Boundary element methods for impedance transmission conditions

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Motivation

Transformator inside a conducting casing.

Cable with a foil shield.

Protection of integrated circuits.

Thin conducting sheets

▷ for electromagnetic shielding,
▷ for mechanical stability,
▷ as casing for liquids and gases,
▷ with little material wastage.
Eddy current model in 2D (TM polarisation)

\[-\Delta e(x) + i\omega \mu_0 \sigma(x) e(x) = -i\omega \mu_0 j_0(x)\]
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Two important effects of the thin sheet

- **Shielding effect** – in conductors induced currents diminish electromagnetic fields (behind the conductor)
- **Skin effect** – major current flow in a boundary layer (skins of the conductor)
  - Skin depth in solid body \(\delta = \sqrt{\frac{2}{\mu_0 \sigma \omega}}\)
  - Copper at 50 Hz \(\rightarrow \delta \approx 8\text{mm}\)
Eddy current model in 2D (TM polarisation)

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Two important effects of the thin sheet

- **Shielding effect** – in conductors induced currents diminish electromagnetic fields (behind the conductor)
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  - Skin depth in solid body \( \delta = \sqrt{\frac{2}{\mu_0\sigma\omega}} \)
  - Copper at 50 Hz \( \rightarrow \delta \approx 8\text{mm} \)
Meshing problem

Triangle mesh inside the sheet.

Conforming triangle mesh resolving the sheet.

- High number of triangles to achieve conforming mesh of good quality.
- Even more triangles for resolving the sheet.

Replace the thin sheet by an interface.

Replace its behaviour by transmission conditions, e.g.

\[
[e] = 0,
[\partial_n e] = i\omega \mu \sigma d\{e\}
\]

Conforming triangle mesh resolving an interface.

Mesh of interface.

Reduction to interface by boundary integral formulation.
Model reduction using transmission problems

Original problem

\[ L_{\text{ext}}(\omega)e_{\text{ext}} = f \quad \text{in } \Omega^{d}_{\text{ext}} \]
\[ L_{\text{int}}(\omega)e_{\text{int}} = 0 \quad \text{in } \Omega^{d}_{\text{int}} \quad (1) \]

Reduced problem with transmission conditions

\[ L_{\text{ext}}(\omega)e_{\text{ext}} = f \quad \text{in } \Omega^{0}_{\text{ext}} \]
\[ T(\omega, d)e_{\text{ext}} = 0 \quad \text{on } \Gamma \quad (2) \]

- \( e_{\text{ext}} \) is extended from \( \Omega^{d}_{\text{ext}} \) into \( \Omega^{0}_{\text{ext}} \)
- extended \( e_{\text{ext}} \) satisfy some transmission conditions on \( \Gamma \)
- solutions \( e_{\text{ext}} \) of (1) and (2) coincide in \( \Omega^{d}_{\text{ext}} \)
- \( e_{\text{int}} \) in \( \Omega^{d}_{\text{int}} \) can be computed \( a\)-\textit{posteriori}
Model reduction using transmission problems

Original eddy current problem

\[-\Delta e_{\text{ext}}(x) = -i\omega \mu_0 j_0(x) \quad \text{in } \Omega^d_{\text{ext}}\]
\[-\Delta e_{\text{int}}(x) + i\omega \mu_0 \sigma e_{\text{int}}(x) = 0 \quad \text{in } \Omega^d_{\text{int}}\]

Reduced problem with transmission conditions ITC-1-0
(derived by asymptotic expansion)

\[-\Delta \tilde{e}_{\text{ext}}(x) = -i\omega \mu_0 j_0(x) \quad \text{in } \Omega^0_{\text{ext}}\]
\[[\tilde{e}_{\text{ext}}](x) = 0 \quad \text{on } \Gamma\]
\[[\partial_n \tilde{e}_{\text{ext}}](x) - i\omega \mu_0 \sigma d\{\tilde{e}_{\text{ext}}\}(x) = 0 \quad \text{on } \Gamma\]
\[\tilde{e}_{\text{int}}(x, y) = \{\tilde{e}_{\text{ext}}\}(x) \quad \text{in } \Omega^d_{\text{int}}\]

- \(\tilde{e}_{\text{ext}}\) is defined in \(\Omega^0_{\text{ext}}\) and approximates \(e_{\text{ext}}\) in \(\Omega^d_{\text{ext}}\)
- \(\tilde{e}_{\text{int}}\) is \textit{a-posteriori} computed in \(\Omega^d_{\text{int}}\) and approximates \(e_{\text{int}}\)
Reduced problem with transmission conditions ITC-1-0

\[-\Delta \tilde{e}_{\text{ext}}(x) = -i\omega \mu_0 j_0(x) \quad \text{in } \Omega^0_{\text{ext}}\]
\[[\tilde{e}_{\text{ext}}](x) = 0 \quad \text{on } \Gamma\]
\[[\tilde{h}_{x,\text{ext}}](x) + \sigma d \{\tilde{e}_{\text{ext}}\}(x) = 0 \quad \text{on } \Gamma\]

\(\Delta\) goes back to Levi-Civita (1902), see e.g. in the book of Bateman (1915)

In the case of a very thin conducting sheet it is convenient to treat the thickness of the sheet as negligible and regard the tangential components of the magnetic force as discontinuous when a point moves along the normal from one side of the sheet to the other.

\(\Delta\) in review article on “Approximate boundary conditions” by Senior (1981)

\(\Delta\) boundary integral formulation by Knott and Senior (1974) and McWirthner (1982)

\(\Delta\) used in a network approach by Carpenter and Djurovic (1975)

\(\Delta\) used in FEM by Poltz and Romanowski (1983), Rodger and Atkinson (1988), Biro et. al. (1992), Jin et.al. (1992)

\(\Delta\) derived as limit conditions for asymptotically small thickness, \(d \to 0\) while \(\omega \sigma \sim d^{-1}\) and error analysis by K.S., S. Tordeux (2010) \(\to\) ITC-1-0
Asymptotics – frameworks with $\varepsilon$ varying conductivity

$$-\Delta u^\varepsilon + c(\varepsilon) u^\varepsilon = f$$

$c(\varepsilon) = c_0$ ... compare sheets of constant material

$c(\varepsilon) = \frac{c_1}{\varepsilon}$ ... compare sheets of constant shielding

$c(\varepsilon) = \frac{c_2}{\varepsilon^2}$ ... compare sheet of constant skin depth

Shielding efficiency $SE(x_j) = 20 \log_{10} \frac{|H_0(x_j)|}{|H(x_j)|}$
Asymptotics – frameworks with $\varepsilon$ varying conductivity

\[-\Delta u^\varepsilon + c(\varepsilon) u^\varepsilon = f\]

$c(\varepsilon) = c_0$ \ldots compare sheets of constant material

$c(\varepsilon) = \frac{c_1}{\varepsilon}$ \ldots compare sheets of constant shielding

$c(\varepsilon) = \frac{c_2}{\varepsilon^2}$ \ldots compare sheet of constant skin depth

Lemma (Asymptotics for $c(\varepsilon) = \frac{c_0}{\varepsilon^\alpha}$)

Let $\alpha \in [0, 2]$, $\Gamma$ being $C^{1,1}$ continuous. Then

\[\|\{u^\varepsilon\}_-^{\varepsilon^\frac{\alpha}{2}}\|_{L^2(\Gamma)} \leq C \varepsilon^{\max(0, \alpha - 1)} \quad \|\{\partial_n u^\varepsilon\}_-^{\varepsilon^\frac{\alpha}{2}}\|_{L^2(\Gamma)} \leq C \varepsilon^{\max(0, 1 - \alpha)}\]

