Rellich estimates in L^p for elliptic systems

P. Auscher[1]

[1]Université Paris-Sud, France

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en l'honneur des 65 ans de Martin Costabel

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- joint work with Mihalis Mourgoglou, available on arXiv.
- earlier work with Andreas Axelsson, Alan McIntosh, Steve Hofmann

Systems

$\Omega = \mathbf{R}^{n+1}_+$. Same analysis works in unit ball and every domain obtained by bilipschitz change of variables.

Points $\mathbf{x} = (t, x)$, $t > 0, x \in \mathbf{R}^n$.

Measurable, bounded, with $M_{m\times m}(\mathbf{C})$-valued coefficients $A_{i,j}, i, j = 0, \dots n$, $m \geq 1$.

Weak solution: $u \in W^{1,2}_{loc}(\Omega; \mathbf{C}^m)$ and $Lu = 0$ holds in $\mathcal{D}'(\Omega; \mathbf{C}^m)$: with summation convention

$$\text{Re} \int_\Omega A^{\alpha,\beta}_{i,j} \partial_j u^\beta \, \overline{\partial_i \varphi^\alpha} = 0, \quad \forall \varphi \in C^\infty_0(\Omega; \mathbf{C}^m).$$

Short notation: $A^{\alpha,\beta}_{i,j} \partial_j u^\beta \, \overline{\partial_i \varphi^\alpha} = A\nabla u \cdot \nabla \overline{\varphi}$ and $Lu = \mathrm{div} A \nabla u$ in Ω.

$i = 0$ corresponds to the vertical direction, $i = 1, \dots, n$ to the horizontal directions.

Energy space

- $\mathcal{E} := \dot{H}^1(\Omega; \mathbf{C}^m) =: \{u \in \mathcal{D}'(\Omega; \mathbf{C}^m) : \|\nabla u\|_2 < \infty\}$.
- $\mathcal{E}/\mathbf{C}^m$ Banach space.
- $C_0^\infty(\overline{\Omega}; \mathbf{C}^m)$ dense in $\mathcal{E}$ and $\mathcal{E} \subset C([0,\infty); L^2_{\mathrm{loc}}(\mathbf{R}^n; \mathbf{C}^m))$.
- Trace of $\mathcal{E}$ on $\mathbf{R}^n$: $\mathcal{T} = \dot{H}^{1/2}(\mathbf{R}^n; \mathbf{C}^m)$. $\mathcal{T}$ contains $C_0^\infty(\mathbf{R}^n)$, dense. $\mathcal{T} \subset L^2_{\mathrm{loc}}(\mathbf{R}^n; \mathbf{C}^m))$. Dual $\dot{H}^{-1/2}(\mathbf{R}^n; \mathbf{C}^m)$: space of distributions.
- $\mathcal{E}_0 = \{u \in \mathcal{E} : Tr(u) \in \mathbf{C}^m\}$. $C_0^\infty(\Omega; \mathbf{C}^m)$ dense in $\mathcal{E}_0$.
- When $u \in \mathcal{E}$, $Lu = 0$ (energy solution) means $\int_\Omega A\nabla u \cdot \nabla\overline{\varphi} = 0, \quad \forall \varphi \in \mathcal{E}_0$.

Conormal derivative

If $u \in \mathcal{E}$ and $Lu = 0$, then there exists a unique distribution $g \in \mathcal{T}'$ such that

$$\int_\Omega A\nabla u \cdot \nabla\overline{\varphi} = \langle g, \varphi_0\rangle, \quad \forall \varphi \in \mathcal{E}.$$

Notation: $g = \partial_{\nu_A} u|_{t=0} = \partial_{\nu_A} u_0$.

Green's formula: If $u, w \in \mathcal{E}$ and $Lu = 0 = L^* w$, then

$$\langle u_0, \partial_{\nu_{A^*}} w_0\rangle = \langle \partial_{\nu_A} u_0, w_0\rangle. \tag{1}$$

Rellich

Comparison between $\|\nabla_{\tan} u_0\|$ and $\|\partial_{\nu_A} u_0\|$ for energy solutions in some appropriate norm on the boundary.

Alternately, study of boundedness of the Dirichlet to Neumann operator or of the Neumann to Dirichlet operator.

Dirichlet problem

Ellipticity: Assume for some $\lambda > 0$,

$$\mathrm{Re} \int_\Omega A\nabla g \cdot \nabla \overline{g} \geq \lambda \int_\Omega |\nabla g|^2, \quad \forall g \in \mathcal{E}_0. \tag{2}$$

Let $f \in \mathcal{T}$. Then, there is a unique energy solution $u \in \mathcal{E}$ of the equation $\mathrm{div} A\nabla u = 0$ with $u|_{t=0} = f$ (equality also in $L^2_{\mathrm{loc}}(\mathbf{R}^n; \mathbf{C}^m)$, hence uniqueness). Moreover,

$$\|\nabla u\|_2 \sim \|f\|_{\mathcal{T}} \sim \|\nabla_{\tan} f\|_{\mathcal{T}'}.$$

Hence, for any energy solution when A satisfies (2)

$$\|\partial_{\nu_A} u_0\|_{\mathcal{T}'} \lesssim \|\nabla_{\tan} u_0\|_{\mathcal{T}'}$$

notation: u has smooth Dirichlet datum if $u_0 \in C_0^\infty$.

