Boundary element methods for impedance transmission conditions

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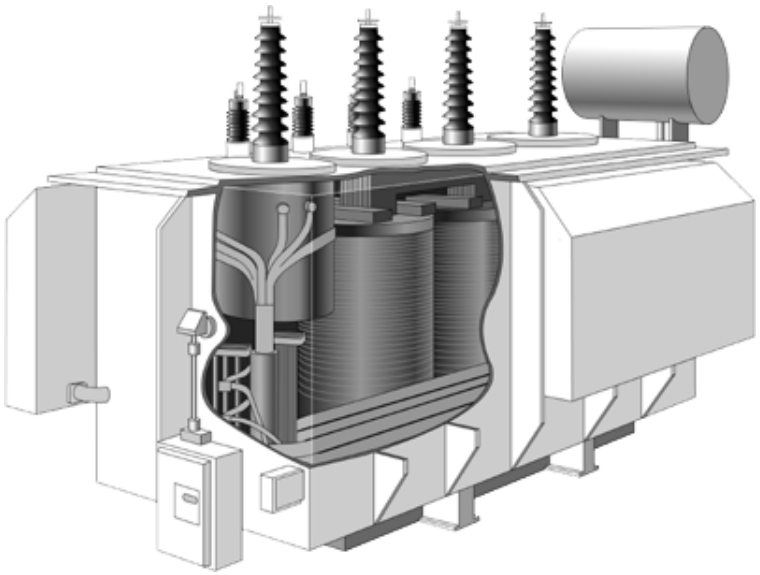
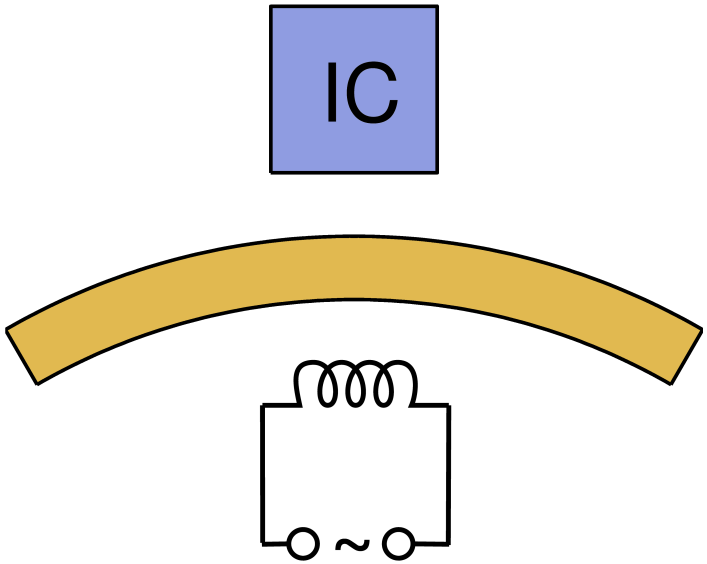
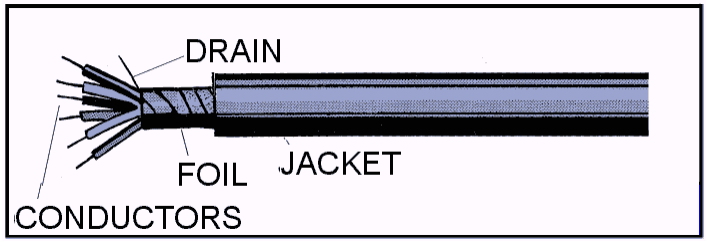
DFG Research Center MATHEON
Mathematics for key technologies



Transformator inside a conducting casing.



Protection of integrated circuits.



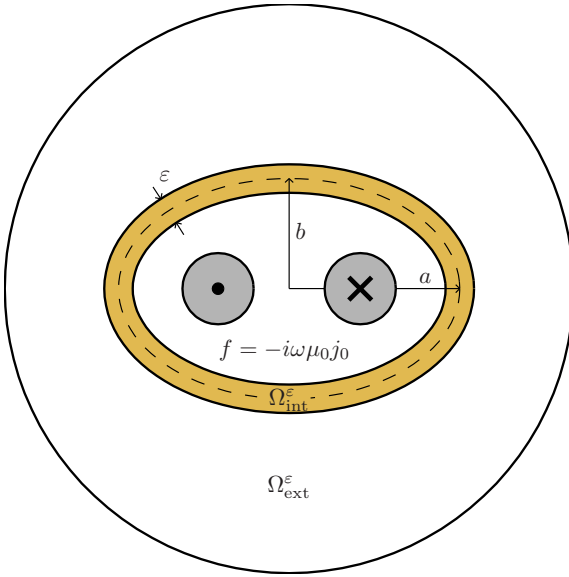
Cable with a foil shield.

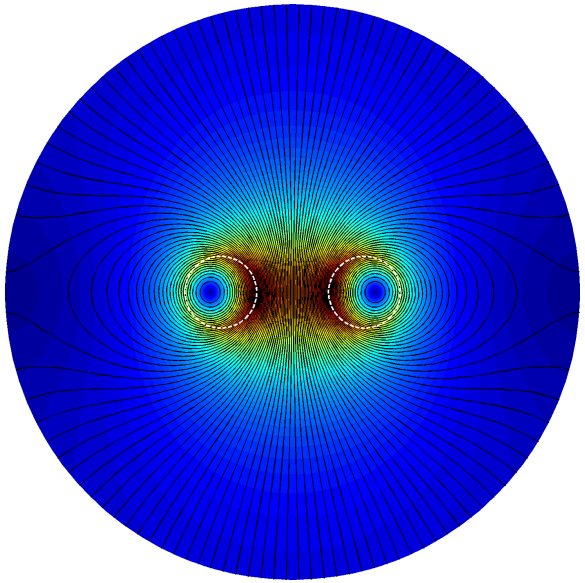
Thin conducting sheets

- for electromagnetic shielding,
- for mechanical stability,
- as casing for liquids and gases,
- with little material wastage.

Eddy current model in 2D (TM polarisation)

$$-\Delta e(\mathbf{x}) + i\omega\mu_0\sigma(\mathbf{x})\, e(\mathbf{x}) = -i\omega\mu_0 j_0(\mathbf{x})$$



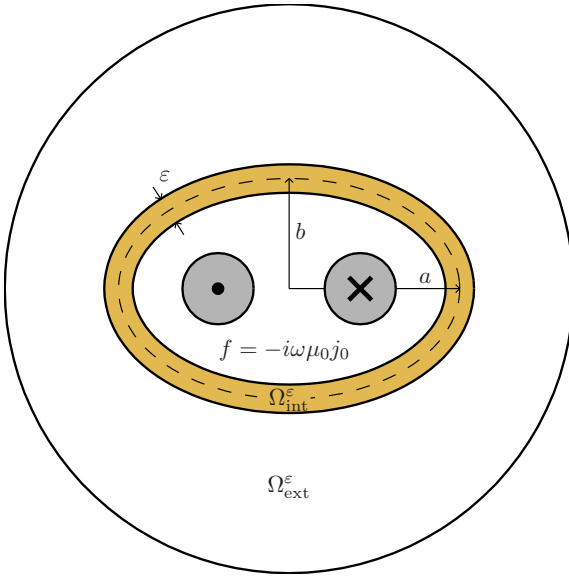


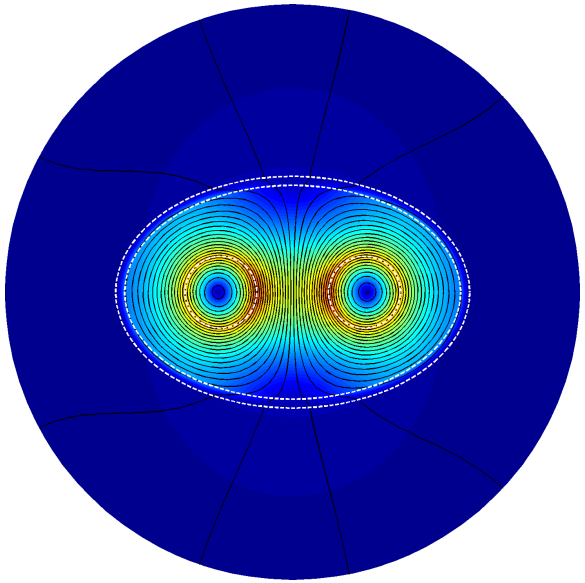
Eddy current model in 2D (TM polarisation)

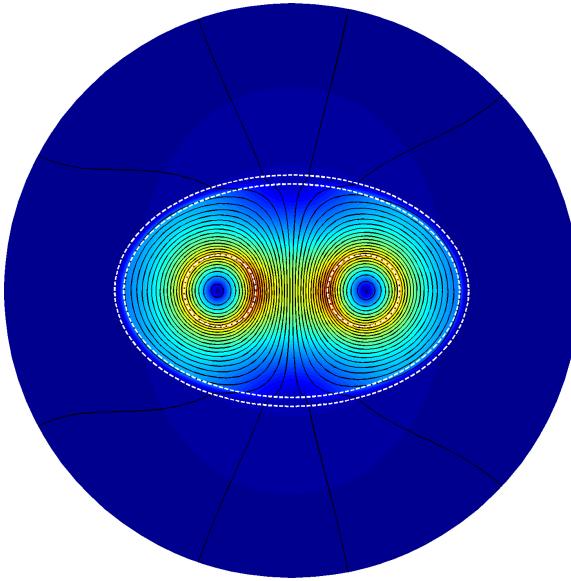
$$-\Delta e(\mathbf{x}) + i\omega\mu_0\sigma(\mathbf{x})\, e(\mathbf{x}) = -i\omega\mu_0 j_0(\mathbf{x})$$

Two important effects of the thin sheet

- Shielding effect – in conductors induced currents diminish electromagnetic fields (behind the conductor)
- Skin effect – major current flow in a boundary layer (skins of the conductor)
  - Skin depth in solid body $\delta = \sqrt{\frac{2}{\mu_0\sigma\omega}}$
  - Copper at 50 Hz $\rightarrow \delta \approx 8\mathrm{mm}$







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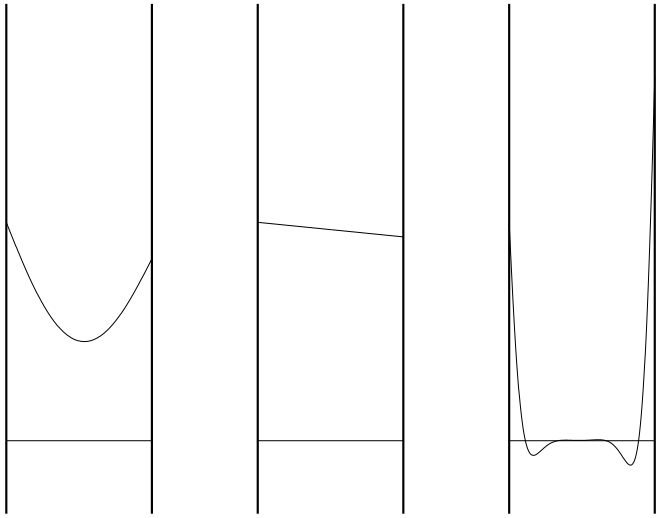
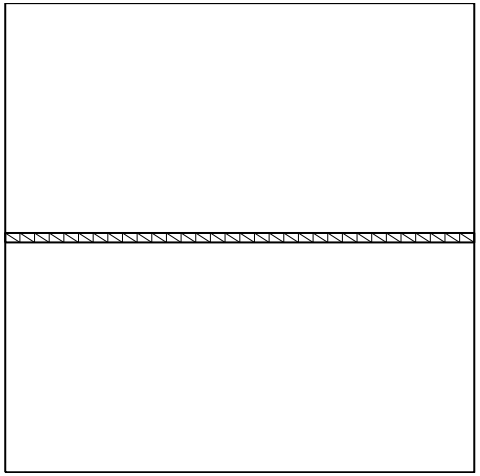
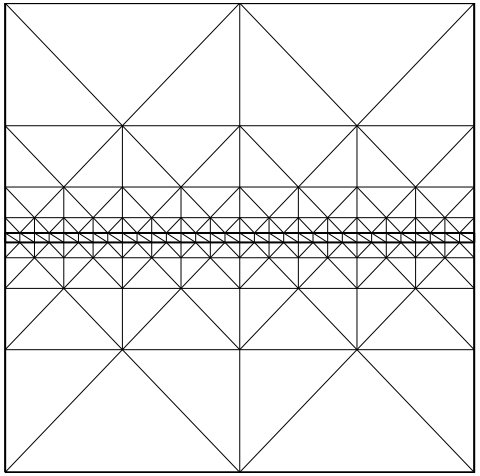


$d \sim \delta$ $\quad$ $d \ll \delta$ $\quad$ $d \gg \delta$

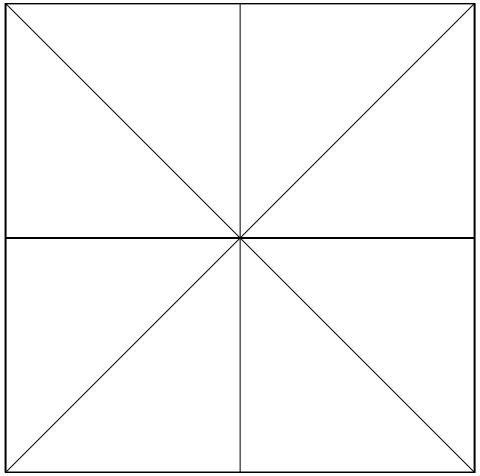


Triangle mesh inside the sheet.



Conforming triangle mesh resolving the sheet.

- High number of triangles to achieve conforming mesh of good quality.
- Even more triangles for resolving the sheet.



