$.

Definition 3 *We call a kernel $\mathscr{K}(t)$ in (12) admissible iff:*

i. $\mathscr{K}(t)$ has only a finite number of poles $c_0, \ldots, c_\ell$ which belong to the positive semi-axes, i.e. $\arg c_0 = \cdots = \arg c_\ell = 0$;

ii. The corresponding multiplicities are one $m_0 = \cdots = m_\ell = 1$;

iii. The points $c_{\ell+1}, c_{\ell+2}, \ldots$ does not condense to any point of the positive semi-axes and their real parts are uniformly bounded

$$\varliminf_{j\to\infty} c_j \notin [0, \infty), \qquad \sup_{j=\ell+1,\ell+2,\ldots} \operatorname{Re} c_j \leqslant K < \infty. \tag{13}$$

Example 4 *The function*

$$\mathscr{K}(t) = \exp\left(\frac{1}{t-c}\right), \qquad \operatorname{Re} c < 0 \quad \textit{or} \quad \operatorname{Im} c \neq 0$$

is an example of the admissible kernel which also satisfies the condition of the next Corollary 5.

More trivial examples of operators with admissible kernels, which also satisfies the condition of the next Corollary 5 are operators which we encounter in all applications known to me and, in general, any finite sum in (12).

Corollary 5 *Let conditions*

$$\beta := \frac{1+\gamma}{p}, \qquad 1 < p < \infty, \qquad -1 < \gamma < p-1. \tag{14}$$

hold, $\mathscr{K}(t)$ *in* (12) *be an admissible kernel and*

$$K_\beta := \frac{\pi}{|\sin \pi\beta|} \sum_{j=0}^{\infty} 2^{m_j} \binom{\beta-1}{m_j} |c_j|^{\beta-m_j} |d_j| < \infty. \tag{15}$$

Then the Mellin convolution $\mathfrak{M}^0_{a_\beta}$ *with the admissible meromorphic kernel* $\mathscr{K}(t)$ *in* (12) *is bounded in the Lebesgue space* $\mathbb{L}_p(\mathbb{R}^+, t^\gamma)$ *and its norm is estimated by the constant* $\|\mathfrak{M}^0_{a_\beta}\| \leqslant M K_\beta$ *with some* $M > 0$.

In particular, for a meromorphic kernel with simple poles $m_0 = m_1 = \cdots = 1$ *we get the estimate*

$$|\mathfrak{M}^0_{a_\beta}\| \leqslant M K_\beta = \frac{2M\pi |c_j|^{\beta-1}}{|\sin \pi\beta|} \sum_{j=0}^{\infty} |d_j|.$$

The next Theorem 6 solves the problem of the boundedness.

Theorem 6 *Let conditions* (14) *hold* $s \in \mathbb{R}$ *and* $m^0 := \sup_{j=0,1,\ldots} m_j < \infty$. *The Mellin convolution operator* $\mathfrak{M}^0_{a_\beta}$ *with an admissible kernel* $\mathscr{K}$ *(see* (12)*) is bounded in the Bessel potential spaces*

$$\mathfrak{M}^0_a \ : \ \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+), \tag{16}$$

provided the condition (15) *holds. The condition on the parameter* p *can be relaxed to* $1 \leqslant p \leqslant \infty$, *provided the admissible kernel* $\mathscr{K}$ *in* (12) *has no poles on positive semi-axes* $\alpha_j = \arg c_j \neq 0$ *for all* $j = 0, 1, \ldots$.

To next problem is lifting: we need to lift a Mellin convolution operator

$$\mathfrak{M}_a \ : \ \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \longrightarrow \mathbb{H}^s_p(\mathbb{R}^+)$$

get

$$\Lambda^s_- \mathfrak{M}_a \Lambda^{-s}_+ \ : \ \widetilde{\mathbb{L}}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$$

and find out:

i. what happens if we drug the operator Λ^s_- through $\mathfrak{M}_a$ to collide it with Λ^{-s}_+?

ii. can to investigate the resulting operator?

It is relatively easy to trace what happens with the model Mellin convolution operator

$$\boldsymbol{K}^m_c\varphi(t) := \int_0^\infty \frac{\tau^{m-1}\varphi(\tau)d\tau}{(t-c\,\tau)^m} \tag{17}$$

with the meromorphic kernel

$$\mathscr{K}^m_c(t) := \frac{1}{(t-c)^m}, \qquad c \neq 0, \quad 0 \leqslant |\arg c| < \pi, \quad m = 1, 2, \ldots \tag{18}$$

if we drug the Bessel potentials with integer power $m = 1, 2, \ldots$

$$\boldsymbol{\Lambda}^m_{\pm\gamma} = W_{\lambda^m_{\pm\gamma}} = \left(i\frac{d}{dt} \pm \gamma\right)^m = \sum_{k=0}^m \left(\begin{array}{c} m \\ k \end{array}\right) i^k(\pm\gamma)^{m-k}\frac{d^k}{dt^k}.$$

through it, since

$$\frac{d}{dt}\boldsymbol{K}_c^m\varphi(t) = \int_0^\infty \frac{d}{dt}\frac{\tau^{m-1}}{(t-c\,\tau)^m}\varphi(\tau)d\tau$$
$$= -\sum_{j=0}^{m-1}(-c)^{-1-j}\left(\boldsymbol{K}_c^{m-j}\frac{d}{dt}\varphi\right)(t).$$

To write formula for the operator of order n is a matter of simple algebra, but will not elaborate on this particular result any more and in the next Theorem 7 and Theorem 8 solve the problem of lifting for meromorphic kernels.