Borderline case $\alpha = 1$ of asymptotic constant shielding

\begin{itemize}
  \item only non-trivial limit problem (ITC-1-0) for $\varepsilon \to 0$
  \item for $\alpha > 1$ the limit $\varepsilon \to 0$ is PEC $u^0 = 0$ on $\Gamma$
  \item for $\alpha < 1$ the limit $\varepsilon \to 0$ is “no sheet”: $[u^0] = [\partial_n u^0] = 0$
\end{itemize}
Original eddy current problem

\[-\Delta e_{\text{ext}}(x) = -i\omega\mu_0 j_0(x) \quad \text{in } \Omega^d_{\text{ext}}\]
\[-\Delta e_{\text{int}}(x) + i\omega\mu_0 \sigma e_{\text{int}}(x) = 0 \quad \text{in } \Omega^d_{\text{int}}\]

Reduced problem with transmission conditions ITC-1-0

\[-\Delta \tilde{e}_{\text{ext}}(x) = -i\omega\mu_0 j_0(x) \quad \text{in } \Omega^0_{\text{ext}}\]

\[[\tilde{e}_{\text{ext}}](x) = 0 \quad \text{on } \Gamma\]

\[[\partial_y \tilde{e}_{\text{ext}}](x) - i\omega\mu_0 \sigma d\{\tilde{e}_{\text{ext}}\}(x) = 0 \quad \text{on } \Gamma\]

\[\tilde{e}_{\text{int}}(x, y) = \{\tilde{e}_{\text{ext}}\}(x) \quad \text{in } \Omega^d_{\text{int}}\]

Lemma (Order zero in its asymptotic regime, K.S. and S.Tordeux (2010))

Let \(\omega\sigma \sim d^{-1}\), then \(\|e_{\text{ext}} - \tilde{e}_{\text{ext}}\|_{H^1(\Omega^d_{\text{ext}})} \leq C d\) and \(\|e_{\text{int}} - \tilde{e}_{\text{int}}\|_{H^1(\Omega^d_{\text{int}})} \leq C \sqrt{d}\).

Lemma (Robustness for low and high frequency, K.S. and A.Chernov (2012))

Independently of \(\omega\sigma\) it holds \(\|e_{\text{ext}} - \tilde{e}_{\text{ext}}\|_{H^1(\Omega^d_{\text{ext}})} \leq C(d)\) where \(\lim_{d \to 0} C(d) = 0\).
Reduced problem with transmission conditions ITC-1-0

\[-\Delta \tilde{e}_{\text{ext}}(x) = -i\omega \mu_0 j_0(x) \quad \text{in } \Omega^0_{\text{ext}}\]

\[\left[\tilde{e}_{\text{ext}}\right](x) = 0 \quad \text{on } \Gamma\]

\[\left[\partial_y \tilde{e}_{\text{ext}}\right](x) - i\omega \mu_0 \sigma d \{\tilde{e}_{\text{ext}}\}(x) = 0 \quad \text{on } \Gamma\]

Lemma (Robustness w.r.t. to the frequency, K.S. and A.Chernov (2012))

Independently of \(\omega \sigma\) it holds
\[
\|e_{\text{ext}} - \tilde{e}_{\text{ext}}\|_{H^1(\Omega^d_{\text{ext}})} \leq C(d) \text{ where } \lim_{d \to 0} C(d) = 0.
\]

In fact \(\|e_{\text{ext}} - \tilde{e}_{\text{ext}}\|_{H^1(\Omega^d_{\text{ext}})} \leq C d\).
Reduced problem with transmission conditions ITC-1-0

\[-\Delta \tilde{e}_{\text{ext}}(x) = -i\omega \mu_0 j_0(x) \quad \text{in } \Omega^0_{\text{ext}}\]

\[[\tilde{e}_{\text{ext}}](x) = 0 \quad \text{on } \Gamma\]

\[[\partial_y \tilde{e}_{\text{ext}}](x) - i\omega \mu_0 \sigma d\{\tilde{e}_{\text{ext}}\}(x) = 0 \quad \text{on } \Gamma\]

Lemma (Robustness w.r.t. to the frequency, K.S. and A.Chernov (2012))

*Independently of* $\omega \sigma$ *it holds* $\|e_{\text{ext}} - \tilde{e}_{\text{ext}}\|_{H^1(\Omega^d_{\text{ext}})} \leq C(d)$ *where* $\lim_{d \to 0} C(d) = 0$.

In fact $\|e_{\text{ext}} - \tilde{e}_{\text{ext}}\|_{H^1(\Omega^d_{\text{ext}})} \leq C d$.

▷ and some extra accuracy for large skin depth (low frequency)

Skin depth $\delta \sim 1/\sqrt{\omega \sigma}$
ITC-1-1

\[-\Delta\tilde{e}_{\text{ext}}(x) = -i\omega\mu_0 j_0(x) \quad \text{in } \Omega^0_{\text{ext}}\]

\[\tilde{e}_{\text{ext}}(t) = 0, \quad \text{on } \Gamma\]

\[\tilde{e}_{\text{int}}(s, t) = \{\tilde{e}_{\text{ext}}\}(t) \left(1 - \frac{i\omega\mu_0 \sigma}{2} \left(\frac{s}{d}\right)^2 + \frac{1}{4}\right) + \{\partial_n\tilde{e}_{\text{ext}}\}(t) s \quad \text{in } \Omega^d_{\text{int}}\]

- First order accurate for \(\sigma\omega \sim d^{-1}\):

\[\|e_{\text{ext}} - \tilde{e}_{\text{ext}}\|_{H^1(\Omega^d_{\text{ext}})} \leq C d^2\]

\[\|e_{\text{int}} - \tilde{e}_{\text{int}}\|_{H^1(\Omega^d_{\text{int}})} \leq C d^{3/2}\]

- Robust in \(\sigma\omega\): \(\|e_{\text{ext}} - \tilde{e}_{\text{ext}}\|_{H^1(\Omega^d_{\text{ext}})} \leq C d\)

- same structure as ITC-1-0, accuracy improved for low and intermediate shielding
Transmission conditions ITC-1-1

ITC-1-1

\[ [\tilde{e}_{\text{ext}}] (t) = 0, \]

\[ [\partial_n \tilde{e}_{\text{ext}}] (t) - im\mu_0 \sigma d (1 - \frac{im\mu_0 \sigma}{6} d^2) \{\tilde{e}_{\text{ext}}\} (t) = 0 \]

Comparison with “intuitive” low-frequency shielding element by Nakata et.al. (1990)

Shielding element

\[ [\tilde{e}_{\text{ext}}] (t) = 0, \]

\[ [\partial_n \tilde{e}_{\text{ext}}] (t) - d (im\mu_0 \sigma - \partial_t^2) \{\tilde{e}_{\text{ext}}\} (t) = 0 \]
Transmission conditions ITC-1-2

\[ \tilde{e}_{\text{ext}}(t) - d^3 \frac{i \omega \mu_0 \sigma \kappa(t)}{24} \{ \tilde{e}_{\text{ext}} \}(t) - \frac{i \omega \mu_0 \sigma}{12} d^3 \{ \partial_n \tilde{e}_{\text{ext}} \}(t) = 0, \]

\[ \{ \partial_n \tilde{e}_{\text{ext}} \}(t) + i \omega \mu_0 \sigma d \left( 1 - \frac{i \omega \mu_0 \sigma}{6} d^2 - \frac{d^2}{12} \left( \frac{7 \omega^2 \mu_0^2 \sigma^2}{20} d^2 + \partial_t^2 \right) \right) \{ \tilde{e}_{\text{ext}} \}(t) + d^3 \frac{i \omega \mu_0 \sigma \kappa(t)}{24} \{ \partial_n \tilde{e}_{\text{ext}} \}(t) = 0, \]