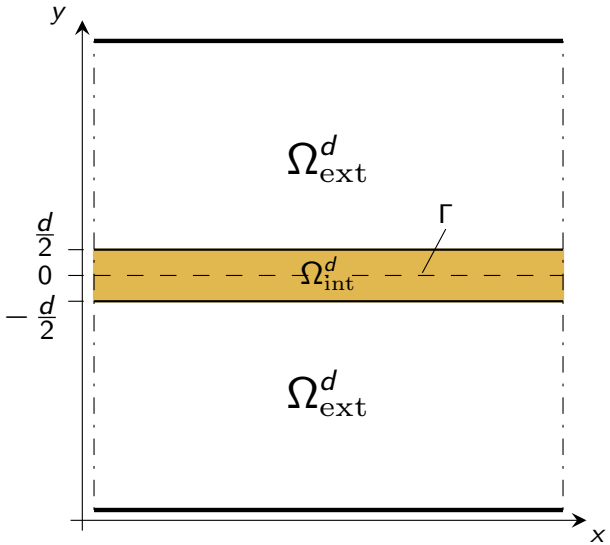
Conforming triangle mesh resolving an interface.

- Replace the thin sheet by an interface.
- Replace its behaviour by transmission conditions, *e. g.*

$$[e] = 0,$$
$$[\partial_n e] = \mathrm{i}\omega\mu\sigma\, d\{e\}$$

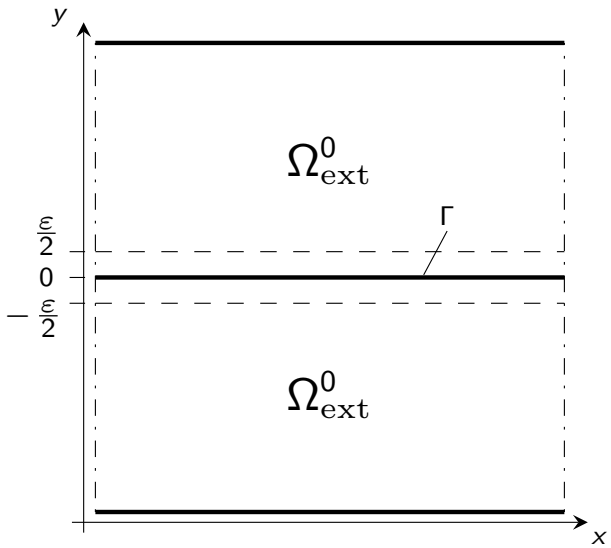
Mesh of interface.

- Reduction to interface by boundary integral formulation.



Original problem

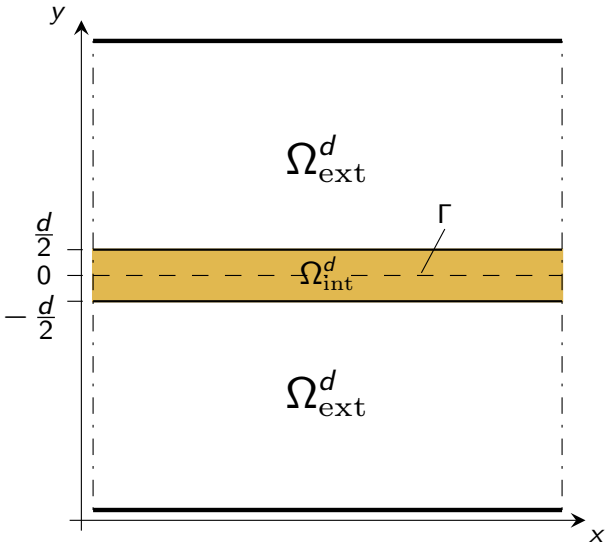
$$\begin{aligned} L_{\text{ext}}(\omega)e_{\text{ext}} &= f \quad \text{in } \Omega^d_{\text{ext}} \\ L_{\text{int}}(\omega)e_{\text{int}} &= 0 \quad \text{in } \Omega^d_{\text{int}} \end{aligned} \tag{1}$$



Reduced problem with transmission conditions

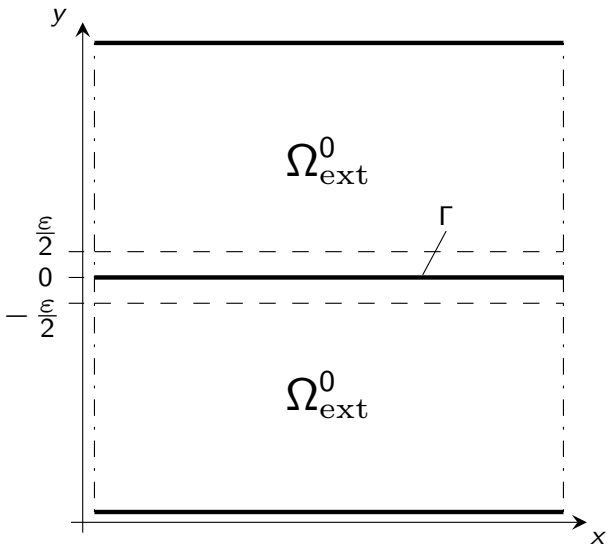
$$\begin{aligned} L_{\text{ext}}(\omega)e_{\text{ext}} &= f \quad \text{in } \Omega^0_{\text{ext}} \\ T(\omega, d)e_{\text{ext}} &= 0 \quad \text{on } \Gamma \end{aligned} \tag{2}$$

- e_{ext} is extended from Ω^d_{ext} into Ω^0_{ext}
- extended e_{ext} satisfy some transmission conditions on Γ
- solutions e_{ext} of (1) and (2) coincide in Ω^d_{ext}
- e_{int} in Ω^d_{int} can be computed *a-posteriori*



Original eddy current problem

$$
\begin{aligned}
-\Delta e_{\text{ext}}(\mathbf{x}) &= -i\omega\mu_0 j_0(\mathbf{x}) && \text{in } \Omega^d_{\text{ext}} \\
-\Delta e_{\text{int}}(\mathbf{x}) + i\omega\mu_0\sigma\, e_{\text{int}}(\mathbf{x}) &= 0 && \text{in } \Omega^d_{\text{int}}
\end{aligned}
$$



Reduced problem with transmission conditions ITC-1-0

(derived by asymptotic expansion)

$$
\begin{aligned}
-\Delta \widetilde{e}_{\text{ext}}(\mathbf{x}) &= -\mathrm{i}\omega\mu_0 j_0(\mathbf{x}) && \text{in } \Omega^0_{\text{ext}} \\
[\widetilde{e}_{\text{ext}}](x) &= 0 && \text{on } \Gamma \\
[\partial_n \widetilde{e}_{\text{ext}}](x) - \mathrm{i}\omega\mu_0\sigma\, d\{\widetilde{e}_{\text{ext}}\}(x) &= 0 && \text{on } \Gamma \\
\hline
\widetilde{e}_{\text{int}}(x,y) &= \{\widetilde{e}_{\text{ext}}\}(x) && \text{in } \Omega^d_{\text{int}}
\end{aligned}
$$

- ▷ $\widetilde{e}_{\text{ext}}$ is defined in Ω^0_{ext} and approximates e_{ext} in Ω^d_{ext}
- ▷ $\widetilde{e}_{\text{int}}$ is *a-posteriori* computed in Ω^d_{int} and approximates e_{int}

Reduced problem with transmission conditions ITC-1-0

$$
\begin{aligned}
-\Delta \widetilde{e}_{\mathrm{ext}}(\mathbf{x}) &= -\mathrm{i}\omega\mu_0 j_0(\mathbf{x}) && \text{in } \Omega^0_{\mathrm{ext}} \\
[\widetilde{e}_{\mathrm{ext}}](x) &= 0 && \text{on } \Gamma \\
[\widetilde{h}_{x,\mathrm{ext}}](x) + \sigma\, d\{\widetilde{e}_{\mathrm{ext}}\}(x) &= 0 && \text{on } \Gamma
\end{aligned}
$$

- goes back to Levi-Civita (1902), see *e.g.* in the book of Bateman (1915)

> In the case of a very thin conducting sheet it is convenient to treat the thickness of the sheet as negligible and regard the tangential components of the magnetic force as discontinuous when a point moves along the normal from one side of the sheet to the other.

- in review article on "Approximate boundary conditions" by Senior (1981)
- boundary integral formulation by Knott and Senior (1974) and McWirther (1982)
- used in a network approach by Carpenter and Djurovic (1975)
- used in FEM by Poltz and Romanowski (1983), Rodger and Atkinson (1988), Biro *et. al.* (1992), Jin *et.al.* (1992)
- derived as limit conditions for asymptotically small thickness, $d \to 0$ while $\omega\sigma \sim d^{-1}$ and error analysis by K.S., S. Tordeux (2010) $\to$ ITC-1-0

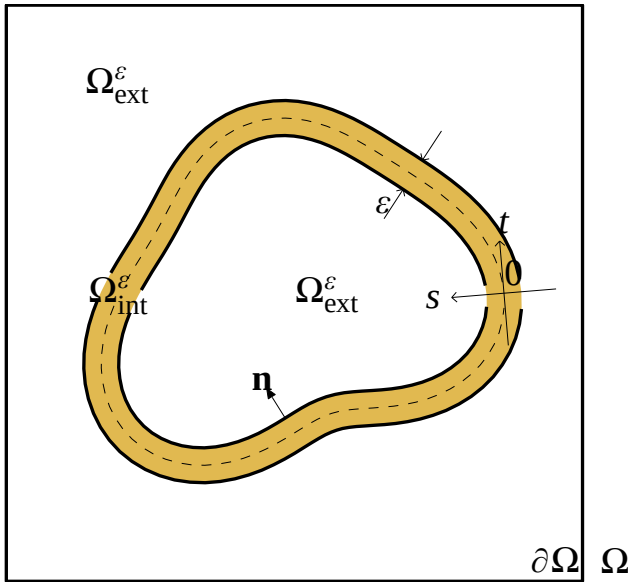
Asymptotics – frameworks with ε varying conductivity

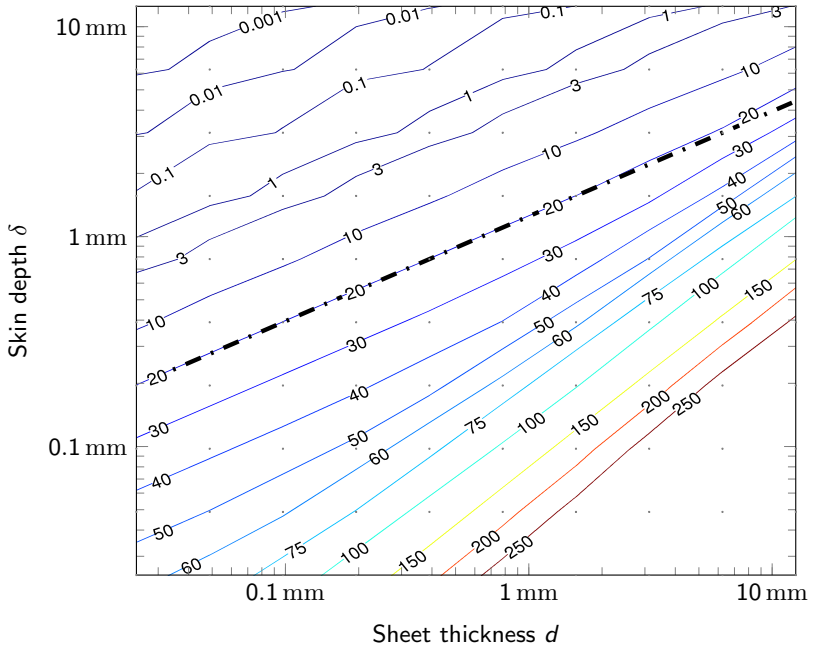
$$-\Delta u^\varepsilon + c(\varepsilon)\, u^\varepsilon = f$$

$c(\varepsilon) = c_0$ ... compare sheets of constant material

$c(\varepsilon) = \frac{c_1}{\varepsilon}$ ... compare sheets of constant shielding

$c(\varepsilon) = \frac{c_2}{\varepsilon^2}$ ... compare sheet of constant skin depth





Shielding efficieny $SE(\mathbf{x}_j) = 20 \log_{10} \frac{|\mathbf{H}_0(\mathbf{x}_j)|}{|\mathbf{H}(\mathbf{x}_j)|}$

Asymptotics – frameworks with ε varying conductivity

$$-\Delta u^\varepsilon + c(\varepsilon)\, u^\varepsilon = f$$

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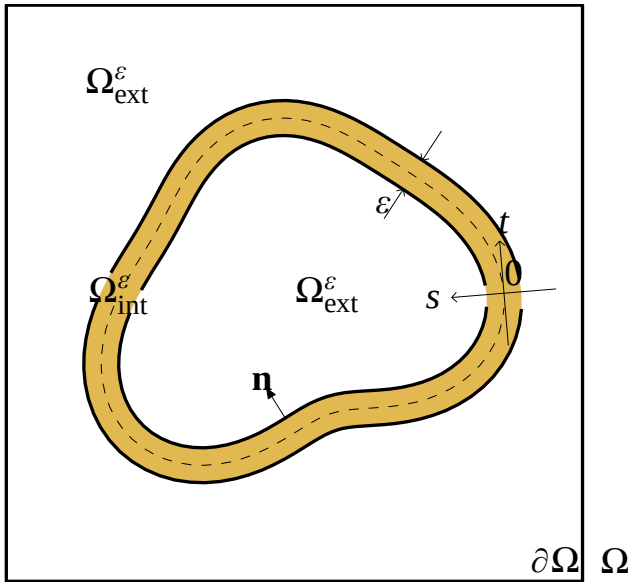

Lemma (Asymptotics for $c(\varepsilon) = \frac{c_0}{\varepsilon^\alpha}$)