Neumann problem

Ellipticity: Assume for some $\lambda > 0$,

$$\mathrm{Re} \int_\Omega A\nabla g \cdot \nabla \overline{g} \geq \lambda \int_\Omega |\nabla g|^2, \quad \forall g \in \mathcal{E}. \tag{3}$$

Let $g \in \mathcal{T}'$. Then, there is a energy solution $u \in \mathcal{E}$, unique modulo constants, of the equation $\mathrm{div} A\nabla u = 0$ with $\partial_{\nu_A} u|_{t=0} = g$. Moreover,

$$\|\nabla u\|_2 \sim \|g\|_{\mathcal{T}'}.$$

Hence, for any energy solution when A satisfies (3)

$$\|\nabla_{\tan} u_0\|_{\mathcal{T}'} \lesssim \|\partial_{\nu_A} u_0\|_{\mathcal{T}'}$$

Notation: u has smooth Neumann datum if $g = \partial_{\nu_A} u_0 \in C_0^\infty$ with $\int g = 0$.

BVP problems in L^p, $1 < p < \infty$

Typical problems in harmonic analysis (for example for the Laplace equation).

• (Dir, A, p): Solve $Lu = 0$ with $\|\widetilde{N}(u)\|_p < \infty$ and $u_0 = f$ given in $L^p(\mathbf{R}^n; \mathbf{C}^m)$.

• (Reg, A, p): Solve $Lu = 0$ with $\|\widetilde{N}(\nabla u)\|_p < \infty$ and $\nabla_{\tan} u_0 = \nabla_{\tan} f$, f given in $\dot{W}^{1,p}(\mathbf{R}^n; \mathbf{C}^m)$.

• (Neu, A, p): Solve $Lu = 0$ with $\|\widetilde{N}(\nabla u)\|_p < \infty$ and $\partial_{\nu_A} u_{|t=0} = g$ given in $L^p(\mathbf{R}^n; \mathbf{C}^m)$.

$\widetilde{N}(h)$ is non-tangential maximal interior control of h defined in Ω: it comes up quite naturally.

Not always solvable nor well-posed. No comprehensive theory at this time. To solve, Rellich inequalities are needed (not enough).

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Whitney ball:

$$W(t,x) := [(1-c_0)t, (1+c_0)t] \times B(x; c_1 t),$$

for fixed $c_0 \in (0,1)$, $c_1 > 0$.

$$\widetilde{N}(h)(x) := \sup_{t>0} t^{-(n+1)/2} \|h\|_{L_2(W(t,x))}$$

Duality in Rellich estimates for the Dirichlet problem

Theorem

Let $A(\mathbf{x})$ be a bounded measurable matrix with the Gårding inequality (2). Let $1 < p < \infty$. The following are equivalent.

1. *There exists $C_p < \infty$ such that for any $u \in \mathcal{E}$ solution of* $\operatorname{div} A \nabla u = 0$, $\|\partial_{\nu_A} u_0\|_p \le C_p \|\nabla_{\tan} u_0\|_p$.
2. *There exists $C_{p'} < \infty$ such that for any $w \in \mathcal{E}$ solution of* $\operatorname{div} A^* \nabla w = 0$, $\|\partial_{\nu_{A^*}} w_0\|_{\dot{W}^{-1,p'}} \le C_{p'} \|\nabla_{\tan} w_0\|_{\dot{W}^{-1,p'}}$.

Moreover, in any direction it suffices the assumption holds for energy solutions with smooth Dirichlet data.

The tangential gradient and conormal derivative at the boundary are distributions in $\mathbf{R}^n$ (in $\mathcal{T}'$). Thus, finiteness of any of the norms above means that the distribution is identified with an element in the considered space which is also embedded in the space of distributions.

Duality in Rellich estimates for the Neumann problem

Theorem

Let $A(\mathbf{x})$ be a bounded measurable matrix with the stronger Gårding inequality (3). Let $1 < p < \infty$. The following are equivalent.

1. *There exists $C_p < \infty$ such that for any $u \in \mathcal{E}$ solution of* $\mathrm{div} A \nabla u = 0$, $\|\nabla_{\tan} u_0\|_p \le C_p \|\partial_{\nu_A} u_0\|_p$.
2. *There exists $C_{p'} < \infty$ such that for any $w \in \mathcal{E}$ solution of* $\mathrm{div} A^* \nabla w = 0$, $\|\nabla_{\tan} w_0\|_{\dot W^{-1,p'}} \le C_{p'} \|\partial_{\nu_{A^*}} w_0\|_{\dot W^{-1,p'}}$.

Moreover, in any direction it suffices the assumption holds for energy solutions with smooth Neumann data.

Conclusion

There is a duality principle between Dirichlet problems in L^p and $\dot{W}^{-1,p'}$ for L and L^* and a similar duality principle for Neumann problems in L^p and $\dot{W}^{-1,p'}$ for L and L^*.