- Second order accurate for \( \sigma \omega \sim d^{-1} \):
  \[ \| e_{\text{ext}} - \tilde{e}_{\text{ext}} \|_{H^1(\Omega_{\text{ext}}^d)} \leq C d^3 \]
  \[ \| e_{\text{int}} - \tilde{e}_{\text{int}} \|_{H^1(\Omega_{\text{int}}^d)} \leq C d^{\frac{5}{2}} \]

- **not** robust in \( \sigma \omega \), for \( \sigma \omega \to \infty \):
  \( \{ \tilde{e}_{\text{ext}} \}, \{ \partial_n \tilde{e}_{\text{ext}} \} \to 0 \), but
  \( e_{\text{ext}} \big|_{\pm \frac{d}{2}} \to 0 \) (PEC), and so
  \[ \lim_{\omega \sigma \to \infty} \| e_{\text{ext}} - \tilde{e}_{\text{ext}} \|_{H^1(\Omega_{\text{ext}}^d)} = C \]

- more general structure than ITC-1-0/ITC-1-1,
  accuracy improved for low and intermediate shielding
Reduced problem with thin sheet conditions

\[-\Delta \tilde{e}_{\text{ext}}(x) = -i\omega \mu_0 j_0(x)\]

\[
[\tilde{e}_{\text{ext}}] = 2\sqrt{-i\omega \mu_0 \sigma^{-1}} \tanh(\sqrt{-i\omega \mu_0 \sigma} \frac{d}{2}) \{\partial_y \tilde{e}_{\text{ext}}\}
\]

\[
[\partial_y \tilde{e}_{\text{ext}}] = 2\sqrt{-i\omega \mu_0 \sigma} \tanh(\sqrt{-i\omega \mu_0 \sigma} \frac{d}{2}) \{\tilde{e}_{\text{ext}}\}
\]

d ▶ first time in the book of Tozoni and Mayergoyz (1974)

d ▶ article in english by Mayergoyz and Bedrosian (1995)

d ▶ Boundary integral equation by Krähenbühl and Muller (1993), Igarashi et.al. (1998)

d ▶ FEM by Guérin et.al. (1995)

▶ Structure is more complicated than ITC-1-0, but about the same accuracy

▶ ... also more complicated than ITC-1-1, but less accurate in low-frequency

▶ \[\|e - \tilde{e}\|_{H^1(\Omega_{\text{ext}}^d)} = O(d)\ \text{indep. of } \omega \sigma \text{ (robust)}\]
Reduced problem with thin sheet conditions

\[-\Delta \tilde{e}_{\text{ext}}(x) = -i\omega\mu_0 j_0(x)\]

\[
\begin{align*}
[\tilde{e}_{\text{ext}}] &= 2\sqrt{-i\omega\mu_0\sigma}^{-1} \tanh(\sqrt{-i\omega\mu_0\sigma} \frac{d}{2}) \{\partial_y \tilde{e}_{\text{ext}}\} \\
[\partial_y \tilde{e}_{\text{ext}}] &= 2\sqrt{-i\omega\mu_0\sigma} \tanh(\sqrt{-i\omega\mu_0\sigma} \frac{d}{2}) \{\tilde{e}_{\text{ext}}\}
\end{align*}
\]

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Reduced problem with thin sheet conditions

\[-\Delta \tilde{e}_{ext}(x) = -i\omega \mu_0 j_0(x)\]

\[
\tilde{e}_{ext} = 2\sqrt{-i\omega \mu_0 \sigma}^{-1} \tanh(\sqrt{-i\omega \mu_0 \sigma} \frac{d}{2}) \{ \partial_y \tilde{e}_{ext} \}
\]

\[
\partial_y \tilde{e}_{ext} = 2\sqrt{-i\omega \mu_0 \sigma} \tanh(\sqrt{-i\omega \mu_0 \sigma} \frac{d}{2}) \{ \tilde{e}_{ext} \}
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- Structure is more complicated than ITC-1-0, but about the same accuracy
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- \[\| e - \tilde{e} \|_{H^1(\Omega_{d_{ext}})} = O(d) \text{ indep. of } \omega \sigma \text{ (robust)}\]
Reduced problem with thin sheet conditions

\[-\Delta \tilde{e}_{\text{ext}}(x) = -i\omega \mu_0 j_0(x)\]

\[
[\tilde{e}_{\text{ext}}] = 2\sqrt{-i\omega \mu_0 \sigma}^{-1} \tanh(\sqrt{-i\omega \mu_0 \sigma} \frac{d}{2}) \{\partial_y \tilde{e}_{\text{ext}}\}
\]

\[
[\partial_y \tilde{e}_{\text{ext}}] = 2\sqrt{-i\omega \mu_0 \sigma} \tanh(\sqrt{-i\omega \mu_0 \sigma} \frac{d}{2}) \{\tilde{e}_{\text{ext}}\}
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- Structure is more complicated than ITC-1-0, but about the same accuracy
- ... also more complicated than ITC-1-1, but less accurate in low-frequency
- \[\|e - \tilde{e}\|_{H^1(\Omega^d_{\text{ext}})} = O(d)\] indep. of \(\omega \sigma\) (robust)
Transmission conditions by asymptotic expansion with $\omega \sigma \sim 1/\varepsilon^2$

Thin sheet conditions $\delta := \sqrt{2/(\mu_0 \sigma \omega)}$

\[
\begin{align*}
[\tilde{e}_{\text{ext}}] &= (1 + i) \delta \tanh\left(\frac{1-i}{2} d \delta\right)\{\partial_n \tilde{e}_{\text{ext}}\} \\
[\partial_n \tilde{e}_{\text{ext}}] &= 2(1 - i) \delta^{-1} \tanh\left(\frac{1-i}{2} d \delta\right)\{\tilde{e}_{\text{ext}}\}
\end{align*}
\]

Transmission conditions by asymptotic expansion for $\omega \sigma \sim 1/\varepsilon^2$

Trivial limit for $\varepsilon \to 0$ \quad $\tilde{e}|_{\Gamma} = 0$ (PEC)

New family of transmission conditions ITC-2-N

- include $\sinh\left(\frac{1-i}{2} d \delta\right)$, $\cosh\left(\frac{1-i}{2} d \delta\right)$ as thin sheet conditions

ITC-2-0

\[
[\tilde{e}_{\text{ext}}] = 0
\]

\[
[\partial_y \tilde{e}_{\text{ext}}] = \frac{2(1 - i) \delta^{-1} \sinh\left(\frac{1-i}{2} d \delta\right)}{\cosh\left(\frac{1-i}{2} d \delta\right) - \frac{1-i}{2} d \delta \sinh\left(\frac{1-i}{2} d \delta\right)}\{\tilde{e}_{\text{ext}}\}
\]

- same structure than ITC-1-0 and ITC-1-1, accuracy as ITC-1-1
Transmission conditions by asymptotic expansion with $\omega \sigma \sim 1/\varepsilon^2$

Thin sheet conditions $\delta := \sqrt{2/(\mu_0 \sigma \omega)}$

$$
\begin{align*}
[\tilde{e}_{\text{ext}}] &= (1 + i) \delta \tanh(\frac{1-i}{2} \frac{d}{\delta}) \{\partial_n \tilde{e}_{\text{ext}}\} \\
[\partial_n \tilde{e}_{\text{ext}}] &= 2(1 - i) \delta^{-1} \tanh(\frac{1-i}{2} \frac{d}{\delta}) \{\tilde{e}_{\text{ext}}\}
\end{align*}
$$