Let $\alpha \in [0,2]$, Γ being $C^{1,1}$ continuous. Then

$$\|\{u^\varepsilon\}^{+\frac{\varepsilon}{2}}_{-\frac{\varepsilon}{2}}\|_{L^2(\Gamma)} \le C\, \varepsilon^{\max(0,\alpha-1)} \qquad \|[\partial_n u^\varepsilon]^{+\frac{\varepsilon}{2}}_{-\frac{\varepsilon}{2}}\|_{L^2(\Gamma)} \le C\, \varepsilon^{\max(0,1-\alpha)}$$

Borderline case $\alpha = 1$ of asymptotic constant shielding

- only non-trivial limit problem (ITC-1-0) for $\varepsilon \to 0$
- for $\alpha > 1$ the limit $\varepsilon \to 0$ is PEC $u^0 = 0$ on Γ
- for $\alpha < 1$ the limit $\varepsilon \to 0$ is "no sheet": $[u^0] = [\partial_n u^0] = 0$

Original eddy current problem

$$
\begin{aligned}
-\Delta e_{\text{ext}}(\mathbf{x}) &= -i\omega\mu_0 j_0(\mathbf{x}) && \text{in } \Omega^d_{\text{ext}} \\
-\Delta e_{\text{int}}(\mathbf{x}) + i\omega\mu_0\sigma\, e_{\text{int}}(\mathbf{x}) &= 0 && \text{in } \Omega^d_{\text{int}}
\end{aligned}
$$

Reduced problem with transmission conditions ITC-1-0

$$
\begin{aligned}
-\Delta \widetilde{e}_{\text{ext}}(\mathbf{x}) &= -\mathrm{i}\omega\mu_0 j_0(\mathbf{x}) && \text{in } \Omega^0_{\text{ext}} \\
[\widetilde{e}_{\text{ext}}](x) &= 0 && \text{on } \Gamma \\
[\partial_y \widetilde{e}_{\text{ext}}](x) - \mathrm{i}\omega\mu_0\sigma\, d\{\widetilde{e}_{\text{ext}}\}(x) &= 0 && \text{on } \Gamma \\
\hline
\widetilde{e}_{\text{int}}(x,y) &= \{\widetilde{e}_{\text{ext}}\}(x) && \text{in } \Omega^d_{\text{int}}
\end{aligned}
$$

Lemma (Order zero in its asymptotic regime, K.S. and S.Tordeux (2010))

Let $\omega\sigma \sim d^{-1}$, *then* $\|e_{\text{ext}} - \widetilde{e}_{\text{ext}}\|_{H^1(\Omega^d_{\text{ext}})} \leq C\, d$ *and* $\|e_{\text{int}} - \widetilde{e}_{\text{int}}\|_{H^1(\Omega^d_{\text{int}})} \leq C\sqrt{d}$.

Lemma (Robustness for low and high frequency, K.S. and A.Chernov (2012))

Independently of $\omega\sigma$ *it holds* $\|e_{\text{ext}} - \widetilde{e}_{\text{ext}}\|_{H^1(\Omega^d_{\text{ext}})} \leq C(d)$ *where* $\lim_{d\to 0} C(d) = 0$.

Reduced problem with transmission conditions ITC-1-0

$$
\begin{aligned}
-\Delta \widetilde{e}_{\mathrm{ext}}(\mathbf{x}) &= -\mathrm{i}\omega\mu_0 j_0(\mathbf{x}) && \text{in } \Omega^0_{\mathrm{ext}} \\
[\widetilde{e}_{\mathrm{ext}}](x) &= 0 && \text{on } \Gamma \\
[\partial_y \widetilde{e}_{\mathrm{ext}}](x) - \mathrm{i}\omega\mu_0\sigma\, d\{\widetilde{e}_{\mathrm{ext}}\}(x) &= 0 && \text{on } \Gamma
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Lemma (Robustness w.r.t. to the frequency, K.S. and A.Chernov (2012))

Independently of $\omega\sigma$ it holds $\|e_{\mathrm{ext}} - \widetilde{e}_{\mathrm{ext}}\|_{H^1(\Omega^d_{\mathrm{ext}})} \leq C(d)$ where $\lim_{d\to 0} C(d) = 0$.

In fact $\|e_{\mathrm{ext}} - \widetilde{e}_{\mathrm{ext}}\|_{H^1(\Omega^d_{\mathrm{ext}})} \leq C\, d$.

Reduced problem with transmission conditions ITC-1-0

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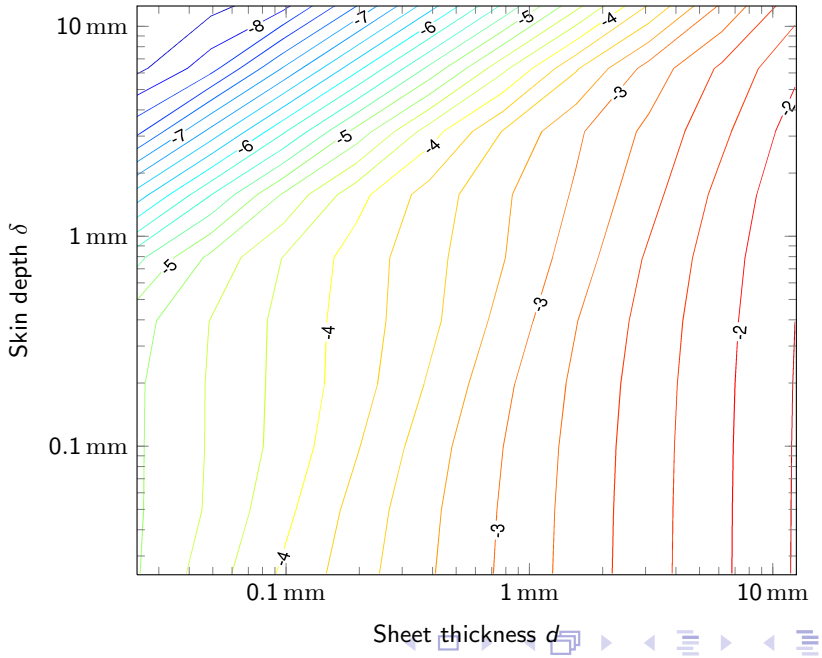
In fact $\|e_{\mathrm{ext}} - \widetilde{e}_{\mathrm{ext}}\|_{H^1(\Omega^d_{\mathrm{ext}})} \leq C\, d$.

- ▷ and some extra accuracy for large skin depth (low frequency)



Skin depth $\delta \sim 1/\sqrt{\omega\sigma}$

ITC-1-1

$$-\Delta \widetilde{e}_{\text{ext}}(\mathbf{x}) = -\mathrm{i}\omega\mu_0 j_0(\mathbf{x}) \quad \text{in } \Omega^0_{\text{ext}}$$

$$\left[\widetilde{e}_{\text{ext}}\right](t) = 0, \quad \text{on } \Gamma$$

$$\left[\partial_n \widetilde{e}_{\text{ext}}\right](t) - \mathrm{i}\omega\mu_0\sigma\, d\left(1 - \frac{\mathrm{i}\omega\mu_0\sigma}{6} d^2\right)\left\{\widetilde{e}_{\text{ext}}\right\}(t) = 0, \quad \text{on } \Gamma$$

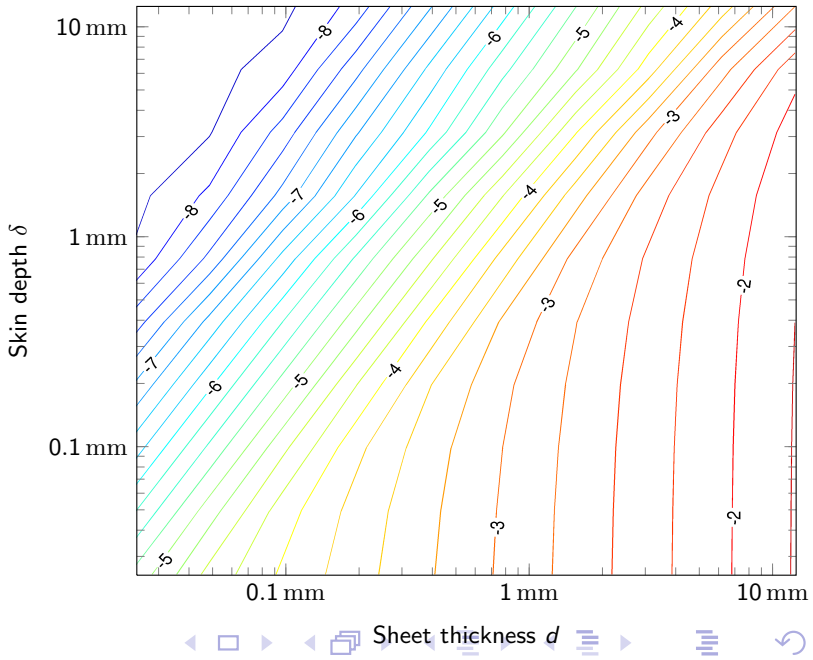
$$\widetilde{e}_{\text{int}}(s,t) = \{\widetilde{e}_{\text{ext}}\}(t)\left(1 - \frac{\mathrm{i}\omega\mu_0\sigma\, d}{2}\left(\frac{s}{d}\right)^2 + \frac{1}{4}\right) + \{\partial_n \tilde{e}_{\text{ext}}\}(t)\, s \quad \text{in } \Omega^d_{\text{int}}$$

- First order accurate for $\sigma\omega \sim d^{-1}$:

$$\|e_{\text{ext}} - \widetilde{e}_{\text{ext}}\|_{H^1(\Omega^d_{\text{ext}})} \leq C\, d^2$$

$$\|e_{\text{int}} - \widetilde{e}_{\text{int}}\|_{H^1(\Omega^d_{\text{int}})} \leq C\, d^{\frac{3}{2}}$$

- Robust in $\sigma\omega$: $\|e_{\text{ext}} - \widetilde{e}_{\text{ext}}\|_{H^1(\Omega^d_{\text{ext}})} \leq C\, d$
- same structure as ITC-1-0, accuracy improved for low and intermediate shielding



ITC-1-1

$$\left[\widetilde{e}_{\mathrm{ext}}\right](t) = 0,$$

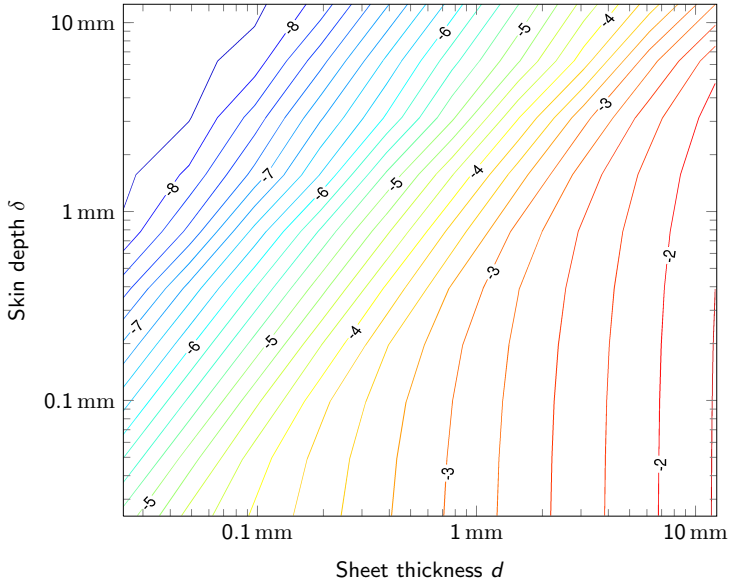
$$\left[\partial_n \widetilde{e}_{\mathrm{ext}}\right](t) - \mathrm{i}\omega\mu_0\sigma\, d\left(1 - \frac{\mathrm{i}\omega\mu_0\sigma}{6} d^2\right)\left\{\widetilde{e}_{\mathrm{ext}}\right\}(t) = 0$$

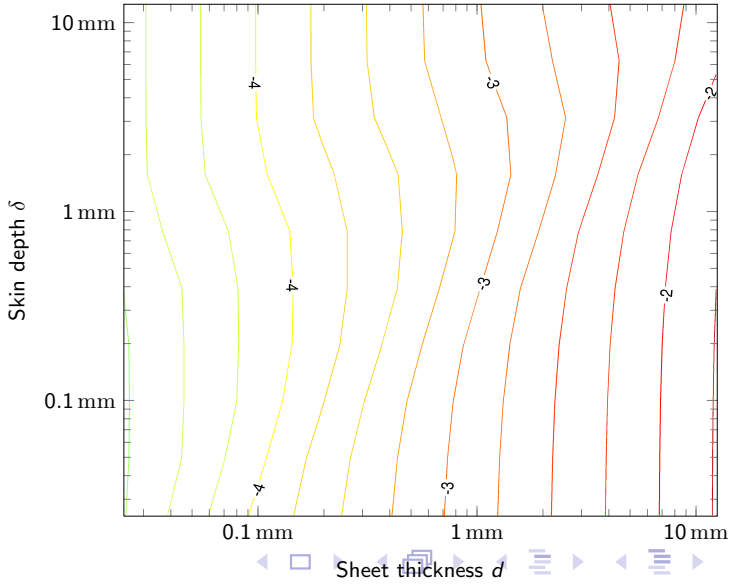
Comparison with "intuitive" low-frequency shielding element by Nakata *et.al.* (1990)

Shielding element

$$\left[\widetilde{e}_{\mathrm{ext}}\right](t) = 0,$$

$$\left[\partial_n \widetilde{e}_{\mathrm{ext}}\right](t) - d\left(\mathrm{i}\omega\mu_0\sigma - \partial_t^2\right)\left\{\widetilde{e}_{\mathrm{ext}}\right\}(t) = 0$$