First what results after drugging a Bessel potential through a model Mellin convolution operator

Theorem 7 *Let* $0 < \arg\gamma < \pi$, $0 < |\arg c| < \pi$, $0 < |\arg(\gamma\, c)| < \pi$, $r, s \in \mathbb{R}$, $m = 1, 2, \ldots$, $1 < p < \infty$. *Then*

$$\Lambda^s_{-\gamma}\boldsymbol{K}^m_c\varphi = \begin{cases} c^{-s}\boldsymbol{K}^m_c\Lambda^s_{-\gamma\, c}\varphi & \text{if} \quad -\pi < \arg\gamma\, c < 0, \\ c^{-s}\widetilde{\boldsymbol{K}}^m_c\Lambda^s_{-\gamma\, c}\varphi & \text{if} \quad 0 < \arg\gamma\, c < \pi, \end{cases} \tag{19}$$

for arbitrary $\varphi \in \widetilde{\mathbb{H}}^r_p(\mathbb{R}^+)$, *where*

$$\widetilde{\boldsymbol{K}}^m_c\psi(t) := \int_{-\infty}^{\infty}\frac{\tau^m\psi(\tau)d\tau}{(t - c\,\tau)^m}, \qquad \psi \in \mathbb{L}_p(\mathbb{R}). \tag{20}$$

Now what we get after lifting procedure.

Theorem 8 *Let* $0 < \arg\gamma < \pi$, $0 < |\arg c| < \pi$, $0 < |\arg(i\gamma\, c| < \pi$, $r, s \in \mathbb{R}$, $m = 1, 2, \ldots$, $1 < p < \infty$. *Then the operator*

$$K^m_c \; : \; \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+) \tag{21a}$$

is lifted equivalently to the following operator

$$\Lambda^s_{-\gamma}\boldsymbol{K}^m_c\Lambda^{-s}_{\gamma} \;:\; \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+), \tag{21b}$$

where

$$\Lambda^s_{-\gamma}\boldsymbol{K}^m_c\Lambda^{-s}_{\gamma} = \begin{cases} c^{-s}\boldsymbol{K}^m_c W_{g^s_{\gamma,c}} = c^{-s}\boldsymbol{K}^m_c + T_1 \\ \qquad\qquad\qquad \textit{if} \quad -\pi < \arg\gamma\, c < 0, \\ c^{-s}\widetilde{\boldsymbol{K}}^m_c W^0_{g^s_{\gamma,c}} \\ = c^{-s}\boldsymbol{K}^m_c W_{\mathfrak{s}^s_\beta} + c^{-s}\boldsymbol{K}^m_{-c} H_{\mathfrak{s}^s_\beta} + T_2 \\ \qquad\qquad\qquad \textit{if} \quad 0 < \arg\gamma\, c < \pi, \end{cases} \tag{21c}$$

$$g^s_{\gamma,c}(\xi) := \left(\frac{\xi - c\gamma}{\xi + \gamma}\right)^s,$$

$$\mathfrak{s}^s_\beta(\xi) := e^{\pi s i}\left[\cos(\pi\, s) - i\sin(\pi\, s)\coth\pi(i\beta - \xi)\right], \tag{21d}$$

$\beta := 1/p$*, and the operators* $T_1, T_2 \;:\; \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$ *are compact.*

2. ALGEBRA GENERATED BY MELLIN, FOURIER CONVOLUTION AND HANKEL OPERATORS

Thus, by lifting a Mellin convolution operator, see (21a),

$$K^m_c \ : \ \widetilde{\mathbb{H}}^s_p(\mathbb{R}^+) \to \mathbb{H}^s_p(\mathbb{R}^+)$$

to $\mathbb{L}_p(\mathbb{R}^+)$ **space, see** (21c),

$$\Lambda^s_{-\gamma}\boldsymbol{K}^m_c\Lambda^{-s}_{\gamma} \ : \ \mathbb{L}_p(\mathbb{R}^+) \to \mathbb{L}_p(\mathbb{R}^+)$$

we get rather complex operators, see (21d),

$$\Lambda^s_{-\gamma}\boldsymbol{K}^m_c\Lambda^{-s}_{\gamma} = c^{-s}\boldsymbol{K}^m_c W_{g^s_{\gamma,c}} = c^{-s}\boldsymbol{K}^m_c + T_1 \qquad \text{or}$$

$$c^{-s}\widetilde{\boldsymbol{K}}^m_c W^0_{g^s_{\gamma,c}} = c^{-s}\boldsymbol{K}^m_c W_{\mathfrak{s}^s_\beta} + c^{-s}\boldsymbol{K}^m_{-c} H_{\mathfrak{s}^s_\beta} + T_2,$$

containing Mellin, Fourier and Hankel operators.

Therefore we should consider the Banach algebra $\mathfrak{A}_p(\mathbb{R}^+)$ **of operators**

$$\mathbf{A} := \sum_{j=1}^{m}[\mathfrak{M}^0_{a_j} W_{b_j} + \mathfrak{M}^0_{d_j} H_{g_j}], \qquad (22)$$

in the weighted Lebesgue space $\mathbb{L}_p(\mathbb{R}^+, x^\alpha)$ generated by:

I. Mellin convolution operators $\mathfrak{M}^0_{a_j}$, $\mathfrak{M}^0_{d_j}$ with continuous $N\times N$ matrix symbols $a_j, d_j \in C\mathfrak{M}_p(\overline{\mathbb{R}})$;

II. Fourier convolution operators W_{b_j} with $N \times N$ matrix symbols $b_j \in C\mathfrak{M}_p(\overline{\mathbb{R}} \setminus \{0\}) := C\mathfrak{M}_p(\overline{\mathbb{R}}^- \cup \overline{\mathbb{R}}^+)$;

III. Hankel operators H_{g_j} with $N \times N$ matrix symbols $g_j \in C\mathfrak{M}_p(\overline{\mathbb{R}} \setminus \{0\}) := C\mathfrak{M}_p(\overline{\mathbb{R}}^- \cup \overline{\mathbb{R}}^+)$.