Transmission conditions by asymptotic expansion for $\omega \sigma \sim 1/\varepsilon^2$

Trivial limit for $\varepsilon \to 0$ $\tilde{e}|_{\Gamma} = 0$ (PEC)

New family of transmission conditions ITC-2-N

- include $\sinh(\frac{1-i}{2} \frac{d}{\delta})$, $\cosh(\frac{1-i}{2} \frac{d}{\delta})$ as thin sheet conditions

ITC-2-1

$$
\begin{align*}
[\tilde{e}_{\text{ext}}] &= \frac{\kappa d}{2} \left( 1 - 2(1 - i) \delta^{-1} \tanh(\frac{1-i}{2} \frac{d}{\delta}) \right) \{\tilde{e}_{\text{ext}}\} - d \left( 1 - 2(1 - i) \delta^{-1} \tanh(\frac{1-i}{2} \frac{d}{\delta}) \right) \{\partial_n \tilde{e}_{\text{ext}}\} \\
[\partial_n \tilde{e}_{\text{ext}}] &= \frac{2(1 - i) \delta^{-1} \sinh(\frac{1-i}{2} \frac{d}{\delta})}{\cosh(\frac{1-i}{2} \frac{d}{\delta}) - \frac{1-i}{2} \frac{d}{\delta} \sinh(\frac{1-i}{2} \frac{d}{\delta})} \{\tilde{e}_{\text{ext}}\} - \frac{\kappa d}{2} \left( 1 - 2(1 - i) \delta^{-1} \tanh(\frac{1-i}{2} \frac{d}{\delta}) \right) \{\partial_n \tilde{e}_{\text{ext}}\}
\end{align*}
$$

- same structure than ITC-1-2 (but without $\partial_t^2$)

Transmission conditions by asymptotic expansion for $\omega \sigma \sim 1/\varepsilon^2$

Trivial limit for $\varepsilon \to 0$ \[ \tilde{e}|_{\Gamma} = 0 \text{ (PEC)} \]

New family of transmission conditions ITC-2-N

- include $\sinh\left(\frac{1-i}{\delta}\right)$, $\cosh\left(\frac{1-i}{\delta}\right)$ as thin sheet conditions

\[
\text{ITC-2-1} \\
[\tilde{e}_{\text{ext}}] = \frac{\kappa d}{2} \left( 1 - 2(1-i)\delta^{-1} \tanh\left(\frac{1-i}{2}\delta\right) \right) \{\tilde{e}_{\text{ext}}\} - d \left( 1 - 2(1-i)\delta^{-1} \tanh\left(\frac{1-i}{2}\delta\right) \right) \{\partial_n \tilde{e}_{\text{ext}}\}
\]

\[
\left[ \partial_n \tilde{e}_{\text{ext}} \right] = \frac{2(1-i)\delta^{-1} \sinh\left(\frac{1-i}{2}\delta\right)}{\cosh\left(\frac{1-i}{2}\delta\right) - \frac{1-i}{2}\delta \sinh\left(\frac{1-i}{2}\delta\right)} \{\tilde{e}_{\text{ext}}\} - \frac{\kappa d}{2} \left( 1 - 2(1-i)\delta^{-1} \tanh\left(\frac{1-i}{2}\delta\right) \right) \{\partial_n \tilde{e}_{\text{ext}}\}
\]

- same structure than ITC-1-2 (but without $\partial_t^2$)
- robust in $\omega \sigma$

\[ \|e - \tilde{e}\|_{H^1(\Omega_{\text{ext}}^d)} = O(d^2) \]

Assume $\Gamma$ to be a closed Lipschitz curve.

- Transmission conditions of **Type I** have form (e.g., ITC-1-0, ITC-1-1, ITC-2-0)
  
  \[-\Delta U = F \quad \text{in } \mathbb{R}^2 \setminus \Gamma\]

  \[
  [\gamma_1 U] - \beta_1 \{\gamma_0 U\} = 0 \quad \text{on } \Gamma
  \]

  \[
  [\gamma_0 U] = 0 \quad \text{on } \Gamma
  \]

- Transmission conditions of **Type II** have form (e.g., shielding element by Nakata et al.)
  
  \[
  [\gamma_1 U] - (\beta_1 - \beta_2 \partial^2) \{\gamma_0 U\} = 0 \quad \text{on } \Gamma,
  \]

  \[
  [\gamma_0 U] = 0 \quad \text{on } \Gamma
  \]

- Transmission conditions of **Type III** have form (e.g., thin sheet conditions)
  
  \[
  [\gamma_1 U] - \beta_1 \{\gamma_0 U\} = 0 \quad \text{on } \Gamma,
  \]

  \[
  [\gamma_0 U] - \beta_3 \{\gamma_1 U\} = 0 \quad \text{on } \Gamma,
  \]

- Transmission conditions of **Type IV** have form (e.g., ITC-1-2, ITC-2-1)
  
  \[
  [\gamma_1 U] - (\beta_1 - \beta_2 \partial^2) \{\gamma_0 U\} + \beta_4 \kappa \{\gamma_1 U\} = 0 \quad \text{on } \Gamma,
  \]

  \[
  [\gamma_0 U] - \beta_4 \kappa \{\gamma_0 U\} - \beta_3 \{\gamma_1 U\} = 0 \quad \text{on } \Gamma
  \]

- $\beta_1$ may be small or large, assume $\beta_2$, $\beta_3$, $\beta_4$ to be small
Assume a closed Lipschitz curve $\Gamma$.

$$-\Delta U = F \quad \text{in } \mathbb{R}^2 \setminus \Gamma$$

**Representation formula**

$$U = -S [\gamma_1 U] + D [\gamma_0 U] + N F$$

with

$$(S \phi)(x) := \int_{\Gamma} G(x - y) \phi(y) dy$$

$$(D \delta)(x) := \int_{\Gamma} \gamma_{1,y} G(x - y) \delta(y) dy$$

$$(N F)(x) := \int_{\mathbb{R}^2} G(x - y) F(y) dy$$

$$G(x - y) = -\frac{1}{2\pi} \log(|x - y|)$$

**Boundary integral operators**

$$V := \{ \gamma_0 S \cdot \} : H^{-1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma),$$

$$K := \{ \gamma_0 D \cdot \} : H^{1/2+s}(\Gamma) \to H^{1/2+s}(\Gamma),$$

$$K' := \{ \gamma_1 S \cdot \} : H^{-1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma),$$

$$W := -\{ \gamma_1 D \cdot \} : H^{1/2+s}(\Gamma) \to H^{-1/2+s}(\Gamma),$$

**Taking mean traces**

$$\{ \gamma_0 U \} = -V [\gamma_1 U] + K [\gamma_0 U] + \gamma_0 NF$$

$$\{ \gamma_1 U \} = -K' [\gamma_1 U] - W [\gamma_0 U] + \gamma_1 NF$$
Boundary integral equations for type I transmission conditions

- Transmission conditions of Type I have form (e.g., ITC-1-0, ITC-1-1, ITC-2-0)
  \[
  [\gamma_1 U] - \beta_1 \{\gamma_0 U\} = 0 \quad \text{on } \Gamma \\
  [\gamma_0 U] = 0 \quad \text{on } \Gamma
  \]

- Representation of mean traces by jump traces using boundary integral operators
  \[
  \{\gamma_0 U\} = -V [\gamma_1 U] + K [\gamma_0 U] + \gamma_0 NF \\
  \{\gamma_1 U\} = -K' [\gamma_1 U] - W [\gamma_0 U] + \gamma_1 NF
  \]