ITC-1-2

$$[\widetilde{e}_{\mathrm{ext}}](t) - d^3 \frac{\mathrm{i}\omega\mu_0\sigma\kappa(t)}{24}\{\widetilde{e}_{\mathrm{ext}}\}(t) - \frac{\mathrm{i}\omega\mu_0\sigma}{12} d^3 \{\partial_n \widetilde{e}_{\mathrm{ext}}\}(t) = 0,$$

$$[\partial_n \widetilde{e}_{\mathrm{ext}}](t) + \mathrm{i}\omega\mu_0\sigma d\Big(1 - \frac{\mathrm{i}\omega\mu_0\sigma}{6} d^2 - \frac{d^2}{12}\big(\frac{7\omega^2\mu_0^2\sigma^2}{20} d^2 + \partial_t^2\big)\Big)\{\widetilde{e}_{\mathrm{ext}}\}(t) + d^3 \frac{\mathrm{i}\omega\mu_0\sigma\kappa(t)}{24}\{\partial_n \widetilde{e}_{\mathrm{ext}}\}(t) = 0,$$

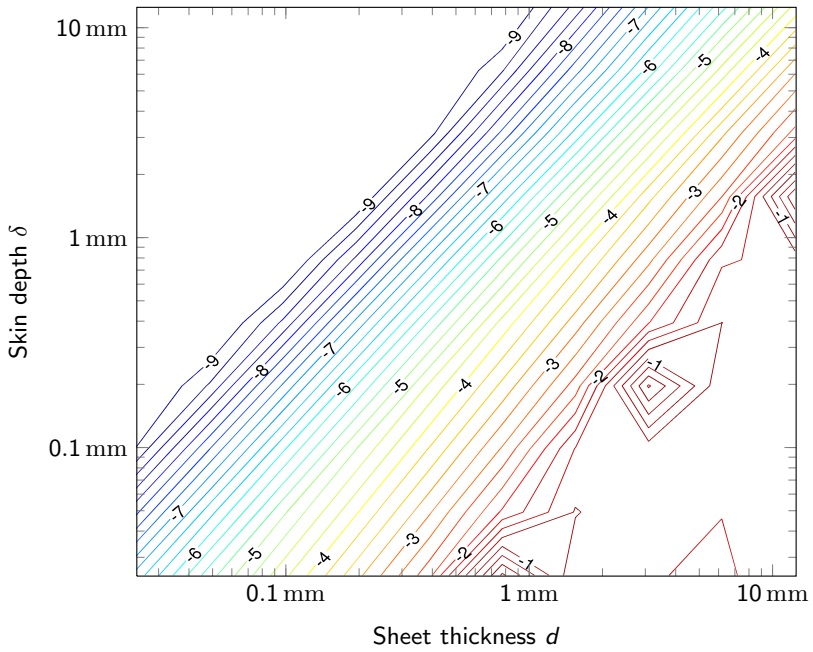
- Second order accurate for $\sigma\omega \sim d^{-1}$:

$$\|e_{\mathrm{ext}} - \widetilde{e}_{\mathrm{ext}}\|_{H^1(\Omega^d_{\mathrm{ext}})} \leq C\, d^3$$

$$\|e_{\mathrm{int}} - \widetilde{e}_{\mathrm{int}}\|_{H^1(\Omega^d_{\mathrm{int}})} \leq C\, d^{\frac{5}{2}}$$

- not robust in $\sigma\omega$, for $\sigma\omega \to \infty$:
  $\{\widetilde{e}_{\mathrm{ext}}\}, \{\partial_n \widetilde{e}_{\mathrm{ext}}\} \to 0$, but
  $e_{\mathrm{ext}}|_{\pm\frac{d}{2}} \to 0$ (PEC), and so

$$\lim_{\omega\sigma\to\infty} \|e_{\mathrm{ext}} - \widetilde{e}_{\mathrm{ext}}\|_{H^1(\Omega^d_{\mathrm{ext}})} = C$$



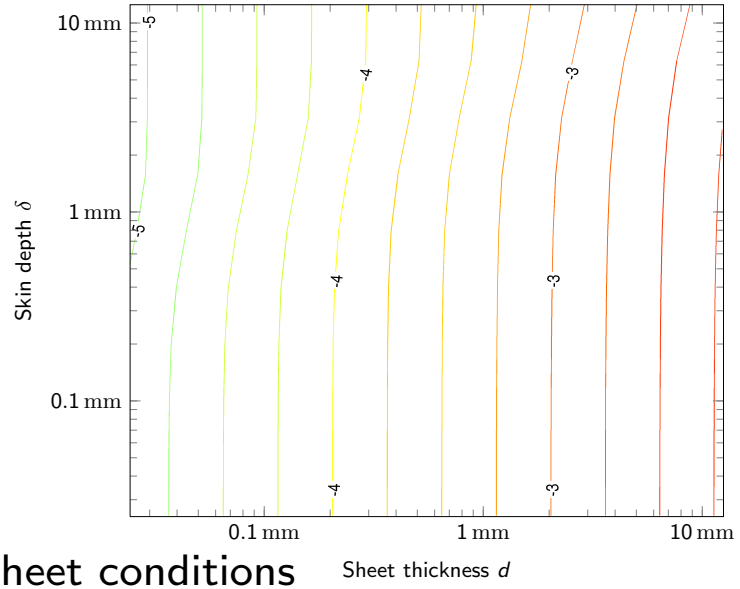
- more general structure than ITC-1-0/ITC-1-1, accuracy improved for low and intermediate shielding

Reduced problem with thin sheet conditions

$$
\begin{aligned}
-\Delta \widetilde{e}_{\mathrm{ext}}(\mathbf{x}) &= -\mathrm{i}\omega\mu_0 j_0(\mathbf{x}) \\
\left[\widetilde{e}_{\mathrm{ext}}\right] &= 2\sqrt{-\mathrm{i}\omega\mu_0\sigma}^{-1} \tanh\left(\sqrt{-\mathrm{i}\omega\mu_0\sigma}\,\tfrac{d}{2}\right)\left\{\partial_y \widetilde{e}_{\mathrm{ext}}\right\} \\
\left[\partial_y \widetilde{e}_{\mathrm{ext}}\right] &= 2\sqrt{-\mathrm{i}\omega\mu_0\sigma} \quad \tanh\left(\sqrt{-\mathrm{i}\omega\mu_0\sigma}\,\tfrac{d}{2}\right)\left\{\widetilde{e}_{\mathrm{ext}}\right\}
\end{aligned}
$$

- first time in the book of Tozoni and Mayergoyz (1974)
- article in english by Mayergoyz and Bedrosian (1995)
- Boundary integral equation by Krähenbühl and Muller (1993), Igarashi *et.al.* (1998)
- FEM by Guérin *et.al.* (1995)

- Structure is more complicated than ITC-1-0, but about the same accuracy
- ... also more complicated than ITC-1-1, but less accurate in low-frequency
- $\|e - \widetilde{e}\|_{H^1(\Omega^d_{\mathrm{ext}})} = O(d)$ indep. of $\omega\sigma$(robust)



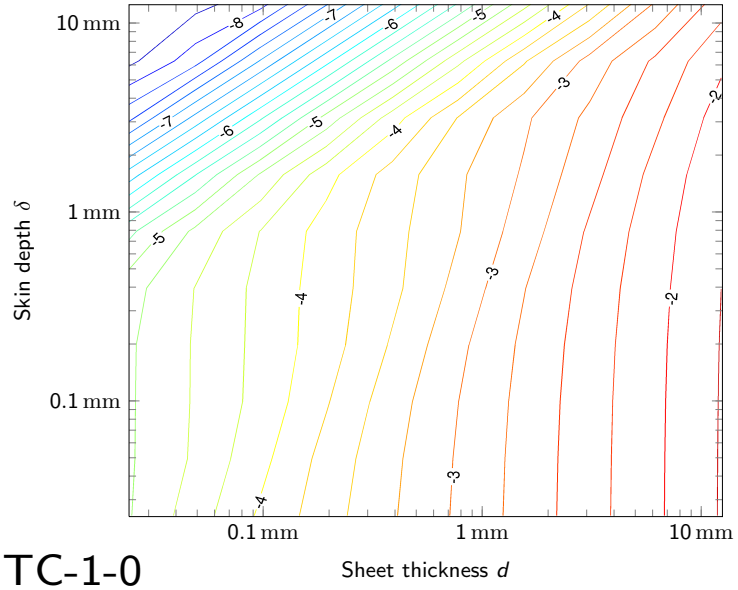
Thin sheet conditions

Reduced problem with thin sheet conditions

$$
\begin{aligned}
-\Delta \widetilde{e}_{\mathrm{ext}}(\mathbf{x}) &= -\mathrm{i}\omega\mu_0 j_0(\mathbf{x}) \\
\left[\widetilde{e}_{\mathrm{ext}}\right] &= 2\sqrt{-\mathrm{i}\omega\mu_0\sigma}^{-1} \tanh\left(\sqrt{-\mathrm{i}\omega\mu_0\sigma}\,\tfrac{d}{2}\right)\left\{\partial_y \widetilde{e}_{\mathrm{ext}}\right\} \\
\left[\partial_y \widetilde{e}_{\mathrm{ext}}\right] &= 2\sqrt{-\mathrm{i}\omega\mu_0\sigma} \quad \tanh\left(\sqrt{-\mathrm{i}\omega\mu_0\sigma}\,\tfrac{d}{2}\right)\left\{\widetilde{e}_{\mathrm{ext}}\right\}
\end{aligned}
$$

- first time in the book of Tozoni and Mayergoyz (1974)
- article in english by Mayergoyz and Bedrosian (1995)
- Boundary integral equation by Krähenbühl and Muller (1993), Igarashi *et.al.* (1998)
- FEM by Guérin *et.al.* (1995)

- Structure is more complicated than ITC-1-0, but about the same accuracy
- ... also more complicated than ITC-1-1, but less accurate in low-frequency
- $\|e - \widetilde{e}\|_{H^1(\Omega^d_{\mathrm{ext}})} = O(d)$ indep. of $\omega\sigma$(robust)



ITC-1-0

Reduced problem with thin sheet conditions

$$
\begin{aligned}
-\Delta \widetilde{\mathbf{e}}_{\mathrm{ext}}(\mathbf{x}) &= -\mathrm{i}\omega\mu_0 j_0(\mathbf{x}) \\
\left[\widetilde{\mathbf{e}}_{\mathrm{ext}}\right] &= 2\sqrt{-\mathrm{i}\omega\mu_0\sigma}^{-1} \tanh\left(\sqrt{-\mathrm{i}\omega\mu_0\sigma}\,\tfrac{d}{2}\right)\left\{\partial_y \widetilde{\mathbf{e}}_{\mathrm{ext}}\right\} \\
\left[\partial_y \widetilde{\mathbf{e}}_{\mathrm{ext}}\right] &= 2\sqrt{-\mathrm{i}\omega\mu_0\sigma} \quad \tanh\left(\sqrt{-\mathrm{i}\omega\mu_0\sigma}\,\tfrac{d}{2}\right)\left\{\widetilde{\mathbf{e}}_{\mathrm{ext}}\right\}
\end{aligned}
$$

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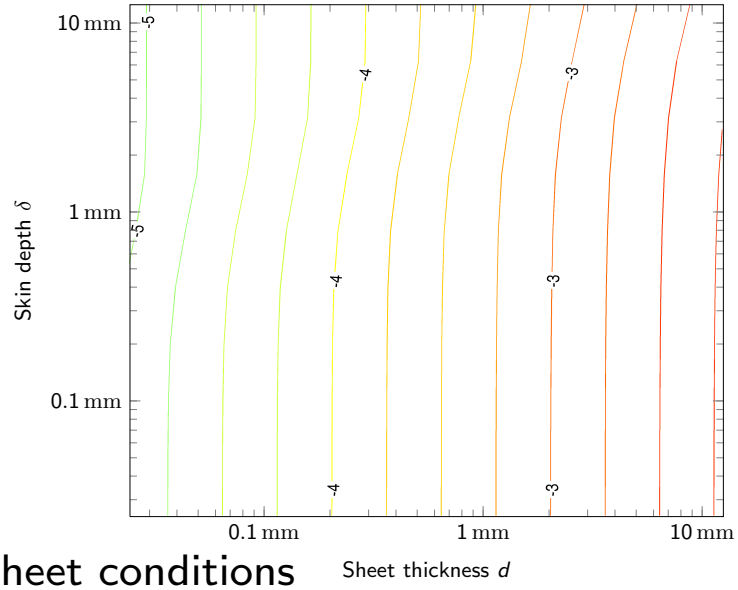

Thin sheet conditions

Reduced problem with thin sheet conditions

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\begin{aligned}
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\left[\partial_y \widetilde{\mathsf{e}}_{\mathrm{ext}}\right] &= 2\sqrt{-\mathrm{i}\omega\mu_0\sigma} \quad \tanh\left(\sqrt{-\mathrm{i}\omega\mu_0\sigma}\,\tfrac{d}{2}\right)\left\{\widetilde{\mathsf{e}}_{\mathrm{ext}}\right\}
\end{aligned}
$$