No assumption on A but bounded + elliptic. Systems OK.

One can formulate similar results for $\frac{n}{n+1} < p \leq 1$ using Hardy and Hardy-Sobolev spaces and their duals.

Problem: Rellich not enough. Need also identification of the space of solutions corresponding to $\|\nabla_{\tan} u_0\|_X + \|\partial_{\nu_A} u_0\|_X$.

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Application 1

Theorem

(Auscher-Axelsson) Assume $A(\mathbf{x}) = A(t,x) = A(x)$ *is independent of t, with the Garding inequality*

$$\text{Re} \int_{\mathbf{R}^n} A(x)f(x) \cdot \overline{f(x)}\, dx \geq \lambda \int_{\mathbf{R}^n} |f(x)|^2 dx,\ \forall\, f \in \mathcal{H}_0, \qquad (4)$$

where $f \in \mathcal{H}_0$ *means* $f = (f_\perp, f_\parallel) \in L^2(\mathbf{R}^n; \mathbf{C}^m \otimes (\mathbf{C}^m)^n)$ *with* $\mathrm{curl}_{\tan} f_\parallel = 0$. *Then, for any weak solution, when the LHS is finite,*

$$\|\widetilde{N}(\nabla u)\|_2 \sim \|\nabla_{\tan} u_0\|_2 + \|\partial_{\nu_A} u_0\|_2.$$

$$\iint_\Omega t|\nabla u(t,x)|^2\, dtdx \sim \|\nabla_{\tan} u_0\|^2_{\dot{W}^{-1,2}} + \|\partial_{\nu_A} u_0\|^2_{\dot{W}^{-1,2}}.$$

Note: This holds also for $A(t,x)$ satisfying (4) uniformly in t and a certain Carleson type condition measuring t-regularity.

Application 1: continued

Theorem

(Jerison-Kenig and Kenig-Pipher for real equations, Auscher-Axelsson-McIntosh for systems) If A is t-independent with (4) and $A = A^*$, then for any weak solution with $\|\widetilde{N}(\nabla u)\|_2 < \infty$ or any energy solution

$$\|\nabla_{\tan} u_0\|_2 \sim \|\partial_{\nu_A} u_0\|_2.$$

Furthermore, (Dir′, A, 2), (Reg, A, 2), (Neu, A, 2) are well-posed.

Here, (Dir′, A, 2) is formulated as follows: $Lu = 0$, $u_0 = f$ given in $L^2(\mathbf{R}^n; \mathbf{C}^m)$ with interior control $\iint_\Omega t|\nabla u(t,x)|^2\, dtdx < \infty$ instead of $\|\widetilde{N}(u)\|_2 < \infty$.

This comparison follows from an integral identity in the spirit of the identities discovered by Rellich for eigenvalue problems. Self-adjointness is essential.

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Theorem

(Jerison-Kenig and Kenig-Pipher for real equations, Auscher-Axelsson-McIntosh for systems) If A is t-independent with (4) *and* $A = A^*$, *then for any weak solution with* $\|\widetilde{N}(\nabla u)\|_2 < \infty$ *or any energy solution*

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This comparison follows from an integral identity in the spirit of the identities discovered by Rellich for eigenvalue problems.
Self-adjointness is essential.

Application 1: continued

Theorem

(Auscher-McIntosh-Mourgoglou) If A is t-independent with (4) *and A block lower-triangular ($A_{0,j} = 0$ for $j = 1, \dots, n$), then for any weak solution with $\|\widetilde{N}(\nabla u)\|_2 < \infty$ or any energy solution*

$$\|\nabla_{\tan} u_0\|_2 \lesssim \|\partial_{\nu_A} u_0\|_2.$$

Furthermore, (Neu, A, 2) is well-posed.

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$$\|\partial_{\nu_A} u_0\|_2 \lesssim \|\nabla_{\tan} u_0\|_2.$$

Furthermore, (Reg, A, 2) and (Dir', A^*, 2) are well-posed.

Application 2

Theorem

(Hofmann-Kenig-Mayboroda-Pipher, 2012 & 2013) If A is t-independent, real with $m = 1$ (equation), then there is a $2 \leq p_0 < \infty$ such that (Dir, A, p) is well-posed and (Reg, A^*, p') is well-posed when $p_0 < p < \infty$.

To prove Dirichlet, they showed for any energy solution

$$\|\partial_{\nu_A} u_0\|_{\dot{W}^{-1,p}} \lesssim \|\nabla_{\tan} u_0\|_{\dot{W}^{-1,p}}. \tag{5}$$

Their derivation of (Reg, A^*, p') is quite technical. Here is a simple (at least conceptually) way for existence: We have (A.-Mourgoglou) for any energy solution of $L^* u = 0$ and $1 < p' \leq 2$,

$$\|\widetilde{N}(\nabla u)\|_{p'} \sim \|\nabla_{\tan} u_0\|_{p'} + \|\partial_{\nu_{A^*}} u_0\|_{p'}.$$

Apply (5) and our duality result:

$$\|\partial_{\nu_{A^*}} u_0\|_{p'} \lesssim \|\nabla_{\tan} u_0\|_{p'}.$$

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