- Boundary integral formulation (second kind) for \( \phi := [\gamma_1 U] \in L^2(\Gamma) \)
  \[
  (Id + \beta_1 V) \phi = \beta_1 \gamma_0 NF
  \]

- Variational formulation
  \[
  \langle \phi, \phi' \rangle + \beta_1 \langle V \phi, \phi' \rangle = \beta_1 \langle \gamma_0 NF, \phi' \rangle
  \]

- \( |\beta_1| \) may be small or large, assume \( 0 \geq \text{arg}(\beta_1) \geq \theta_1^* > -\pi \)
- bilinearform is \( L^2(\Gamma) \)-elliptic with constant \( O(1) \) and \( H^{-1/2}(\Gamma) \)-elliptic with constant \( O(|\beta_1|) \)

**Theorem (K.S. and R. Hiptmair (2013))**

Let \( \Gamma \) be a Lipschitz curve. Then, there exists a constant \( C \) independent of \( |\beta_1| \) such that

\[
\|\phi\|_{L^2(\Gamma)} \leq C|\beta_1|\|\gamma_0 NF\|_{H^{1/2}(\Gamma)}, \quad \|\phi\|_{H^{-1/2}(\Gamma)} \leq C\min(1, |\beta_1|)\|\gamma_0 NF\|_{H^{1/2}(\Gamma)}.
\]

- singularly pertubed BIE for \( |\beta_1| \gg 1 \) (\( \delta \ll d \), high frequency) \( \Rightarrow \) internal layers
Transmission conditions of Type I have form (e.g., ITC-1-0, ITC-1-1, ITC-2-0)
\[
[\gamma_1 U] - \beta_1 \{\gamma_0 U\} = 0 \quad \text{on} \ \Gamma
\]
\[
[\gamma_0 U] = 0 \quad \text{on} \ \Gamma
\]

Boundary integral formulation (second kind) for \(\phi := [\gamma_1 U] \in L^2(\Gamma)\)
\[
(Id + \beta_1 V) \phi = \beta_1 \gamma_0 NF
\]

|\(\beta_1|\) may be small or large, assume \(0 \geq \arg(\beta_1) \geq \theta_1^* > -\pi\)

**Theorem (K.S. and R. Hiptmair (2013))**

Let \(\Gamma\) be a Lipschitz curve. Then, there exists a constant \(C\) independent of \(|\beta_1|\) such that
\[
\|\phi\|_{L^2(\Gamma)} \leq C|\beta_1|\|\gamma_0 NF\|_{H^{1/2}(\Gamma)},
\]
\[
\|\phi\|_{H^{-1/2}(\Gamma)} \leq C \min(1, |\beta_1|)\|\gamma_0 NF\|_{H^{1/2}(\Gamma)}.
\]

Singularly perturbed BIE for \(|\beta_1| \gg 1\) (\(\delta \ll d\), high frequency) \(\Rightarrow\) internal layers

**Theorem (K.S. and R. Hiptmair (2013))**

For \(s \geq 0\) it holds
\[
\|\phi\|_{H^{s+1/2}(\Gamma)} \leq C_s \min(1, |\beta_1|)\|\gamma_0 NF\|_{H^{2s+5/2}(\Gamma)}.
\]

**Proof.**

Using an asymptotic expansion in \(\beta_1^{-1}\), that is \(\phi \sim \phi_0 + \beta_1^{-1} \phi_1 + \beta_1^{-2} \phi_2 + \ldots\), elliptic shift theorems and bootstrapping.
Theorem (K.S. and R. Hiptmair (2013))

Let \( \Gamma \) be a Lipschitz curve. Then, there exists a constant \( C \) independent of \( |\beta_1| \) such that

\[
\| \phi \|_{L^2(\Gamma)} \leq C |\beta_1| \| \gamma_0 NF \|_{H^{1/2}(\Gamma)}, \quad \| \phi \|_{H^{-1/2}(\Gamma)} \leq C \min(1, |\beta_1|) \| \gamma_0 NF \|_{H^{1/2}(\Gamma)}.
\]

Theorem (K.S. and R. Hiptmair (2013))

For \( s \geq 0 \) it holds

\[
\| \phi \|_{H^{s+1/2}(\Gamma)} \leq C_s \min(1, |\beta_1|) \| \gamma_0 NF \|_{H^{2s+5/2}(\Gamma)}.
\]

Proof.

Rewriting BIE

\[
\phi = \beta_1 (-V \phi + \gamma_0 NF)
\]

Estimation in \( H^{1/2}(\Gamma) \)-norm

\[
\| \phi \|_{H^{1/2}(\Gamma)} \leq C |\beta_1| (\| \phi \|_{H^{-1/2}(\Gamma)} + \| \gamma_0 NF \|_{H^{1/2}(\Gamma)}) \leq C |\beta_1| \| \gamma_0 NF \|_{H^{1/2}(\Gamma)}
\]

Estimation in \( H^{s+1/2}(\Gamma) \)-norm (recurrence)

\[
\| \phi \|_{H^{s+1/2}(\Gamma)} \leq C \max(|\beta_1|^{s+1}, |\beta_1|) \| \gamma_0 NF \|_{H^{s+1/2}(\Gamma)}
\]
Boundary element method for type I transmission conditions

Theorem (K.S. and R. Hiptmair (2013))

Let $\Gamma$ be a Lipschitz curve. Then, there exists a constant $C$ independent of $|\beta_1|$ such that

\[
\|\phi\|_{L^2(\Gamma)} \leq C|\beta_1|\|\gamma_0 NF\|_{H^{1/2}(\Gamma)},
\|\phi\|_{H^{-1/2}(\Gamma)} \leq C\min(1, |\beta_1|)\|\gamma_0 NF\|_{H^{1/2}(\Gamma)}.
\]

Theorem (K.S. and R. Hiptmair (2013))

For $s \geq 0$ it holds $\|\phi\|_{H^{s+1/2}(\Gamma)} \leq C_s \min(1, |\beta_1|)\|\gamma_0 NF\|_{H^{2s+5/2}(\Gamma)}$.

Proof.

Ansatz for $|\beta_1| > 1$

\[
\phi = \phi_0 + \beta_1^{-1}\phi_1 + \ldots + \beta_1^{-N}\phi_N + \delta\phi_N
\]

with

\[
V\phi_0 = \gamma_0 NF,
V\phi_{n+1} = -\phi_n,
(Id + \beta_1 V)\delta\phi_N = -\beta_1^{-N}\phi_N.
\]

We estimate

\[
\|\phi_n\|_{H^{s+1/2}(\Gamma)} \leq C_n\|\gamma_0 NF\|_{H^{s+3/2+n}(\Gamma)},
\|\delta\phi_N\|_{H^{s+1/2}(\Gamma)} \leq C|\beta_1|^{s+1-N}\|\phi_N\|_{H^{s+1/2}(\Gamma)} \leq C|\beta_1|^{s+1-N}\|\gamma_0 NF\|_{H^{s+3/2+N}(\Gamma)}
\]
Transmission conditions of Type I (e.g., ITC-1-0, ITC-1-1, ITC-2-0)

Variational formulation: Seek \( \phi \in L^2(\Gamma) \) such that for all \( \phi' \in L^2(\Gamma) \)

\[
\langle \phi, \phi' \rangle + \beta_1 \langle V \phi, \phi' \rangle = \beta_1 \langle \gamma_0 NF, \phi' \rangle
\]

Discretisation on a mesh \( \Gamma_h \)

by piecewise constants \( S_0^{-1}(\Gamma_h) \) or piecewise linear, continuous \( S_1^0(\Gamma_h) \)

Theorem (K.S. and R. Hiptmair (2013))

The discrete variational formulation has a unique solution \( \phi_h \in S_0^{-1}(\Gamma_h)/S_1^0(\Gamma_h) \) \((p = 0, 1)\), and

- \( \| \phi_h \|_{L^2(\Gamma)} \leq C \ | \beta_1 | \ \| F \|_{(H^1(\mathbb{R}^2 \setminus \Gamma))'} \) for Lipschitz \( \Gamma \),
- \( \| \phi_h \|_{L^2(\Gamma)} \leq C \min(1, | \beta_1 |) \ \| F \|_{(H^1(\mathbb{R}^2 \setminus \Gamma))'} \) if \( \Gamma \in C^{1,1} \),
- \( \| \phi_h - \phi \|_{L^2(\Gamma)} \leq C \min(1, | \beta_1 |) h^{p+1} \ \| F \|_{(H^1(\mathbb{R}^2 \setminus \Gamma))'} \), if \( \Gamma \) is smooth enough.