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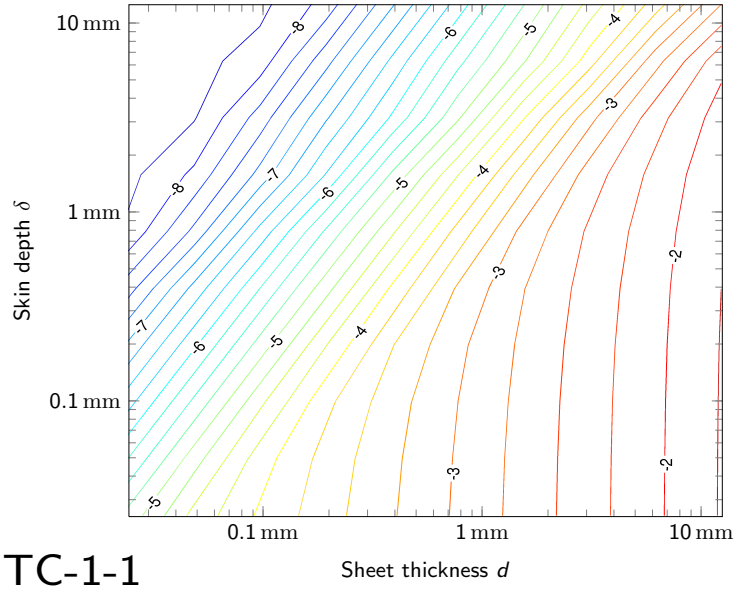

ITC-1-1

Transmission conditions by asymptotic expansion with $\omega\sigma \sim 1/\varepsilon^2$

Thin sheet conditions $\delta := \sqrt{2/(\mu_0\sigma\omega)}$

$$\begin{aligned}\left[\widetilde{e}_{\mathrm{ext}}\right] &= (1+\mathrm{i})\,\delta \quad \tanh\left(\tfrac{1-\mathrm{i}}{2}\tfrac{d}{\delta}\right)\left\{\partial_n \widetilde{e}_{\mathrm{ext}}\right\} \\ \left[\partial_n \widetilde{e}_{\mathrm{ext}}\right] &= 2(1-\mathrm{i})\,\delta^{-1}\tanh\left(\tfrac{1-\mathrm{i}}{2}\tfrac{d}{\delta}\right)\left\{\widetilde{e}_{\mathrm{ext}}\right\}\end{aligned}$$

Transmission conditions by asymptotic expansion for $\omega\sigma \sim 1/\varepsilon^2$

Trivial limit for $\varepsilon \to 0$ $\qquad \widetilde{e}|_\Gamma = 0$ (PEC)

New family of transmission conditions ITC-2-N

- include $\sinh\left(\frac{1-\mathrm{i}}{2}\frac{d}{\delta}\right)$, $\cosh\left(\frac{1-\mathrm{i}}{2}\frac{d}{\delta}\right)$ as thin sheet conditions

ITC-2-0

$$\left[\widetilde{e}_{\mathrm{ext}}\right] = 0$$

$$\left[\partial_y \widetilde{e}_{\mathrm{ext}}\right] = \frac{2(1-\mathrm{i})\,\delta^{-1}\sinh\left(\frac{1-\mathrm{i}}{2}\frac{d}{\delta}\right)}{\cosh\left(\frac{1-\mathrm{i}}{2}\frac{d}{\delta}\right) - \frac{1-\mathrm{i}}{2}\frac{d}{\delta}\sinh\left(\frac{1-\mathrm{i}}{2}\frac{d}{\delta}\right)}\left\{\widetilde{e}_{\mathrm{ext}}\right\}$$

- same structure than ITC-1-0 and ITC-1-1, accuracy as ITC-1-1

Transmission conditions by asymptotic expansion with $\omega\sigma \sim 1/\varepsilon^2$

Thin sheet conditions $\delta := \sqrt{2/(\mu_0\sigma\omega)}$

$$\begin{aligned}\left[\widetilde{e}_{\mathrm{ext}}\right] &= (1+\mathrm{i})\,\delta \quad \tanh\left(\tfrac{1-\mathrm{i}}{2}\tfrac{d}{\delta}\right)\left\{\partial_n \widetilde{e}_{\mathrm{ext}}\right\}\\ \left[\partial_n \widetilde{e}_{\mathrm{ext}}\right] &= 2(1-\mathrm{i})\,\delta^{-1}\tanh\left(\tfrac{1-\mathrm{i}}{2}\tfrac{d}{\delta}\right)\left\{\widetilde{e}_{\mathrm{ext}}\right\}\end{aligned}$$

Transmission conditions by asymptotic expansion for $\omega\sigma \sim 1/\varepsilon^2$

Trivial limit for $\varepsilon \to 0$ $\qquad \widetilde{e}|_\Gamma = 0$ (PEC)

New family of transmission conditions ITC-2-N

- ▷ include $\sinh\left(\frac{1-\mathrm{i}}{2}\frac{d}{\delta}\right)$, $\cosh\left(\frac{1-\mathrm{i}}{2}\frac{d}{\delta}\right)$ as thin sheet conditions

ITC-2-1

$$[\widetilde{e}_{\mathrm{ext}}] = \frac{\kappa d}{2}\left(1 - 2(1-\mathrm{i})\,\delta^{-1}\tanh\left(\tfrac{1-\mathrm{i}}{2}\tfrac{d}{\delta}\right)\right)\{\widetilde{e}_{\mathrm{ext}}\} - d\left(1 - 2(1-\mathrm{i})\,\delta^{-1}\tanh\left(\tfrac{1-\mathrm{i}}{2}\tfrac{d}{\delta}\right)\right)\{\partial_n \widetilde{e}_{\mathrm{ext}}\}$$

$$[\partial_n \widetilde{e}_{\mathrm{ext}}] = \frac{2(1-\mathrm{i})\,\delta^{-1}\sinh\left(\frac{1-\mathrm{i}}{2}\frac{d}{\delta}\right)}{\cosh\left(\frac{1-\mathrm{i}}{2}\frac{d}{\delta}\right) - \frac{1-\mathrm{i}}{2}\frac{d}{\delta}\sinh\left(\frac{1-\mathrm{i}}{2}\frac{d}{\delta}\right)}\{\widetilde{e}_{\mathrm{ext}}\} - \frac{\kappa d}{2}\left(1 - 2(1-\mathrm{i})\,\delta^{-1}\tanh\left(\tfrac{1-\mathrm{i}}{2}\tfrac{d}{\delta}\right)\right)\{\partial_n \widetilde{e}_{\mathrm{ext}}\}$$

- ▷ same structure than ITC-1-2 (but without ∂_t^2)

K.S. and A. Chernov, SIAM J. Appl. Math., accepted, 2013.

Transmission conditions by asymptotic expansion with $\omega\sigma \sim 1/\varepsilon^2$

Transmission conditions by asymptotic expansion for $\omega\sigma \sim 1/\varepsilon^2$

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- ▷ include $\sinh(\frac{1-\mathrm{i}}{2}\frac{d}{\delta})$, $\cosh(\frac{1-\mathrm{i}}{2}\frac{d}{\delta})$ as thin sheet conditions

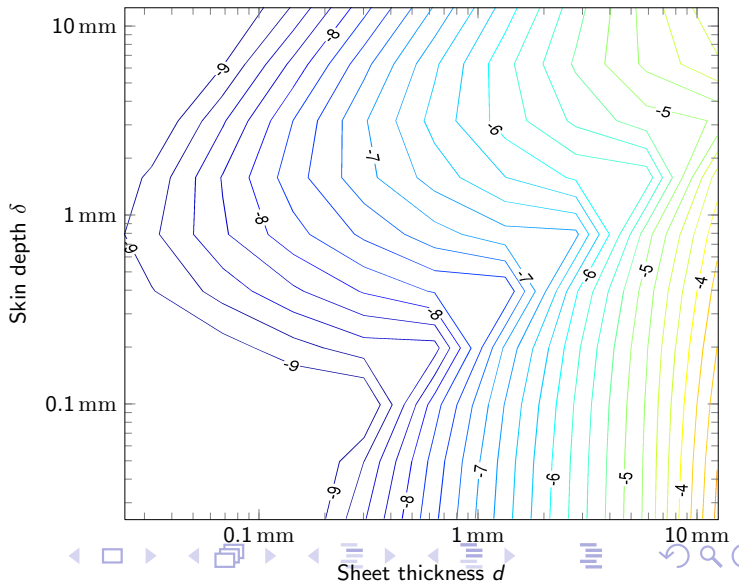
ITC-2-1

$$[\widetilde{e}_{\mathrm{ext}}] = \frac{\kappa d}{2}\left(1 - 2(1-\mathrm{i})\,\delta^{-1}\tanh(\tfrac{1-\mathrm{i}}{2}\tfrac{d}{\delta})\right)\{\widetilde{e}_{\mathrm{ext}}\} - d\left(1 - 2(1-\mathrm{i})\,\delta^{-1}\tanh(\tfrac{1-\mathrm{i}}{2}\tfrac{d}{\delta})\right)\{\partial_n\widetilde{e}_{\mathrm{ext}}\}$$

$$[\partial_n\widetilde{e}_{\mathrm{ext}}] = \frac{2(1-\mathrm{i})\,\delta^{-1}\sinh(\frac{1-\mathrm{i}}{2}\frac{d}{\delta})}{\cosh(\frac{1-\mathrm{i}}{2}\frac{d}{\delta}) - \frac{1-\mathrm{i}}{2}\frac{d}{\delta}\sinh(\frac{1-\mathrm{i}}{2}\frac{d}{\delta})}\{\widetilde{e}_{\mathrm{ext}}\} - \frac{\kappa d}{2}\left(1 - 2(1-\mathrm{i})\,\delta^{-1}\tanh(\tfrac{1-\mathrm{i}}{2}\tfrac{d}{\delta})\right)\{\partial_n\widetilde{e}_{\mathrm{ext}}\}$$

- ▷ same structure than ITC-1-2 (but without ∂_t^2)
- ▷ robust in $\omega\sigma$

$$\|e - \widetilde{e}\|_{H^1(\Omega^d_{\mathrm{ext}})} = O(d^2)$$



K.S. and A. Chernov,
SIAM J. Appl. Math., accepted, 2013.

Assume Γ to be a closed Lipschitz curve.

- Transmission conditions of Type I have form (*e. g.*, ITC-1-0, ITC-1-1, ITC-2-0)

$$\begin{aligned} -\Delta U &= F && \text{in } \mathbb{R}^2 \backslash \Gamma \\ [\gamma_1 U] - \beta_1 \{\gamma_0 U\} &= 0 && \text{on } \Gamma \\ [\gamma_0 U] &= 0 && \text{on } \Gamma \end{aligned}$$

- Transmission conditions of Type II have form (*e. g.*, shielding element by Nakata *et.al.*)

$$\begin{aligned} [\gamma_1 U] - (\beta_1 - \beta_2 \partial_\Gamma^2) \{\gamma_0 U\} &= 0 && \text{on } \Gamma, \\ [\gamma_0 U] &= 0 && \text{on } \Gamma \end{aligned}$$

- Transmission conditions of Type III have form (*e. g.*, thin sheet conditions)

$$\begin{aligned} [\gamma_1 U] - \beta_1 \{\gamma_0 U\} &= 0 && \text{on } \Gamma, \\ [\gamma_0 U] - \beta_3 \{\gamma_1 U\} &= 0 && \text{on } \Gamma, \end{aligned}$$

- Transmission conditions of Type IV have form (*e. g.*, ITC-1-2, ITC-2-1)

$$\begin{aligned} [\gamma_1 U] - (\beta_1 - \beta_2 \partial_\Gamma^2) \{\gamma_0 U\} + \beta_4 \kappa \{\gamma_1 U\} &= 0 && \text{on } \Gamma, \\ [\gamma_0 U] \qquad - \beta_4 \kappa \{\gamma_0 U\} \quad - \beta_3 \{\gamma_1 U\} &= 0 && \text{on } \Gamma, \end{aligned}$$

- β_1 may be small or large, assume β_2, β_3, β_4 to be small

Assume a closed Lipschitz curve Γ.

$$-\Delta U = F \quad \text{in } \mathbb{R}^2 \backslash \Gamma$$

Representation formula

$$U = -S\,[\gamma_1 U] + D\,[\gamma_0 U] + N\,F$$

with

$$(S\,\phi)(x) := \int_\Gamma G(x-y)\phi(y)\mathrm{d}y \qquad (D\,\delta)(x) := \int_\Gamma \gamma_{1,y} G(x-y)\delta(y)\mathrm{d}y$$

$$(N\,F)(x) := \int_{\mathbb{R}^2} G(x-y)F(y)\mathrm{d}y \qquad G(x-y) = -\frac{1}{2\pi}\log(|x-y|)$$

Boundary integral operators

$$V := \{\gamma_0 S\,\cdot\} : H^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma), \qquad K := \{\gamma_0 D\,\cdot\} : H^{1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma),$$

$$K' := \{\gamma_1 S\,\cdot\} : H^{-1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma), \quad W := -\{\gamma_1 D\,\cdot\} : H^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma),$$

Taking mean traces

$$\{\gamma_0 U\} = -V\,[\gamma_1 U] + K\,[\gamma_0 U] + \gamma_0 NF$$
$$\{\gamma_1 U\} = -K'\,[\gamma_1 U] - W\,[\gamma_0 U] + \gamma_1 NF$$

Boundary integral equations for type I transmission conditions

- Transmission conditions of Type I have form (*e. g.*, ITC-1-0, ITC-1-1, ITC-2-0)

$$\begin{aligned} [\gamma_1 U] - \beta_1 \{\gamma_0 U\} &= 0 \quad \text{on } \Gamma \\ [\gamma_0 U] &= 0 \quad \text{on } \Gamma \end{aligned}$$

Representation of mean traces by jump traces using boundary integral operators

$$\begin{aligned} \{\gamma_0 U\} &= -V\ [\gamma_1 U] + K\,[\gamma_0 U] + \gamma_0 NF \\ \{\gamma_1 U\} &= -K'\,[\gamma_1 U] - W\,[\gamma_0 U] + \gamma_1 NF \end{aligned}$$

Boundary integral formulation (second kind) for $\phi := [\gamma_1 U] \in L^2(\Gamma)$

$$(Id + \beta_1 V)\,\phi = \beta_1 \gamma_0 NF$$

Variational formulation

$$\langle \phi, \phi' \rangle + \beta_1 \langle V\phi, \phi' \rangle = \beta_1 \langle \gamma_0 NF, \phi' \rangle$$

- $|\beta_1|$ may be small or large, assume $0 \geq \arg(\beta_1) \geq \theta_1^\star > -\pi$
- bilinearform is $L^2(\Gamma)$-elliptic with constant $O(1)$ and $H^{-1/2}(\Gamma)$-elliptic with constant $O(|\beta_1|)$

Theorem (K.S. and R. Hiptmair (2013))

Let Γ be a Lipschitz curve. Then, there exists a constant C independent of $|\beta_1|$ such that

$$\|\phi\|_{L^2(\Gamma)} \leq C|\beta_1| \|\gamma_0 NF\|_{H^{1/2}(\Gamma)}, \qquad \|\phi\|_{H^{-1/2}(\Gamma)} \leq C \min(1, |\beta_1|) \|\gamma_0 NF\|_{H^{1/2}(\Gamma)}.$$

- singularly pertubed BIE for $|\beta_1| \gg 1$ ($\delta \ll d$, high frequency) $\Rightarrow$ internal layers

Boundary integral equations for type I transmission conditions

- Transmission conditions of Type I have form (*e. g.*, ITC-1-0, ITC-1-1, ITC-2-0)

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Theorem (K.S. and R. Hiptmair (2013))

For $s \geq 0$ it holds $\|\phi\|_{H^{s+1/2}(\Gamma)} \leq C_s \min(1, |\beta_1|) \|\gamma_0 NF\|_{H^{2s+5/2}(\Gamma)}$.