The algebra of $N \times N$ matrix $\mathbb{L}_p$-multipliers $C\mathfrak{M}_p(\overline{\mathbb{R}} \setminus \{0\})$ consists of those piecewise-continuous $N \times N$ matrix multipliers $b \in \mathfrak{M}_p(\mathbb{R}) \cap PC(\overline{\mathbb{R}})$ which are continuous on the semi-axis $\mathbb{R}^-$ and $\mathbb{R}^+$ but might have finite jump discontinuities at 0 and at the infinity.

This and more general algebras, with piecewise-continuous coefficients, were studied in the papers by R. Duduchava in 1974-1987. part of results were obtained independetly by G. Thelen's in PhD dissertation

and papers from 1985.

Note that the algebra $\mathfrak{A}_p(\mathbb{R}^+)$ is actually a subalgebra of the Banach algebra $\mathfrak{F}_p(\mathbb{R}^+)$ generated by the Fourier convolution operators W_a with piecewise-constant symbols $a(\xi)$ in the space $\mathbb{L}_p(\mathbb{R}^+)$. Hankel and Mellin convolution operators belong to this algebra and we include them in generators just to write their symbols explicitly. This is very helpful in forthcoming applications to BVPs (see § 3).

Let $\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ denote the ideal of all compact operators in $\mathbb{L}_p(\mathbb{R}^+)$. Since the quotient algebra $\mathfrak{F}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ is commutative in the scalar case $N = 1$, the following is true.

Corollary 9 *The quotient algebra $\mathfrak{A}_p(\mathbb{R}^+)/\mathfrak{S}(\mathbb{L}_p(\mathbb{R}^+))$ is commutative in the scalar case $N = 1$.*

To describe the symbol of the operator (22), consider the infinite clockwise oriented "rectangle" $\mathfrak{R} := \Gamma_1 \cup \Gamma_2^- \cup \Gamma_2^+ \cup \Gamma_3$**, (cf. Fig. 2)**

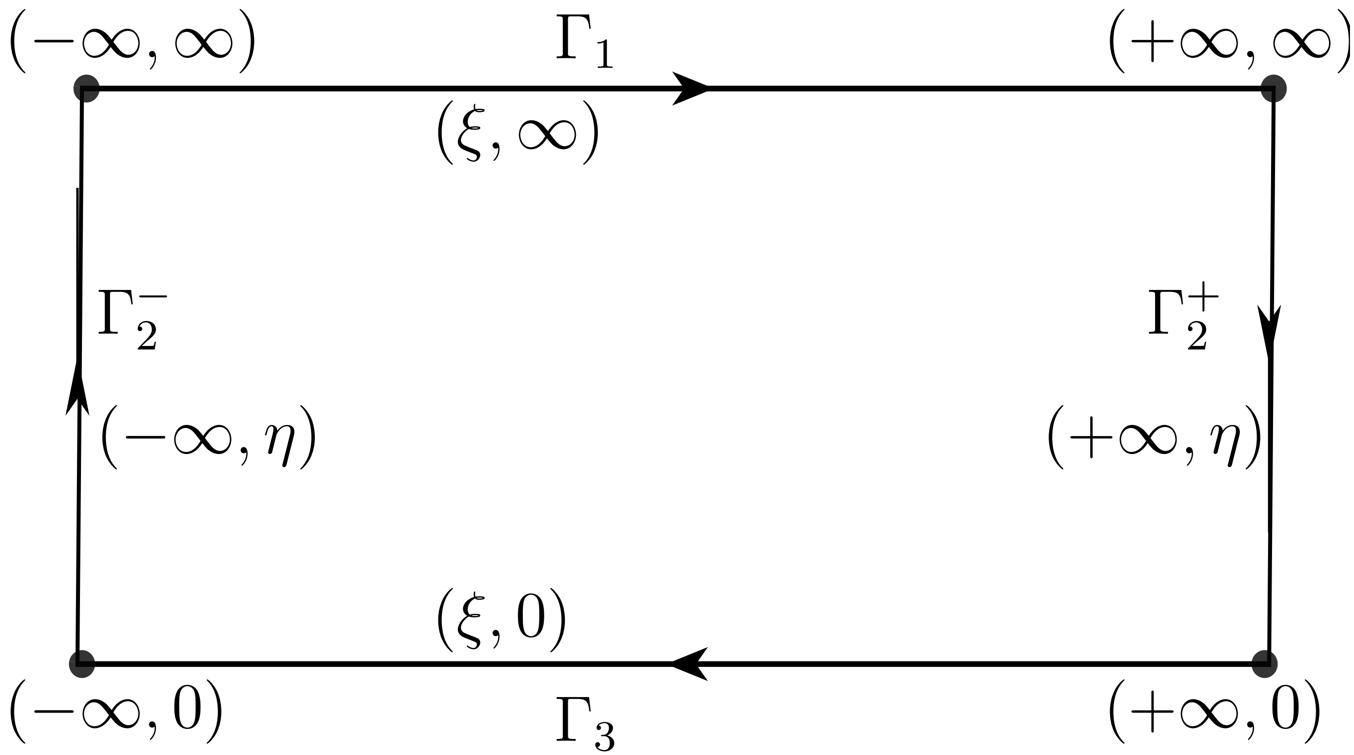

Fig. 2. The domain $\mathfrak{R}$ of definition of the symbol $\mathscr{A}_p(\xi, \eta)$.