Proof.

Lax-Milgram + Cea’s lemma.

For \( | \beta_1 | \gg 1 \) Cea’s lemma gives additional \( | \beta_1 | \), use discrete version of bootstrapping.

Decomposition \( \phi = \phi_0 + \delta \phi_0 \) where \( \phi_0 = V^{-1} \gamma_0 NF \)

\( \phi_h = \phi_{0,h} + \delta \phi_{0,h} \) where \( \langle V \phi_{0,h}, \phi' \rangle = \langle \gamma_0 NF, \phi' \rangle \)
Transmission conditions of Type I have form (e.g., ITC-1-0, ITC-1-1, ITC-2-0)

\[
[\gamma_1 U] - \beta_1 \{\gamma_0 U\} = 0 \quad \text{on } \Gamma
\]

\[
[\gamma_0 U] = 0 \quad \text{on } \Gamma
\]

Variational formulation: Seek \( \phi \in L^2(\Gamma) \) such that for all \( \phi' \in L^2(\Gamma) \)

\[
\langle \phi, \phi' \rangle + \beta_1 \langle V \phi, \phi' \rangle = \beta_1 \langle \gamma_0 NF, \phi' \rangle
\]

Discretisation on a mesh \( \Gamma_h \)

- by piecewise constants \( S_0^{-1}(\Gamma_h) \) or piecewise linear, continuous \( S_1^0(\Gamma_h) \)
Transmission conditions of **Type I** have form (e.g., ITC-1-0, ITC-1-1, ITC-2-0)

\[
\begin{align*}
[\gamma_1 U] - \beta_1 \{\gamma_0 U\} &= 0 \quad \text{on } \Gamma \\
\{\gamma_0 U\} &= 0 \quad \text{on } \Gamma
\end{align*}
\]

**Variational formulation**: Seek \( \phi \in L^2(\Gamma) \) such that for all \( \phi' \in L^2(\Gamma) \)

\[
\langle \phi, \phi' \rangle + \beta_1 \langle V\phi, \phi' \rangle = \beta_1 \langle \gamma_0 NF, \phi' \rangle
\]

**Discretisation on a mesh** \( \Gamma_h \)

- by piecewise constants \( S_{-1}^0(\Gamma_h) \) or piecewise linear, continuous \( S_1^0(\Gamma_h) \)
Transmission conditions of Type II have form (e.g., shielding element by Nakata et al.)

\[
\begin{align*}
\{ \gamma_1 U \} - (\beta_1 - \beta_2 \partial^2_r) \{ \gamma_0 U \} &= 0 \quad \text{on } \Gamma, \\
\{ \gamma_0 U \} &= 0 \quad \text{on } \Gamma
\end{align*}
\]

Representation of mean traces by jump traces using boundary integral operators

\[
\begin{align*}
\{ \gamma_0 U \} &= -V \{ \gamma_1 U \} + K \{ \gamma_0 U \} + \gamma_0 NF \\
\{ \gamma_1 U \} &= -K' \{ \gamma_1 U \} - W \{ \gamma_0 U \} + \gamma_1 NF
\end{align*}
\]

Boundary integral equation as mixed formulation (1st kind)

for \( \phi := [\gamma_1 U] \in H^{-1/2}(\Gamma) \), \( u := \{ \gamma_0 U \} \in H^1(\Gamma) \)

\[
V \phi + u = \gamma_0 N f
\]

\[
-\phi + \beta_1 u - \beta_2 \partial^2_r u = 0
\]

singly perturbed BIE for \( \beta_1 \gg 1 \) (high frequency) or \( \beta_2 \ll 1 \) (always)
Transmission conditions of Type II have form (e.g., shielding element by Nakata et al.)

\[
[\gamma_1 U] - (\beta_1 - \beta_2 \partial_r^2) \{\gamma_0 U\} = 0 \quad \text{on } \Gamma,
\]

\[
[\gamma_0 U] = 0 \quad \text{on } \Gamma
\]

Representation of mean traces by jump traces using boundary integral operators

\[
\{\gamma_0 U\} = -V [\gamma_1 U] + K [\gamma_0 U] + \gamma_0 NF
\]

\[
\{\gamma_1 U\} = -K' [\gamma_1 U] - W [\gamma_0 U] + \gamma_1 NF
\]

Boundary integral equation as mixed formulation (1st kind)

for \( \phi := [\gamma_1 U] \in H^{-1/2}(\Gamma) \), \( u := \{\gamma_0 U\} \in H^1(\Gamma) \)

\[
V \phi + u = \gamma_0 N f
\]

\[-\phi + \beta_1 u - \beta_2 \partial_r^2 u = 0 \quad \Rightarrow \quad \begin{pmatrix} V & I_d \\ -I_d & \beta_1 I_d - \beta_2 \partial_r^2 \end{pmatrix} \begin{pmatrix} \phi \\ u \end{pmatrix} = \begin{pmatrix} \gamma_0 N f \\ 0 \end{pmatrix}
\]

Variational formulation

\[
\langle V \phi, \phi' \rangle_\Gamma + \langle u, \phi' \rangle_\Gamma = \langle \gamma_0 N f, \phi' \rangle_\Gamma
\]

\[-\langle \phi, u' \rangle_\Gamma + \beta_1 \langle u, u' \rangle_\Gamma + \beta_2 \langle \partial_r u, \partial_r u' \rangle_\Gamma = 0
\]

\(\triangleright\) singularly perturbed BIE for \( \beta_1 \gg 1 \) (high frequency) or \( \beta_2 \ll 1 \) (always)
Transmission conditions of Type II have form (e.g., shielding element by Nakata et al.)

\[
\begin{align*}
[\gamma_1 U] - (\beta_1 - \beta_2 \partial^2) \{\gamma_0 U\} &= 0 \quad \text{on } \Gamma, \\
[\gamma_0 U] &\quad = 0 \quad \text{on } \Gamma
\end{align*}
\]

Boundary integral equation as mixed formulation (1st kind)

for \( \phi := [\gamma_1 U] \in H^{-1/2}(\Gamma) \), \( u := \{\gamma_0 U\} \in H^1(\Gamma) \)

\[
\begin{align*}
V \phi + u &\quad = \gamma_0 N f \\
-\phi + \beta_1 u - \beta_2 \partial^2 u &\quad = 0
\end{align*}
\]

⇒ \( \begin{pmatrix} V & \text{Id} \\ -\text{Id} & \beta_1 \text{Id} - \beta_2 \partial^2 \end{pmatrix} \begin{pmatrix} \phi \\ u \end{pmatrix} = \begin{pmatrix} \gamma_0 N f \\ 0 \end{pmatrix} \)

singly perturbed BIE for \( \beta_1 \gg 1 \) (high frequency) or \( \beta_2 \ll 1 \) (always)