Proof.

Using an asymptotic expansion in β_1^{-1}, that is $\phi \sim \phi_0 + \beta_1^{-1}\phi_1 + \beta_1^{-2}\phi_2 + \ldots$, elliptic shift theorems and bootstrapping. □

Theorem (K.S. and R. Hiptmair (2013))

Let Γ be a Lipschitz curve. Then, there exists a constant C independent of $|\beta_1|$ such that

$$\|\phi\|_{L^2(\Gamma)} \leq C|\beta_1|\|\gamma_0 NF\|_{H^{1/2}(\Gamma)}, \qquad \|\phi\|_{H^{-1/2}(\Gamma)} \leq C\min(1,|\beta_1|)\|\gamma_0 NF\|_{H^{1/2}(\Gamma)}.$$

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Proof.

Rewriting BIE

$$\phi = \beta_1\left(-V\phi + \gamma_0 NF\right)$$

Estimation in $H^{1/2}(\Gamma)$-norm

$$\|\phi\|_{H^{1/2}(\Gamma)} \leq C|\beta_1|(\|\phi\|_{H^{-1/2}(\Gamma)} + \|\gamma_0 NF\|_{H^{1/2}(\Gamma)}) \leq C|\beta_1|\|\gamma_0 NF\|_{H^{1/2}(\Gamma)}$$

Estimation in $H^{s+1/2}(\Gamma)$-norm (recurrence)

$$\|\phi\|_{H^{s+1/2}(\Gamma)} \leq C\max(|\beta_1|^{s+1},|\beta_1|)\|\gamma_0 NF\|_{H^{s+1/2}(\Gamma)}$$

Theorem (K.S. and R. Hiptmair (2013))

Let Γ be a Lipschitz curve. Then, there exists a constant C independent of $|\beta_1|$ such that

$$\|\phi\|_{L^2(\Gamma)} \leq C|\beta_1|\|\gamma_0 NF\|_{H^{1/2}(\Gamma)}, \qquad \|\phi\|_{H^{-1/2}(\Gamma)} \leq C\min(1,|\beta_1|)\|\gamma_0 NF\|_{H^{1/2}(\Gamma)}.$$

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For $s \geq 0$ it holds $\|\phi\|_{H^{s+1/2}(\Gamma)} \leq C_s \min(1,|\beta_1|)\|\gamma_0 NF\|_{H^{2s+5/2}(\Gamma)}$.

Proof.

Ansatz for $|\beta_1| > 1$

$$\phi = \phi_0 + \beta_1^{-1}\phi_1 + \ldots + \beta_1^{-N}\phi_N + \delta\phi_N$$

with

$$V\phi_0 = \gamma_0 NF, \qquad V\phi_{n+1} = -\phi_n, \qquad (Id + \beta_1 V)\delta\phi_N = -\beta_1^{-N}\phi_N.$$

We estimate

$$\|\phi_n\|_{H^{s+1/2}(\Gamma)} \leq C_n\|\gamma_0 NF\|_{H^{s+3/2+n}(\Gamma)},$$
$$\|\delta\phi_N\|_{H^{s+1/2}(\Gamma)} \leq C|\beta_1|^{s+1-N}\|\phi_N\|_{H^{s+1/2}(\Gamma)} \leq C|\beta_1|^{s+1-N}\|\gamma_0 NF\|_{H^{s+3/2+N}(\Gamma)}$$

- Transmission conditions of Type I (*e. g.*, ITC-1-0, ITC-1-1, ITC-2-0)

Variational formulation : Seek $\phi \in L^2(\Gamma)$ such that for all $\phi' \in L^2(\Gamma)$

$$\langle \phi, \phi' \rangle + \beta_1 \langle V\phi, \phi' \rangle = \beta_1 \langle \gamma_0 NF, \phi' \rangle$$

Discretisation on a mesh Γ_h

- by piecewise constants $S_0^{-1}(\Gamma_h)$ or piecewise linear, continuous $S_1^0(\Gamma_h)$

Theorem (K.S. and R. Hiptmair (2013))

The discrete variational formulation has a unique solution $\phi_h \in S_0^{-1}(\Gamma_h)/S_1^0(\Gamma_h)$ ($p = 0, 1$), and

$$\|\phi_h\|_{L^2(\Gamma)} \leq C \qquad |\beta_1| \quad \|F\|_{(H^1(\mathbb{R}^2\setminus\Gamma))'} \qquad \textit{for Lipschitz } \Gamma,$$

$$\|\phi_h\|_{L^2(\Gamma)} \leq C \min(1, |\beta_1|) \quad \|F\|_{(H^1(\mathbb{R}^2\setminus\Gamma))'} \qquad \textit{if } \Gamma \in C^{1,1},$$

$$\|\phi_h - \phi\|_{L^2(\Gamma)} \leq C \min(1, |\beta_1|)\, h^{p+1} \|F\|_{(H^1(\mathbb{R}^2\setminus\Gamma))'}, \qquad \textit{if } \Gamma \textit{ is smooth enough.}$$

Proof.

Lax-Milgram + Cea's lemma.
For $|\beta_1| \gg 1$ Cea's lemma gives additional $|\beta_1|$, use discrete version of bootstrapping.
Decomposition $\phi = \phi_0 + \delta\phi_0$ where $\phi_0 = V^{-1}\gamma_0 NF$

$$\phi_h = \phi_{0,h} + \delta\phi_{0,h} \text{ where } \langle V\phi_{0,h}, \phi' \rangle = \langle \gamma_0 NF, \phi' \rangle$$

□

Boundary element method for type I transmission conditions

- Transmission conditions of Type I have form (*e. g.*, ITC-1-0, ITC-1-1, ITC-2-0)

$$[\gamma_1 U] - \beta_1 \{\gamma_0 U\} = 0 \quad \text{on } \Gamma$$
$$[\gamma_0 U] = 0 \quad \text{on } \Gamma$$

Variational formulation : Seek $\phi \in L^2(\Gamma)$ such that for all $\phi' \in L^2(\Gamma)$

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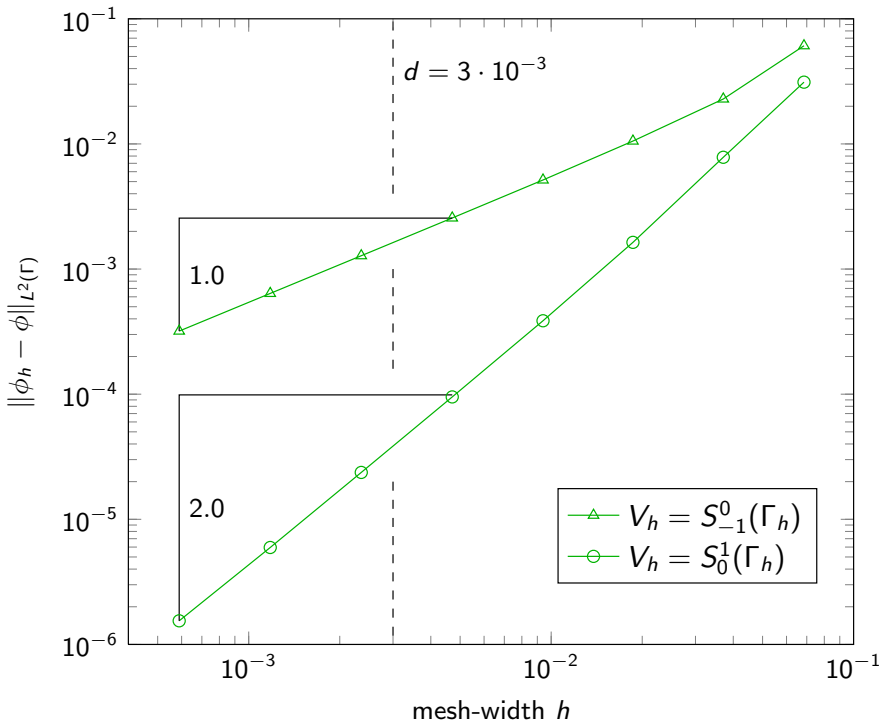
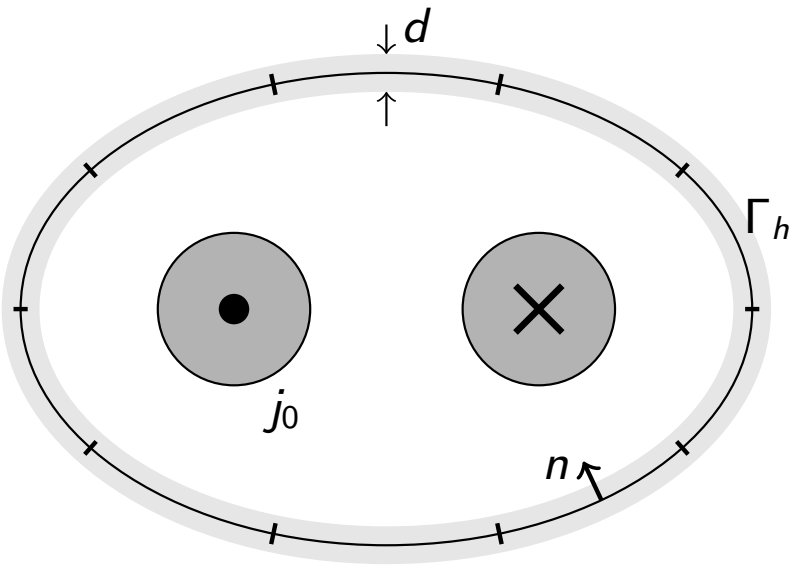
Discretisation on a mesh Γ_h

- by piecewise constants $S_0^{-1}(\Gamma_h)$ or piecewise linear, continuous $S_1^0(\Gamma_h)$





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$$[\gamma_1 U] - \beta_1 \{\gamma_0 U\} = 0 \quad \text{on } \Gamma$$
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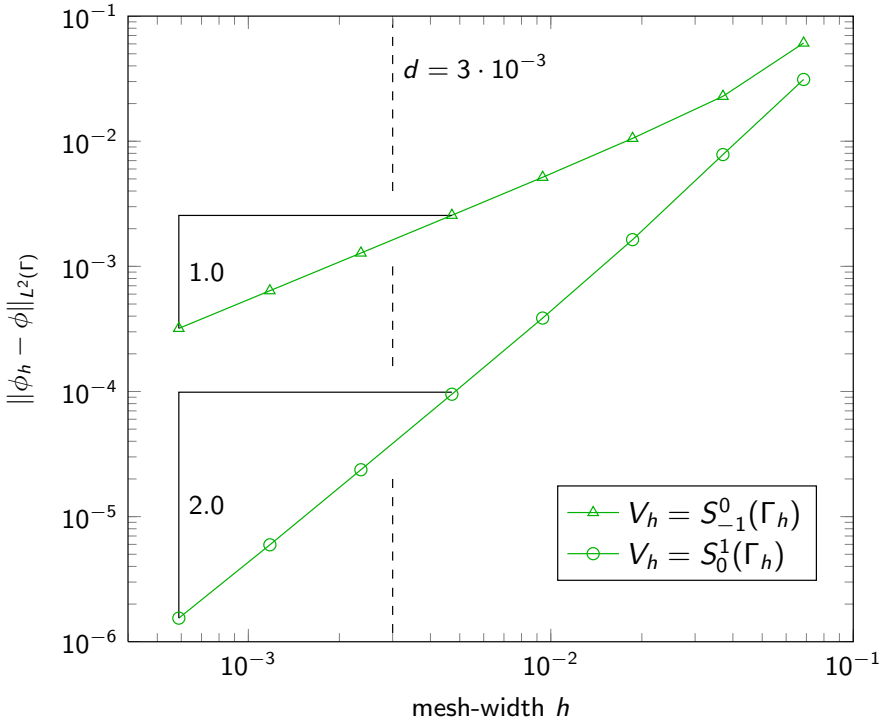