$$\Gamma_1 := \overline{\mathbb{R}} \times \{+\infty\}, \quad \Gamma_2^{\pm} := \{\pm\infty\} \times \overline{\mathbb{R}}^+, \quad \Gamma_3 := \overline{\mathbb{R}} \times \{0\}.$$

The symbol $\mathscr{A}_p(\omega)$ of the operator A in (22) is a function on the set $\mathfrak{R}$:

$$\mathscr{A}_p(\omega) := \tag{23}$$

$$\begin{cases} \sum_{j=1}^{m} [a_j(\xi)(b_j)_p(\infty,\xi) + d_j(\xi)(g_j)_p^{\infty}(\xi)], & \omega = (\xi,\infty) \in \overline{\Gamma_1}, \\ \sum_{j=1}^{m} [a_j(+\infty)b_j(-\eta) + d_j(+\infty)g_j(+\infty)], & \omega = (+\infty,\eta) \in \Gamma_2^+, \\ \sum_{j=1}^{m} [a_j(-\infty)b_j(\eta) + d_j(\infty)g_j(-\infty)], & \omega = (-\infty,\eta) \in \Gamma_2^-, \\ \sum_{j=1}^{m} [a_j(\xi)(b_j)_p(0,\xi) + d_j(\xi)(g_j)_p^{\infty}(\xi)], & \omega = (\xi,0) \in \overline{\Gamma_3}. \end{cases} \tag{24}$$

In (23) we use the notation

$$g_p(\infty,\xi) := \frac{1}{2}[g(+\infty)+g(-\infty)] - \frac{1}{2}[g(+\infty)-g(-\infty)]\coth\,\pi\left(\frac{i}{p}-\xi\right),$$

$$g_p^{\infty}(\xi) := \frac{1}{2}[g(+\infty)+g(-\infty)] - \frac{1}{2}[g(+\infty)-g(-\infty)]\sinh\,\pi\left(\frac{i}{p}-\xi\right),$$

$$g_p(t,\xi) := \frac{1}{2}[g(t+0)+g(t-0)] - \frac{1}{2}[g(t+0)-g(t-0)]\coth\,\pi\left(\frac{i}{p}-\xi\right)$$

where $g \in PC(\overline{\mathbb{R}})$ is a piecewise continuous function and $t, \xi \in \mathbb{R}$.

To make the symbol $\mathscr{A}_p(\omega)$ continuous, we endow the rectangle $\mathfrak{R}$ with a special topology. Thus let us define the distance on the curves $\Gamma_1, \Gamma_2^{\pm}$, Γ_3 and on $\overline{\mathbb{R}}$ by

$$\rho(x,y) := \left|\arg\frac{x-i}{x+i} - \arg\frac{y-i}{y+i}\right| \quad \text{for arbitrary} \quad x, y \in \overline{\mathbb{R}}.$$

In this topology, the length $|\mathfrak{R}|$ of $\mathfrak{R}$ is 6π, and the symbol $\mathscr{A}_p(\omega)$ is continuous everywhere on $\mathfrak{R}$. The image of the function $\det\,\mathscr{A}_p(\omega)$,

$\omega \in \mathfrak{R}$ **(**$\det \mathscr{B}_p(\omega)$**) is a closed curve in the complex plane. It follows from the continuity of the symbol at the angular points of the rectangle** $\mathfrak{R}$ **where the one-sided limits coincide. Thus**

$$\mathscr{A}_p(\pm\infty, \infty) = \sum_{j=1}^{m} [a_j(\pm\infty) b_j(\mp\infty) + d_j(\pm\infty) g_j(\mp\infty)],$$

$$\mathscr{A}_p(\pm\infty, 0) = \sum_{j=1}^{m} [a_j(\pm\infty) b_j(0 \mp 0) + d_j(\pm\infty) g_j(\mp\infty)].$$

Hence, if the symbol of the corresponding operator is elliptic, i.e. if

$$\inf_{\omega \in \mathfrak{R}} \left| \det \mathscr{A}_p(\omega) \right| > 0, \tag{25}$$

the increment of the argument $(1/2\pi) \arg \mathscr{A}_p(\omega)$ **when** ω **ranges through** $\mathfrak{R}$ **in the positive direction is an integer, is called thewinding number or the index and it is denoted by** $\operatorname{ind} \det \mathscr{A}_p$**.**

Theorem 10 *Let* $1 < p < \infty$ *and let* $\mathbf{A}$ *be defined by* (22). *The operator* $\mathbf{A} : \mathbb{L}_p(\mathbb{R}^+) \longrightarrow \mathbb{L}_p(\mathbb{R}^+)$ *is Fredholm if and only if its symbol* $\mathscr{A}_p(\omega)$ *is elliptic. If* $\mathbf{A}$ *is Fredholm, the index of the operator has the value*

$$\mathrm{Ind}\,\mathbf{A} = -\mathrm{ind}\ \det\ \mathscr{A}_p. \tag{26}$$

3. LOCALIZATION AND THE MODEL BOUNDARY VALUE PROBLEM

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Based on results of §§1-2 we study a mixed boundary value problem in a domain with angular points (see again Fig. 1).