Theorem (K.S. and R. Hiptmair (2013))

Let \( 0 \geq \arg(\beta_1) \geq \theta_1^* > -\pi, \beta_2 > 0 \) where \( \text{Re} \beta_2 \geq 0, \text{Im} \beta_2 \leq 0 \). Then, there exists a constant \( C \) independent of \( |\beta_1|, |\beta_2| \) such that

\[
\begin{align*}
\| \phi \|_{H^{-1/2}(\Gamma)} &\quad \leq C \\
\| u \|_{L^2(\Gamma)} &\quad \leq C \min(|\beta_1|^{-1/2}, 1) \| \gamma_0 NF \|_{H^{1/2}(\Gamma)}, \\
| u |_{H^1(\Gamma)} &\quad \leq C |\beta_2|^{-1/2} \| \gamma_0 NF \|_{H^{1/2}(\Gamma)}.
\end{align*}
\]
Transmission conditions of Type II have form (e.g., shielding element by Nakata et al.)

\[
\begin{align*}
\gamma_1 U - (\beta_1 - \beta_2 \partial^2) \{\gamma_0 U\} &= 0 \text{ on } \Gamma, \\
\{\gamma_0 U\} &= 0 \text{ on } \Gamma
\end{align*}
\]

Boundary integral equation as mixed formulation (1st kind)

for \( \phi := [\gamma_1 U] \in H^{-1/2}(\Gamma) \), \( u := \{\gamma_0 U\} \in H^1(\Gamma) \)

\[
\begin{align*}
V \phi + u &= \gamma_0 N f \\
-\phi + \beta_1 u - \beta_2 \partial^2 u &= 0 \\
\Rightarrow \quad \left( \begin{array}{cc}
V & \text{Id} \\
-\text{Id} & \beta_1 \text{Id} - \beta_2 \partial^2
\end{array} \right) \left( \begin{array}{c}
\phi \\
u
\end{array} \right) &= \left( \begin{array}{c}
\gamma_0 N f \\
0
\end{array} \right)
\end{align*}
\]

singly perturbed BIE for \( \beta_1 \gg 1 \) (high frequency) or \( \beta_2 \ll 1 \) (always)

Theorem (K.S. and R. Hiptmair (2013))

Let \( 0 \geq \arg(\beta_1) \geq \theta^*_1 > -\pi \), \( \beta_2 > 0 \) where \( \text{Re} \beta_2 \geq 0 \), \( \text{Im} \beta_2 \leq 0 \) and \( \Gamma \) be smooth enough. Then, there exists a constant \( C \) independent of \( |\beta_1| \), \( |\beta_2| \) such that

\[
\begin{align*}
\|\phi\|_{H^{-1/2}(\Gamma)} &\leq C(\min(1, |\beta_1|) + |\beta_2|) \quad \|F\|_{(H^1(\mathbb{R}^2 \setminus \Gamma))'}, \\
\|u\|_{H^1(\Gamma)} &\leq C \min(|\beta_1|^{-1}, 1) \quad \|F\|_{(H^1(\mathbb{R}^2 \setminus \Gamma))'}, \\
|\phi|_{H^1(\Gamma)} &\leq C \quad \|F\|_{(H^1(\mathbb{R}^2 \setminus \Gamma))'}, \\
|u|_{H^2(\Gamma)} &\leq C(\min(|\beta_1|^{-1}, 1) + |\beta_2|) \|F\|_{(H^1(\mathbb{R}^2 \setminus \Gamma))'}.
\end{align*}
\]
Transmission conditions of Type II have form (e.g., shielding element by Nakata et al.)
\[
\begin{align*}
[\gamma_1 U] - (\beta_1 - \beta_2 \partial^2_\Gamma) \{\gamma_0 U\} &= 0 \quad \text{on } \Gamma, \\
[\gamma_0 U] &= 0 \quad \text{on } \Gamma
\end{align*}
\]

Variational formulation
\[
\langle V \phi, \phi' \rangle_{\Gamma} + \langle u, \phi' \rangle_{\Gamma} = \langle \gamma_0 N f, \phi' \rangle_{\Gamma}
\]
\[
-\langle \phi, u' \rangle_{\Gamma} + \beta_1 \langle u, u' \rangle_{\Gamma} + \beta_2 \langle \partial_\Gamma u, \partial_\Gamma u' \rangle_{\Gamma} = 0
\]

Discretisation on a mesh $\Gamma_h$

- $\phi_h$ and $u_h$ piecewise linear, continuous $S^0_1(\Gamma_h)$
- no stability with $S^{-1}_0(\Gamma_h)$ for $\phi_h$ for $\beta_1, \beta_2 \to 0$
Transmission conditions of Type II have form (e.g., shielding element by Nakata et al.)

\[
[\gamma_1 U] - (\beta_1 - \beta_2 \partial^2) \{\gamma_0 U\} = 0 \quad \text{on } \Gamma,
\]
\[
[\gamma_0 U] = 0 \quad \text{on } \Gamma
\]

Variational formulation

\[
\langle V\phi, \phi' \rangle_{\Gamma} + \langle u, \phi' \rangle_{\Gamma} = \langle \gamma_0 N f, \phi' \rangle_{\Gamma} - \langle \phi, u' \rangle_{\Gamma} + \beta_1 \langle u, u' \rangle_{\Gamma} + \beta_2 \langle \partial_{\Gamma} u, \partial_{\Gamma} u' \rangle_{\Gamma} = 0
\]

Discretisation on a mesh \( \Gamma_h \)

\( \phi_h \) and \( u_h \) piecewise linear, continuous \( S_0^1(\Gamma_h) \)

**Theorem (K.S. and R. Hiptmair (2013))**

*The discrete variational formulation has a unique solution \( (\phi_h, u_h) \in S_0^1(\Gamma_h) \times S_0^1(\Gamma_h) \) and*

\[
\|\phi_h\|_{H^{-1/2}(\Gamma)} \leq C_{\phi,r}(\beta_1, \beta_2) \|F\|_{(H^1(\mathbb{R}^2 \setminus \Gamma))'}, \quad \|u_h\|_{H^1(\Gamma)} \leq C_{u,r}(\beta_1, \beta_2) \|F\|_{(H^1(\mathbb{R}^2 \setminus \Gamma))'}
\]
\[
\|\phi_h - \phi\|_{H^{-1/2}(\Gamma)} \leq C_{\phi,r}(\beta_1, \beta_2) h \|F\|_{(H^1(\mathbb{R}^2 \setminus \Gamma))'}, \quad \|u_h - u\|_{H^1(\Gamma)} \leq C_{u,r}(\beta_1, \beta_2) h \|F\|_{(H^1(\mathbb{R}^2 \setminus \Gamma))'}
\]

**Proof.**

Lax-Milgram + Cea’s lemma. For \( |\beta_2| \ll 1 \) discrete version of bootstrapping.
Transmission conditions of Type II have form (e.g., shielding element by Nakata et al.)

\[
[\gamma_1 U] - (\beta_1 - \beta_2 \partial^2) \{ \gamma_0 U \} = 0 \quad \text{on } \Gamma,
\]
\[
[\gamma_0 U] = 0 \quad \text{on } \Gamma
\]

**Variational formulation**

\[
\langle V\phi, \phi' \rangle_\Gamma + \langle u, \phi' \rangle_\Gamma = \langle \gamma_0 N f, \phi' \rangle_\Gamma
\]
\[
-\langle \phi, u' \rangle_\Gamma + \beta_1 \langle u, u' \rangle_\Gamma + \beta_2 \langle \partial \Gamma u, \partial \Gamma u' \rangle_\Gamma = 0
\]