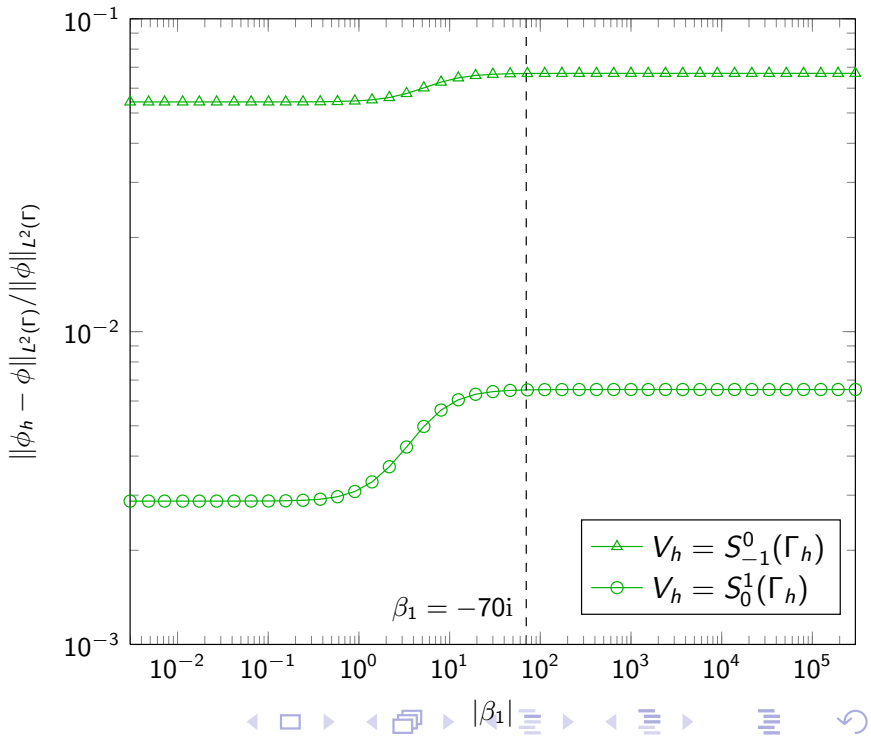

Boundary integral equations for type II transmission conditions

- Transmission conditions of Type II have form (*e. g.*, shielding element by Nakata *et.al.*)

$$\begin{aligned} [\gamma_1 U] - (\beta_1 - \beta_2 \partial_\Gamma^2)\,\{\gamma_0 U\} &= 0 \quad \text{on } \Gamma, \\ [\gamma_0 U] &= 0 \quad \text{on } \Gamma \end{aligned}$$

Representation of mean traces by jump traces using boundary integral operators

$$\begin{aligned} \{\gamma_0 U\} &= -V\ [\gamma_1 U] + K\,[\gamma_0 U] + \gamma_0 NF \\ \{\gamma_1 U\} &= -K'\,[\gamma_1 U] - W\,[\gamma_0 U] + \gamma_1 NF \end{aligned}$$

Boundary integral equation as mixed formulation (1st kind)
for $\phi := [\gamma_1 U] \in H^{-1/2}(\Gamma)$, $u := \{\gamma_0 U\} \in H^1(\Gamma)$

$$\begin{aligned} V\phi + u &= \gamma_0 N f \\ -\phi + \beta_1 u - \beta_2 \partial_\Gamma^2 u &= 0 \end{aligned}$$

- singularly pertubed BIE for $\beta_1 \gg 1$ (high frequency) or $\beta_2 \ll 1$ (always)

Boundary integral equations for type II transmission conditions

- Transmission conditions of Type II have form (*e. g.*, shielding element by Nakata *et.al.*)

$$\begin{aligned} [\gamma_1 U] - (\beta_1 - \beta_2 \partial_\Gamma^2)\,\{\gamma_0 U\} &= 0 \quad \text{on } \Gamma, \\ [\gamma_0 U] &= 0 \quad \text{on } \Gamma \end{aligned}$$

Representation of mean traces by jump traces using boundary integral operators

$$\begin{aligned} \{\gamma_0 U\} &= -V\,[\gamma_1 U] + K\,[\gamma_0 U] + \gamma_0 NF \\ \{\gamma_1 U\} &= -K'\,[\gamma_1 U] - W\,[\gamma_0 U] + \gamma_1 NF \end{aligned}$$

Boundary integral equation as mixed formulation (1st kind)

for $\phi := [\gamma_1 U] \in H^{-1/2}(\Gamma)$, $u := \{\gamma_0 U\} \in H^1(\Gamma)$

$$\begin{aligned} V\phi \quad + u \qquad\qquad &= \gamma_0 N f \\ -\phi + \beta_1 u - \beta_2 \partial_\Gamma^2 u &= 0 \end{aligned} \quad \Rightarrow \quad \begin{pmatrix} V & Id \\ -Id & \beta_1 Id - \beta_2 \partial_\Gamma^2 \end{pmatrix} \begin{pmatrix} \phi \\ u \end{pmatrix} = \begin{pmatrix} \gamma_0 N f \\ 0 \end{pmatrix}$$

Variational formulation

$$\begin{aligned} \langle V\phi, \phi' \rangle_\Gamma \quad + \langle u, \phi' \rangle_\Gamma \qquad\qquad &= \langle \gamma_0 N f, \phi' \rangle_\Gamma \\ -\langle \phi, u' \rangle_\Gamma + \beta_1 \langle u, u' \rangle_\Gamma + \beta_2 \langle \partial_\Gamma u, \partial_\Gamma u' \rangle_\Gamma &= 0 \end{aligned}$$

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Theorem (K.S. and R. Hiptmair (2013))

Let $0 \geq \arg(\beta_1) \geq \theta_1^\star > -\pi$, $\beta_2 > 0$ where $\operatorname{Re}\beta_2 \geq 0$, $\operatorname{Im}\beta_2 \leq 0$. Then, there exists a constant C independent of $|\beta_1|$, $|\beta_2|$ such that

$$\begin{aligned} \|\phi\|_{H^{-1/2}(\Gamma)} &\leq C \qquad\qquad \|\gamma_0 NF\|_{H^{1/2}(\Gamma)}, \\ \|u\|_{L^2(\Gamma)} &\leq C \min(|\beta_1|^{-1/2}, 1) \|\gamma_0 NF\|_{H^{1/2}(\Gamma)}, \\ |u|_{H^1(\Gamma)} &\leq C \quad |\beta_2|^{-1/2} \quad \|\gamma_0 NF\|_{H^{1/2}(\Gamma)}. \end{aligned}$$

Boundary integral equations for type II transmission conditions

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$$\begin{aligned} [\gamma_1 U] - (\beta_1 - \beta_2 \partial_\Gamma^2)\{\gamma_0 U\} &= 0 \quad \text{on } \Gamma, \\ [\gamma_0 U] &= 0 \quad \text{on } \Gamma \end{aligned}$$

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for $\phi := [\gamma_1 U] \in H^{-1/2}(\Gamma)$, $u := \{\gamma_0 U\} \in H^1(\Gamma)$

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Let $0 \geq \arg(\beta_1) \geq \theta_1^\star > -\pi$, $\beta_2 > 0$ *where* $\operatorname{Re}\beta_2 \geq 0$, $\operatorname{Im}\beta_2 \leq 0$ *and* Γ *be smooth enough. Then, there exists a constant* C *independent of* $|\beta_1|$, $|\beta_2|$ *such that*

$$\begin{aligned} \|\phi\|_{H^{-1/2}(\Gamma)} &\leq C(\min(1, |\beta_1|) + |\beta_2|) && \|F\|_{(H^1(\mathbb{R}^2\setminus\Gamma))'}, \\ \|u\|_{H^1(\Gamma)} &\leq C \min(|\beta_1|^{-1}, 1) && \|F\|_{(H^1(\mathbb{R}^2\setminus\Gamma))'}, \\ |\phi|_{H^1(\Gamma)} &\leq C && \|F\|_{(H^1(\mathbb{R}^2\setminus\Gamma))'}, \\ |u|_{H^2(\Gamma)} &\leq C(\min(|\beta_1|^{-1}, 1) + |\beta_2|) && \|F\|_{(H^1(\mathbb{R}^2\setminus\Gamma))'}. \end{aligned}$$

Boundary element method for type II transmission conditions

- Transmission conditions of Type II have form (*e. g.*, shielding element by Nakata *et.al.*)

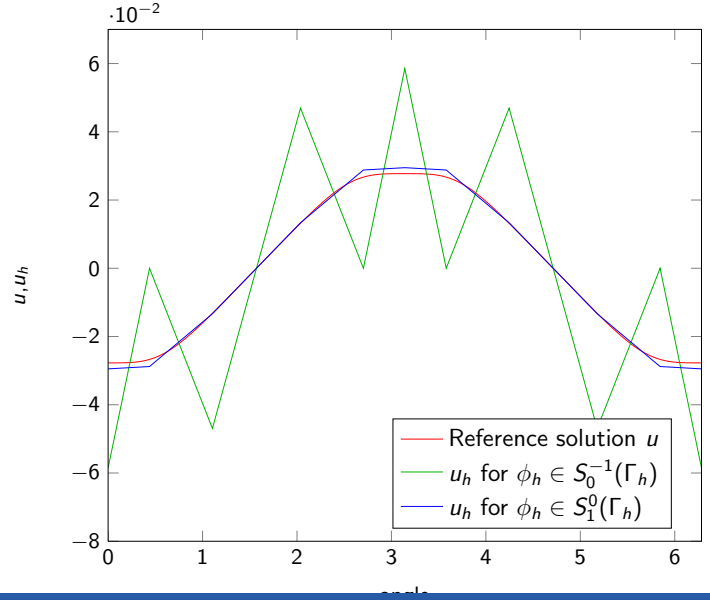
$$
\begin{aligned}
[\gamma_1 U] - (\beta_1 - \beta_2 \partial_\Gamma^2)\{\gamma_0 U\} &= 0 \quad \text{on } \Gamma, \\
[\gamma_0 U] &= 0 \quad \text{on } \Gamma
\end{aligned}
$$

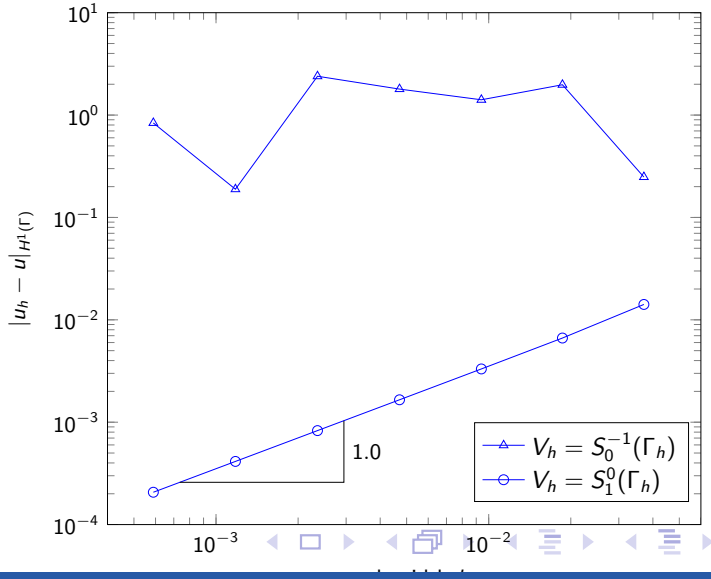
Variational formulation

$$
\begin{aligned}
\langle V\phi, \phi' \rangle_\Gamma \quad + \langle u, \phi' \rangle_\Gamma \quad &= \langle \gamma_0 N f, \phi' \rangle_\Gamma \\
-\langle \phi, u' \rangle_\Gamma + \beta_1 \langle u, u' \rangle_\Gamma + \beta_2 \langle \partial_\Gamma u, \partial_\Gamma u' \rangle_\Gamma &= 0
\end{aligned}
$$

Discretisation on a mesh Γ_h

- ϕ_h and u_h piecewise linear, continuous $S_1^0(\Gamma_h)$
- no stability with $S_0^{-1}(\Gamma_h)$ for ϕ_h for $\beta_1, \beta_2 \to 0$





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Variational formulation

$$\begin{aligned} \langle V\phi, \phi' \rangle_\Gamma \quad + \langle u, \phi' \rangle_\Gamma \quad &= \langle \gamma_0 N f, \phi' \rangle_\Gamma \\ -\langle \phi, u' \rangle_\Gamma + \beta_1 \langle u, u' \rangle_\Gamma + \beta_2 \langle \partial_\Gamma u, \partial_\Gamma u' \rangle_\Gamma &= 0 \end{aligned}$$

Discretisation on a mesh Γ_h

- ϕ_h and u_h piecewise linear, continuous $S_1^0(\Gamma_h)$

Theorem (K.S. and R. Hiptmair (2013))

The discrete variational formulation has a unique solution $(\phi_h, u_h) \in S_1^0(\Gamma_h) \times S_1^0(\Gamma_h)$ *and*

$$\|\phi_h\|_{H^{-1/2}(\Gamma)} \le C_{\phi,r}(\beta_1, \beta_2)\, \|F\|_{(H^1(\mathbb{R}^2\setminus\Gamma))'}, \qquad \|u_h\|_{H^1(\Gamma)} \le C_{u,r}(\beta_1, \beta_2)\, \|F\|_{(H^1(\mathbb{R}^2\setminus\Gamma))'}$$

$$\|\phi_h - \phi\|_{H^{-1/2}(\Gamma)} \le C_{\phi,r}(\beta_1, \beta_2)\, h \|F\|_{(H^1(\mathbb{R}^2\setminus\Gamma))'}, \qquad \|u_h - u\|_{H^1(\Gamma)} \le C_{u,r}(\beta_1, \beta_2)\, h \|F\|_{(H^1(\mathbb{R}^2\setminus\Gamma))'}$$

Proof.