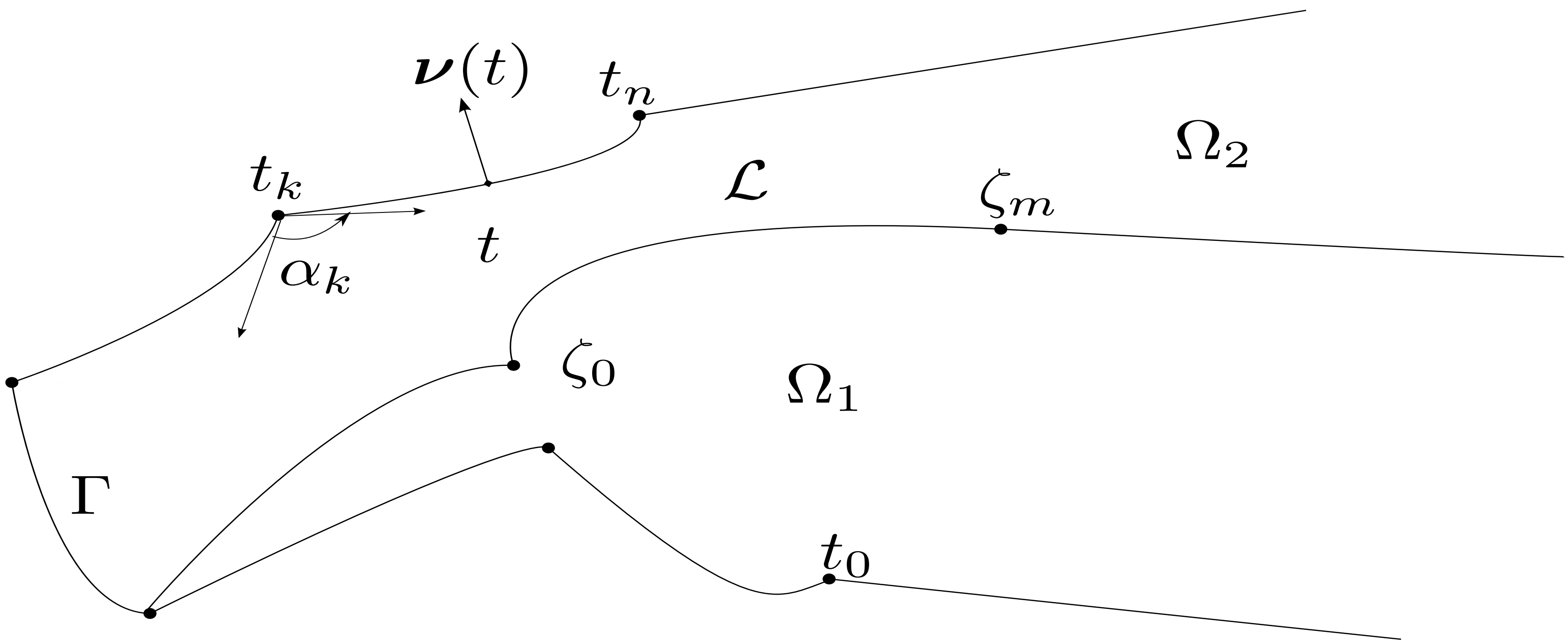


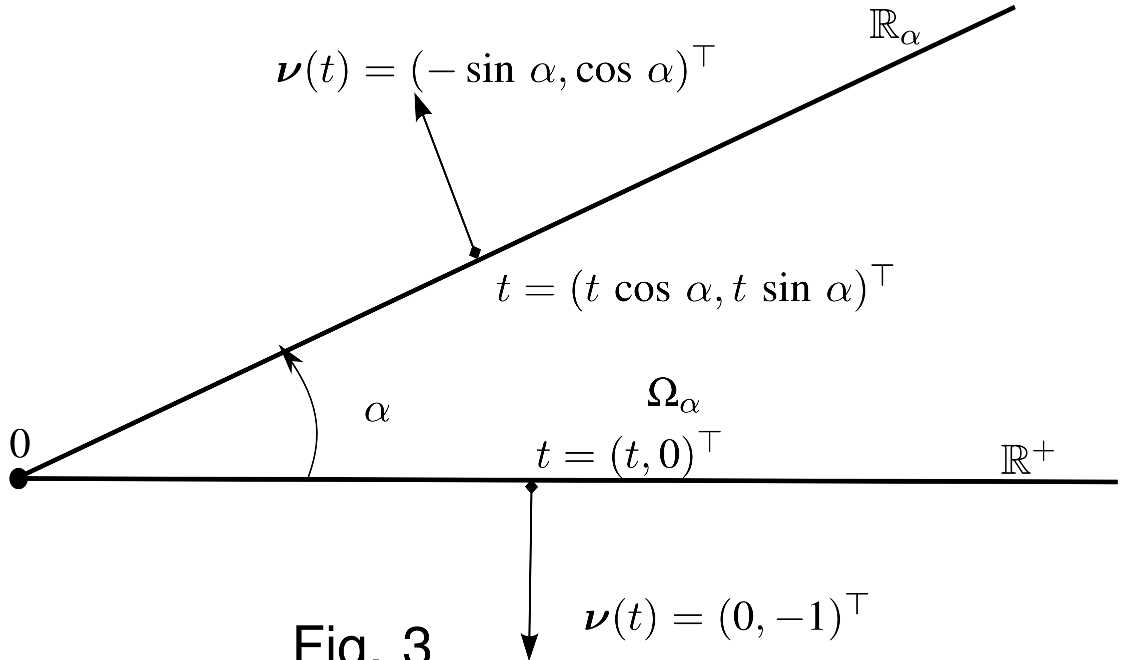
Fig. 1

For this we apply first localization and reduce investigation to several (actually 7 different model problems. Most non-trivial among them are

two series of problems:

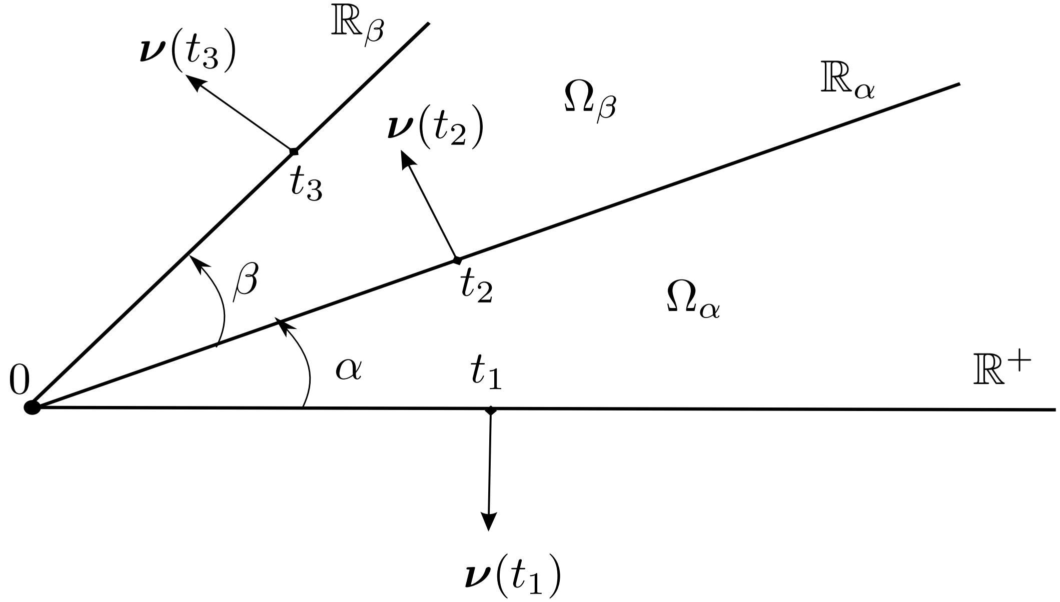
i. **Model problem in the angle of magnitude $\alpha \in (0, 2\pi)$, described on Fig. 3, with different boundary conditions on the boundary rays (Dirichlet-Dirichlet. Neumann-Neumann or Dirichlet-Neumann)**

ii. **Model problem in the double angle of magnitudes $\alpha,\ \beta \in (0, 2\pi)$ and common boundary (interface), the material constants changing the sign, described on Fig. 4, with different combination of boundary conditions on the boundary rays (Dirichlet-Dirichlet-Dirichlet. Neumann-Neumann-Neumann or Dirichlet-Neumann-Dirichlet, Dirichlet-Dirichle-Neumann etc.)**



$\boldsymbol{\nu}(t) = (-\sin\,\alpha, \cos\,\alpha)^\top$
$\mathbb{R}_\alpha$
$t = (t\,\cos\,\alpha, t\,\sin\,\alpha)^\top$
α
Ω_α
0
$t = (t, 0)^\top$
$\mathbb{R}^+$
$\boldsymbol{\nu}(t) = (0, -1)^\top$

Fig. 3



$\mathbb{R}_\beta$
$\boldsymbol{\nu}(t_3)$
Ω_β
$\mathbb{R}_\alpha$
$\boldsymbol{\nu}(t_2)$
t_3
β
t_2
Ω_α
0
α
t_1
$\mathbb{R}^+$
$\boldsymbol{\nu}(t_1)$

Fig. 4

Investigations for model BVP's in double angle are in progress and today we present only results on mixed BVP's for a model angular domain Ω_α of magnitude α enclosed between the positive half-axes $\mathbb{R}^+ = (0, \infty)$ and the beam $\mathbb{R}_\alpha$ rotated at an angle $\alpha \in (0, 2\pi)$ from $\mathbb{R}^+$ (See Fig. 3):

$$\partial\Omega_\alpha = \mathbb{R}^+ \cup \mathbb{R}_\alpha, \quad \mathbb{R}^+ = (0, \infty),$$
$$\mathbb{R}_\alpha := \{te^{i\alpha} = (t \cos\, \alpha, t \sin\, \alpha) \ : \ t \in \mathbb{R}^+\}.$$

We study the following mixed boundary value problem

$$\begin{cases} \Delta u(x) + k^2 u(x) = 0, & x \in \Omega_\alpha, \\ u^+(t) = g(t), & t \in \mathbb{R}^+, \\ (\partial_{\boldsymbol{\nu}} u)^+(t) = h(t), & t \in \mathbb{R}_\alpha, \end{cases} \tag{27}$$

in the model domain, where the normal derivative $\partial_{\boldsymbol{\nu}}$ is defined as follows:

$$\boldsymbol{\nu}(t) = \begin{cases} -\partial_t & \text{for} \quad t = (t, 0) \in \mathbb{R}^+, \\ -\sin\, \alpha\, \partial_{t_1} + \cos\, \alpha\, \partial_{t_2} & \text{for} \quad t = (t_1, t_2) \in \mathbb{R}_\alpha. \end{cases} \tag{28}$$