**Discretisation on a mesh \( \Gamma_h \)**

- \( \phi_h \) and \( u_h \) piecewise linear, continuous \( S^0_1(\Gamma_h) \)

\[
\|u_h - u\|_{H^1(\Gamma)} = 3 \cdot 10^{-3}
\]
\[
\|\Pi_h(\phi_h - \phi)\|_{H^{-1/2}(\Gamma)} = 2 \cdot 10^{-3}
\]
Transmission conditions of Type II have form (e.g., shielding element by Nakata et al.)

\[
[\gamma_1 U] - (\beta_1 - \beta_2 \partial^2) \{ \gamma_0 U \} = 0 \quad \text{on } \Gamma,
\]

\[
[\gamma_0 U] = 0 \quad \text{on } \Gamma
\]

Variational formulation

\[
\langle V \phi, \phi' \rangle_{\Gamma} + \langle u, \phi' \rangle_{\Gamma} = \langle \gamma_0 N f, \phi' \rangle_{\Gamma}
\]

\[-\langle \phi, u' \rangle_{\Gamma} + \beta_1 \langle u, u' \rangle_{\Gamma} + \beta_2 \langle \partial_{\Gamma} u, \partial_{\Gamma} u' \rangle_{\Gamma} = 0
\]

Discretisation on a mesh \( \Gamma_h \)

\( \phi_h \) and \( u_h \) piecewise linear, continuous \( S^0_1(\Gamma_h) \)
Transmission conditions of **Type IV** have form (e.g., ITC-1-2, ITC-2-1)

\[
\begin{align*}
[\gamma_1 U] - (\beta_1 - \beta_2 \partial^2) \{\gamma_0 U\} + \beta_4 \kappa \{\gamma_1 U\} &= 0 \quad \text{on } \Gamma, \\
[\gamma_0 U] - \beta_4 \kappa \{\gamma_0 U\} - \beta_3 \{\gamma_1 U\} &= 0 \quad \text{on } \Gamma,
\end{align*}
\]

**Representation of mean traces by jump traces using boundary integral operators**

\[
\begin{align*}
\{\gamma_0 U\} &= -V [\gamma_1 U] + K [\gamma_0 U] + \gamma_0 NF \\
\{\gamma_1 U\} &= -K' [\gamma_1 U] - W [\gamma_0 U] + \gamma_1 NF
\end{align*}
\]

**Boundary integral equation as mixed formulation**

for \( \phi := [\gamma_1 U] \in H^{-1/2}(\Gamma) \), \( u := \{\gamma_0 U\} \in H^1(\Gamma) \) (or \( H^{1/2}(\Gamma) \) for \( \beta_2 = 0 \)), \( j := [\gamma_0 U] \in H^{1/2}(\Gamma) \)

\[
\begin{pmatrix}
V & \text{Id} & -K \\
-\text{Id} & \beta_1 \text{Id} + \beta_3^{-1} \beta_4^2 \kappa^2 - \beta_2 \partial^2 & -\beta_3^{-1} \beta_4 \kappa \\
K' & -\beta_3^{-1} \beta_4 \kappa & W + \beta_3^{-1} \text{Id}
\end{pmatrix}
\begin{pmatrix}
\phi \\
u \\
j
\end{pmatrix}
= \begin{pmatrix}
\gamma_0 NF \\
0 \\
\gamma_1 NF
\end{pmatrix}
\]

**> all four boundary integral operators involved**

**> singularly pertubed BIE for \( \beta_1 \gg 1 \) (high frequency) or \( 0 \neq \beta_2 \ll 1 \) or \( \beta_3 \ll 1 \) (always)**

**> discretisation by \( S_1^0(\Gamma_h) \) for all three unknowns**
Transmission conditions of Type IV have form (e.g., ITC-1-2, ITC-2-1)

\[
\begin{align*}
[\gamma_1 U] - (\beta_1 - \beta_2 \partial \overline{\Gamma}) \{\gamma_0 U\} + \beta_4 \kappa \{\gamma_1 U\} &= 0 \quad \text{on } \Gamma, \\
[\gamma_0 U] - \beta_4 \kappa \{\gamma_0 U\} - \beta_3 \{\gamma_1 U\} &= 0 \quad \text{on } \Gamma,
\end{align*}
\]

Boundary integral equation as mixed formulation

for \( \phi := [\gamma_1 U] \in H^{-1/2}(\Gamma) \), \( u := \{\gamma_0 U\} \in H^1(\Gamma) \) (or \( H^{1/2}(\Gamma) \) for \( \beta_2 = 0 \)), \( j := [\gamma_0 U] \in H^{1/2}(\Gamma) \)

\[
\begin{pmatrix}
V & I d & -K \\
-I d & \beta_1 I d + \beta_3^{-1} \beta_4^2 \kappa^2 - \beta_2 \partial \overline{\Gamma} & -\beta_3^{-1} \beta_4 \kappa \\
K' & -\beta_3^{-1} \beta_4 \kappa & W + \beta_3^{-1} I d
\end{pmatrix}
\begin{pmatrix}
\phi \\
u \\
j
\end{pmatrix}
= \begin{pmatrix}
\gamma_0 N F \\
0 \\
\gamma_1 N F
\end{pmatrix}
\]

Lemma

If \( \beta_2 = 0 \), \( V_h = W_h = X_h = S_0^0(\Gamma_h) \), and \( \Gamma \in C^{4,1} \), then with \( C = C(\beta_1, \beta_2, \beta_3, \beta_4) \)

\[
\|\phi_h - \phi\|_{H^{-1/2}(\Gamma)} + \|j_h - j\|_{H^{1/2}(\Gamma)} + \|u_h - u\|_{H^{1/2}(\Gamma)} \leq C h^{3/2} \|F\|_{(H^1(\mathbb{R}^2 \setminus \Gamma))'}.
\]

If \( |\beta_2| > 0 \), \( V_h \in \{S_0^{-1}(\Gamma_h), S_0^0(\Gamma_h)\} \), \( W_h = X_h = S_0^0(\Gamma_h) \), and \( \Gamma \in C^{4,1} \), then

\[
\|\phi_h - \phi\|_{H^{-1/2}(\Gamma)} + \|j_h - j\|_{H^{1/2}(\Gamma)} + \|u_h - u\|_{H^1(\Gamma)} \leq C h \|F\|_{(H^1(\mathbb{R}^2 \setminus \Gamma))'}.
\]
Summary

- Model reduction: Thin conducting sheets replaced by transmission conditions $T^\varepsilon(\omega)$
- Boundary integral equation only on the midline of the sheet
- Second kind BIE and BEM for transmission conditions for ITC-1-0, ITC-1-1 and ITC-2-0, well-posedness, error estimates in dependence of parameter
- First kind mixed BIE and BEM for shielding element, well-posedness (stable pairs for $u = \{\gamma_0 U\}$ and $\phi = [\gamma_1 U]$ needed for weak shielding), well-posedness, error estimates
- First kind mixed BIE and BEM for ITC-1-2 and ITC-2-1 for $u = \{\gamma_0 U\}$, $\phi = [\gamma_1 U]$, $j = [\gamma_0 U]$

Extension and Application of the boundary integral equations

- Transmission conditions for Helmholtz equation and small wave-number
- To boundary layers of singular perturbed PDEs (multiscale expansion)
- To thin periodic sheets (surface homogenisation + matched asymptotic expansion)

Open questions and Outlook

- Transmission conditions and boundary integral equations for Maxwell’s equations
- Correction of transmission conditions in case of non-smooth sheets (with kinks)