Lax-Milgram + Cea's lemma. For $|\beta_2| \ll 1$ discrete version of bootstrapping. □

Boundary element method for type II transmission conditions

- Transmission conditions of Type II have form (*e. g.*, shielding element by Nakata *et.al.*)

$$
\begin{aligned}
[\gamma_1 U] - (\beta_1 - \beta_2 \partial_\Gamma^2)\,\{\gamma_0 U\} &= 0 \quad \text{on } \Gamma, \\
[\gamma_0 U] &= 0 \quad \text{on } \Gamma
\end{aligned}
$$

Variational formulation

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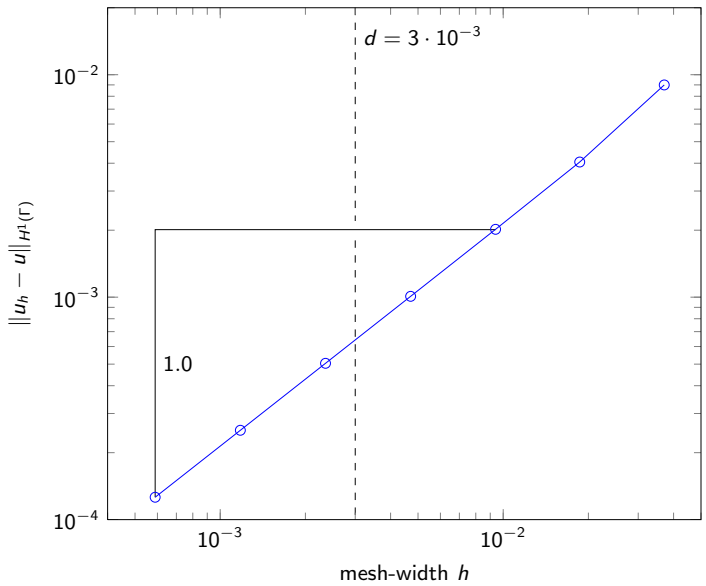
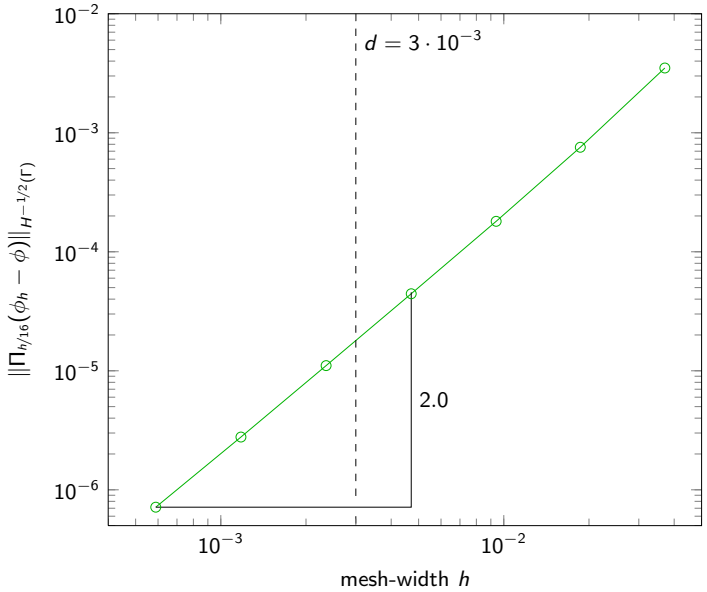
\begin{aligned}
\langle V\phi, \phi' \rangle_\Gamma \quad + \langle u, \phi' \rangle_\Gamma \quad &= \langle \gamma_0 N f, \phi' \rangle_\Gamma \\
-\langle \phi, u' \rangle_\Gamma + \beta_1 \langle u, u' \rangle_\Gamma + \beta_2 \langle \partial_\Gamma u, \partial_\Gamma u' \rangle_\Gamma &= 0
\end{aligned}
$$

Discretisation on a mesh Γ_h

- ϕ_h and u_h piecewise linear, continuous $S_1^0(\Gamma_h)$





Boundary element method for type II transmission conditions

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$$
\begin{aligned}
[\gamma_1 U] - (\beta_1 - \beta_2 \partial_\Gamma^2)\{\gamma_0 U\} &= 0 \quad \text{on } \Gamma, \\
[\gamma_0 U] &= 0 \quad \text{on } \Gamma
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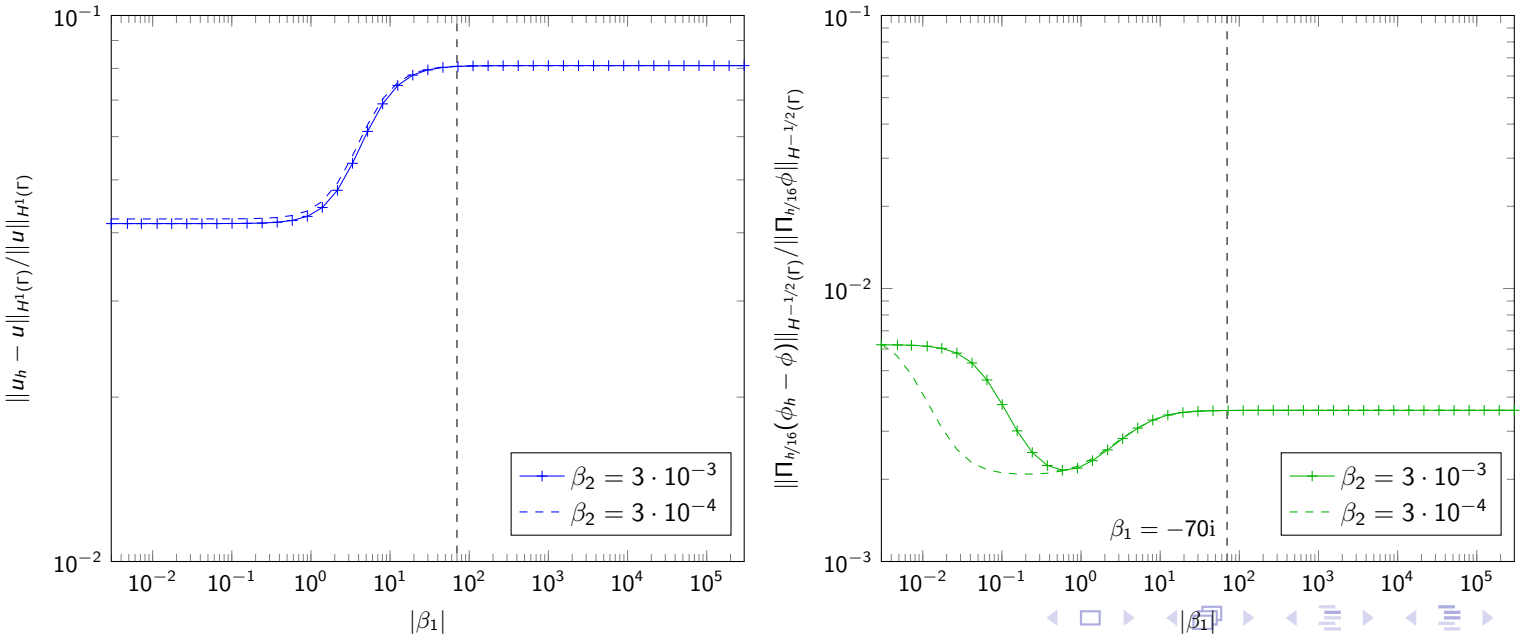
$$

Variational formulation

$$
\begin{aligned}
\langle V\phi, \phi' \rangle_\Gamma + \langle u, \phi' \rangle_\Gamma &= \langle \gamma_0 N f, \phi' \rangle_\Gamma \\
-\langle \phi, u' \rangle_\Gamma + \beta_1 \langle u, u' \rangle_\Gamma + \beta_2 \langle \partial_\Gamma u, \partial_\Gamma u' \rangle_\Gamma &= 0
\end{aligned}
$$

Discretisation on a mesh Γ_h

- ϕ_h and u_h piecewise linear, continuous $S_1^0(\Gamma_h)$



Boundary integral equations for type IV transmission conditions

- Transmission conditions of Type IV have form (*e. g.*, ITC-1-2, ITC-2-1)

$$
\begin{aligned}
[\gamma_1 U] - (\beta_1 - \beta_2 \partial_\Gamma^2)\,\{\gamma_0 U\} + \beta_4 \kappa\,\{\gamma_1 U\} &= 0 \quad \text{on } \Gamma,\\
[\gamma_0 U] \qquad\qquad - \beta_4 \kappa\,\{\gamma_0 U\} \;\; - \beta_3\,\{\gamma_1 U\} &= 0 \quad \text{on } \Gamma,
\end{aligned}
$$

Representation of mean traces by jump traces using boundary integral operators

$$
\begin{aligned}
\{\gamma_0 U\} &= -V\;[\gamma_1 U] + \;K\,[\gamma_0 U] + \gamma_0 NF\\
\{\gamma_1 U\} &= -K'\,[\gamma_1 U] - W\,[\gamma_0 U] + \gamma_1 NF
\end{aligned}
$$

Boundary integral equation as mixed formulation

for $\phi := [\gamma_1 U] \in H^{-1/2}(\Gamma)$, $u := \{\gamma_0 U\} \in H^1(\Gamma)$ (or $H^{1/2}(\Gamma)$ for $\beta_2 = 0$), $j := [\gamma_0 U] \in H^{1/2}(\Gamma)$

$$
\begin{pmatrix} V & Id & -K \\ -Id & \beta_1 Id + \beta_3^{-1}\beta_4^2\kappa^2 - \beta_2\partial_\Gamma^2 & -\beta_3^{-1}\beta_4\kappa \\ K' & -\beta_3^{-1}\beta_4\kappa & W + \beta_3^{-1} Id \end{pmatrix} \begin{pmatrix} \phi \\ u \\ j \end{pmatrix} = \begin{pmatrix} \gamma_0 N F \\ 0 \\ \gamma_1 N F \end{pmatrix}
$$

- all four boundary integral operators involved
- singularly pertubed BIE for $\beta_1 \gg 1$ (high frequency) or $0 \neq \beta_2 \ll 1$ or $\beta_3 \ll 1$ (always)
- discretisation by $S_1^0(\Gamma_h)$ for all three unknowns

Boundary integral equations for type IV transmission conditions

- Transmission conditions of Type IV have form (*e. g.*, ITC-1-2, ITC-2-1)

$$\begin{aligned} [\gamma_1 U] - (\beta_1 - \beta_2 \partial_\Gamma^2)\{\gamma_0 U\} + \beta_4 \kappa \{\gamma_1 U\} &= 0 \quad \text{on } \Gamma, \\ [\gamma_0 U] \qquad\qquad - \beta_4 \kappa \{\gamma_0 U\} - \beta_3 \{\gamma_1 U\} &= 0 \quad \text{on } \Gamma, \end{aligned}$$

Boundary integral equation as mixed formulation

for $\phi := [\gamma_1 U] \in H^{-1/2}(\Gamma)$, $u := \{\gamma_0 U\} \in H^1(\Gamma)$ (or $H^{1/2}(\Gamma)$ for $\beta_2 = 0$), $j := [\gamma_0 U] \in H^{1/2}(\Gamma)$

$$\begin{pmatrix} V & Id & -K \\ -Id & \beta_1 Id + \beta_3^{-1}\beta_4^2\kappa^2 - \beta_2\partial_\Gamma^2 & -\beta_3^{-1}\beta_4\kappa \\ K' & -\beta_3^{-1}\beta_4\kappa & W + \beta_3^{-1} Id \end{pmatrix} \begin{pmatrix} \phi \\ u \\ j \end{pmatrix} = \begin{pmatrix} \gamma_0 N F \\ 0 \\ \gamma_1 N F \end{pmatrix}$$

Lemma

If $\beta_2 = 0$, $V_h = W_h = X_h = S_1^0(\Gamma_h)$, *and* $\Gamma \in C^{4,1}$, *then with* $C = C(\beta_1, \beta_2, \beta_3, \beta_4)$

$$\|\phi_h - \phi\|_{H^{-1/2}(\Gamma)} + \|j_h - j\|_{H^{1/2}(\Gamma)} + \|u_h - u\|_{H^{1/2}(\Gamma)} \le C\, h^{3/2}\, \|F\|_{(H^1(\mathbb{R}^2\setminus\Gamma))'}.$$

If $|\beta_2| > 0$, $V_h \in \{S_0^{-1}(\Gamma_h), S_1^0(\Gamma_h)\}$, $W_h = X_h = S_1^0(\Gamma_h)$, *and* $\Gamma \in C^{4,1}$, *then*

$$\|\phi_h - \phi\|_{H^{-1/2}(\Gamma)} + \|j_h - j\|_{H^{1/2}(\Gamma)} + \|u_h - u\|_{H^{1}(\Gamma)} \le C\, h\, \|F\|_{(H^1(\mathbb{R}^2\setminus\Gamma))'}.$$

Summary

- Model reduction : Thin conducting sheets replaced by transmission conditions $T^\varepsilon(\omega)$
- Boundary integral equation only on the midline of the sheet
- Second kind BIE and BEM for transmission conditions for ITC-1-0, ITC-1-1 and ITC-2-0, well-posedness, error estimates in dependance of parameter
- First kind mixed BIE and BEM for shielding element, well-posedness (stable pairs for $u = \{\gamma_0 U\}$ and $\phi = [\gamma_1 U]$ needed for weak shielding), well-posedness, error estimates
- First kind mixed BIE and BEM for ITC-1-2 and ITC-2-1 for $u = \{\gamma_0 U\}$, $\phi = [\gamma_1 U]$, $j = [\gamma_0 U]$

Extension and Application of the boundary integral equations

- transmission conditions for Helmholtz equation and small wave-number
- to boundary layers of singular pertubed PDEs (multiscale expansion)
- to thin periodic sheets (surface homogenisation + matched asymptotic expansion)

Open questions and Outlook

- Transmission conditions and boundary integral equations for Maxwell's equations
- Correction of transmission conditions in case of non-smooth sheets (with kinks)