We will look for a solution to BVP (27) **both in the classical (finite energy) formulation**

$$\begin{gathered} g \in \mathbb{H}^{1/2}(\mathbb{R}^+), \quad h \in \mathbb{H}^{-1/2}(\mathbb{R}_\alpha), \quad u \in \mathbb{H}^1(\Omega_\alpha) = \mathbb{W}^1(\Omega_\alpha), \\ u(x) = \mathscr{O}(1) \quad \text{as} \quad |x| \to \infty \end{gathered} \tag{29}$$

and in the non-classical formulation

$$\begin{gathered} g \in \mathbb{W}_p^{1-1/p}\mathbb{R}^+), \quad h \in \mathbb{W}_p^{-1/p}(\mathbb{R}_\alpha), \quad u \in \mathbb{H}_p^1(\Omega_\alpha) = \mathbb{W}_p^1(\Omega_\alpha), \\ u(x) = \mathscr{O}(1) \quad \text{as} \quad |x| \to \infty, \quad 1 < p < \infty. \end{gathered} \tag{30}$$

Let $g_0 \in \mathbb{H}^{1/2}(\partial\Omega_\alpha)$ **and** $h_0 \in \mathbb{H}^{-1/2}(\partial\Omega_\alpha)$ **be some fixed extensions of the boundary conditions** $g \in \mathbb{H}^{1/2}(\mathbb{R}^+)$ **and** $h \in \mathbb{H}^{-1/2}(\mathbb{R}_\alpha)$**, initially defined on the parts of the boundary** $\partial\Omega_\alpha = \mathbb{R}^+ \cup \mathbb{R}^\alpha$**. Since the difference between such two extensions belong to the spaces** $\widetilde{\mathbb{H}}^{1/2}(\mathbb{R}_\alpha)$ **and** $\widetilde{\mathbb{H}}^{-1/2}(\mathbb{R}^+)$ **respectively, we seek two unknown functions** $\varphi \in \widetilde{\mathbb{H}}^{1/2}(\mathbb{R}_\alpha)$ **and** $\psi \in \widetilde{\mathbb{H}}^{-1/2}(\mathbb{R}^+)$**, for which the boundary conditions in** (27) **hold on the entire boundary. It is usual to consider** $\widetilde{\mathbb{H}}^s(\mathbb{R}^+)$ **and** $\widetilde{\mathbb{H}}^s(\mathbb{R}_\alpha)$ **as subsets of** $\mathbb{H}^s(\partial\Omega_\alpha)$ **by extending functions from** $\widetilde{\mathbb{H}}^s(\mathbb{R}^+)$ **and** $\widetilde{\mathbb{H}}^s(\mathbb{R}_\alpha)$ **by**

0 to $\mathbb{R}_\alpha$ and to $\mathbb{R}^+$, respectively. Then, if $u(x)$ is a solution to the BVP (27), the following holds:

$$\begin{aligned} u^+(t) = g_0(t) + \varphi(t) &= \begin{cases} g(t) & \text{if} \quad t \in \mathbb{R}^+, \\ g_0(t) + \varphi(t) & \text{if} \quad t \in \mathbb{R}_\alpha, \end{cases} \\ (\partial_{\boldsymbol{\nu}} u)^+(t) = h_0(t) + \psi(t) &= \begin{cases} h_0(t) + \psi(t) & \text{if} \quad t \in \mathbb{R}^+, \\ h(t) & \text{if} \quad t \in \mathbb{R}_\alpha. \end{cases} \end{aligned} \tag{31}$$

By introducing the boundary values of solution (31) to the boundary value problem (27) into the representation formula for a solution, we get the following representation formula

$$u(x) = \boldsymbol{W}[g_0 + \varphi](x) - \boldsymbol{V}_{\mathbb{R}^+}[h_0 + \psi](x), \quad x \in \Omega_\alpha. \tag{32}$$

The known and unknown functions in (31) belong to the following spaces (cf. (27))

$$g_0 \in \mathbb{H}^{1/2}(\partial\Omega_\alpha), \quad h_0 \in \mathbb{H}^{-1/2}(\partial\Omega_\alpha), \quad \varphi \in \widetilde{\mathbb{H}}^{1/2}(\mathbb{R}_\alpha), \quad \psi \in \widetilde{\mathbb{H}}^{-1/2}(\mathbb{R}^+).$$

We will use the parameterizations

$$x = (x_1, x_2) = (t\cos\,\alpha, t\sin\,\alpha)^\top \in \mathbb{R}_\alpha, \quad y = (\tau\cos\,\alpha, \tau\sin\,\alpha)^\top \in \mathbb{R}_\alpha,$$
$$\theta = (\theta, 0) \in \mathbb{R}^+, \quad \text{where} \quad t, \tau, \theta \in \mathbb{R}^+. \tag{33}$$

It is obvious that

$$\text{if} \quad \varphi \in \widetilde{\mathbb{H}}^{1/2}(\mathbb{R}_\alpha) \quad \text{and} \quad \varphi_+(t) := \varphi(t\cos\,\alpha, t\sin\,\alpha), \quad t \in \mathbb{R}^+,$$
$$\text{then} \quad \varphi_+ \in \widetilde{\mathbb{H}}^{1/2}(\mathbb{R}^+), \quad \varphi_0 := \partial_t\varphi_+ \in \widetilde{\mathbb{H}}^{-1/2}(\mathbb{R}^+), \quad \partial_t := \frac{d}{dt}. \tag{34}$$

By using the representation formula (32)**, boundary conditions** (27)**, the celebrated Plemelji formulae for layer potentials, applying the parametrization** (33) **and differentiating with respect to** ∂_t**, this system ac-**

quires the form:

$$\begin{cases} \varphi_0 + \dfrac{1}{2\pi}\left[\boldsymbol{K}^1_{e^{i\alpha}} + \boldsymbol{K}^1_{e^{-i\alpha}}\right]\psi = G_1, \\ \psi - \dfrac{1}{2\pi}\left[\boldsymbol{K}^1_{e^{i\alpha}} + \boldsymbol{K}^1_{e^{-i\alpha}}\right]\varphi_0 = H_1 \quad \text{on} \quad \mathbb{R}^+, \end{cases} \tag{35}$$

where $G_1, H_1 \in \mathbb{H}^{-1/2}(\mathbb{R}^+)$ are known functions. We remind that the operators $\boldsymbol{K}^1_{e^{\pm i\alpha}}$ are Mellin convolutions with meromorphic kernels

$$\boldsymbol{K}^1_{e^{\pm i\alpha}}\varphi(t) := \int_0^\infty \frac{\varphi(\tau)d\tau}{t - e^{\pm i\alpha}\tau}$$

Theorem 11 *A solution* $u \in \mathbb{H}^1(\Omega_\alpha)$ *to the mixed BVP* (27) *is represented by formula* (32)*, where the unknown functions* $\psi, \varphi_0 \in \widetilde{\mathbb{H}}^{-1/2}(\mathbb{R}^+)$ *are solutions to system* (35) *and* $\varphi \in \widetilde{\mathbb{H}}^{-1/2}(\mathbb{R}_\alpha)$ *is recovered from* $\varphi_0(t) = \partial_t \varphi_+(t)$ *(see* (34)*) by the formula*

$$\varphi(x) = \varphi(t\,\cos\,\alpha, t\,\sin\,\alpha) := \varphi_+(t) = \int_0^t \varphi_0(\tau)\,d\tau, \quad t \in \mathbb{R}^+, \quad (36)$$

where

$$x := (t\,\cos\,\alpha, t\,\sin\,\alpha)^\top \in \mathbb{R}_\alpha, \quad \varphi_0 \in \widetilde{\mathbb{H}}^{-1/2}(\mathbb{R}^+) \quad \varphi_+ \in \widetilde{\mathbb{H}}^{1/2}(\mathbb{R}^+).$$

Vice versa: if the functions $\psi, \varphi_0 \in \widetilde{\mathbb{H}}^{-1/2}(\mathbb{R}^+)$ *are solutions to system* (35) *and* $\varphi \in \widetilde{\mathbb{H}}^{1/2}(\mathbb{R}_\alpha)$ *is recovered by formula* (36)*, the function* $u \in \mathbb{H}^1(\Omega_\alpha)$ *represented by formula* (32) *is a solution to the mixed BVP* (27).

Theorem 12 *The mixed BVP* (27) *with classical finite energy constraints* (29) *has a unique solution* $u \in \mathbb{H}^1(\Omega_\alpha)$ *for arbitrary* $\alpha \in (0, 2\pi)$. *The solution is represented by formula* (32), *where the unknown functions* ψ *and* φ_0 *are unique solutions to system* (35) *in the space* $\widetilde{\mathbb{H}}^{-1/2}(\mathbb{R}^+)$ *and* $\varphi \in \widetilde{\mathbb{H}}^{1/2}(\mathbb{R}_\alpha)$ *is recovered from* φ_0 *by formula* (36).

For $1 < p < \infty$, $p \neq 2$, *the mixed BVP* (27) *with constraints* (30) *has a unique solution* $u \in \mathbb{H}^1_p(\Omega_\alpha)$ *provided that*

i. $s \neq \pm\dfrac{1}{2}$;

ii. $s = \pm\dfrac{1}{2}$, *but* $\dfrac{\pi}{2} < \alpha_0 < \dfrac{3\pi}{2}$;

iii. $s = \pm\dfrac{1}{2}$, $0 < \alpha_0 < \dfrac{\pi}{2}$ $\left(or\ \dfrac{3\pi}{2} < \alpha_0 < 2\pi\right)$, *but* α_0 (*respectively,* $2\pi - \alpha_0$) *is not a solution to one of the following transcendental equations* (*is not a forbidden angle*):

$$\sin^2\frac{\pi}{p}-\frac{\left[\cos\,\frac{\pi}{p}\cos\,\left[\frac{\pi}{p}-\alpha\left(\frac{1}{p}\pm 1\right)\right]\mp\cos\,\alpha\left(\frac{1}{p}\mp 1\right)\right]^2}{\sin^2\frac{\pi}{p}}=0,$$

$$\sin^2\frac{\pi}{p}-\frac{\left[\sin\,\frac{\pi}{p}\sin\,\left[\frac{\pi}{p}-\alpha\left(\frac{1}{p}\mp\frac{1}{2}\right)\right]+\cos\,\alpha\left(\frac{1}{p}\mp 1\right)\right]^2}{\sin^2\frac{\pi}{p}}=0.$$

If the solution exists, it is represented by formula (32) *where the unknown functions* ψ *and* φ_0 *are unique solutions to system* (35) *in the space* $\widetilde{\mathbb{W}}_p^{-1/p}(\mathbb{R}^+)$ *and* $\varphi\in\widetilde{\mathbb{W}}_p^{1-1/p}(\mathbb{R}_\alpha)$ *is recovered from* φ_0 *by formula* (36).

Thanks for the attention

&

Happy coming birthday Martin