RENNES, FRANCE
August 26-30, 2013
Jounées Singulières Augmentées en l'honneur de
MARTIN COSTABEL

Singularities of Solutions to Mixed Interface Crack Problems

David Natroshvili
Georgian Technical University, Tbilisi , Georgia

T. Buchukuri, O. Chkadua
A.Razmadze Mathematical Institute
I.Javakhishvili Tbilisi State University

Geometrical description of the composite configuration

$\Omega^{(m)}, \Omega \subset \mathbb{R}^3$, $\partial\Omega \cap \partial\Omega^{(m)} = \overline{\Gamma^{(m)}} = \overline{\Gamma_T^{(m)}} \cup \overline{\Gamma_C^{(m)}}$ - **interface**

$\partial\Omega = \overline{\Gamma_T^{(m)}} \cup \overline{\Gamma_C^{(m)}} \cup \overline{S_N} \cup \overline{S_D}, \quad \partial\Omega^{(m)} = \overline{\Gamma_T^{(m)}} \cup \overline{\Gamma_C^{(m)}} \cup \overline{S_N^{(m)}}.$

$\partial\Gamma_T^{(m)}$, $\partial\Gamma_C^{(m)}$, $\partial\Omega_D$ - **Exceptional (singularity) curves;**

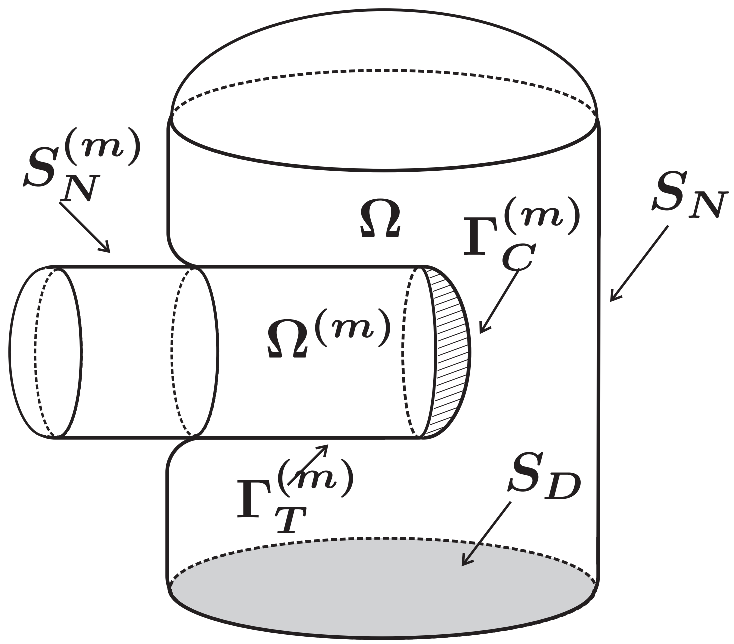

Figure 1: Composite body

$\Omega^{(m)}$ -an anisotropic homogeneous elastic medium (metallic inclusion):
four-dimensional thermo-elastic field $U^{(m)} = (u^{(m)}, \vartheta^{(m)})^\top$:

the displacement vector $u^{(m)} = (u_1^{(m)}, u_2^{(m)}, u_3^{(m)})^\top$,

the temperature function $u_4^{(m)} = \vartheta^{(m)}$;

Ω - an anisotropic homogeneous piezoelectric medium (ceramic matrix):
five-dimensional thermo-electro-elastic field $U = (u, \vartheta, \varphi)^\top$:

the displacement vector $u = (u_1, u_2, u_3)^\top$,

the temperature function $u_4 = \vartheta$,

the electric potential $u_5 = \varphi$ (electric field $E = -\nabla \varphi$).

Thermo-elastic field equations in $\Omega^{(m)}$:

The basic governing equations of the classical thermoelasticity:

Constitutive relations:

$$\sigma_{ij}^{(m)} = c_{ijlk}^{(m)} s_{lk}^{(m)} - \gamma_{ij}^{(m)} \vartheta^{(m)} = c_{ijlk}^{(m)} \partial_l u_k^{(m)} - \gamma_{ij}^{(m)} \vartheta^{(m)},$$

$$\mathcal{S}^{(m)} = \gamma_{ij}^{(m)} s_{ij}^{(m)} + \alpha^{(m)} [T_0^{(m)}]^{-1} \vartheta^{(m)},$$

$$s_{lk}^{(m)} = 2^{-1} (\partial_l u_k^{(m)} + \partial_k u_l^{(m)}), \qquad \partial_j = \frac{\partial}{\partial x_j}.$$

Fourier Law: $q_j^{(m)} = -\varkappa_{jl}^{(m)} \partial_l T^{(m)}, \quad T^{(m)} = T_0^{(m)} + \vartheta^{(m)}.$

Equations of motion: $\partial_i \sigma_{ij}^{(m)} + X_j^{(m)} = \varrho^{(m)} \partial_t^2 u_j^{(m)}.$

Equation of the entropy balance: $T^{(m)} \partial_t \mathcal{S}^{(m)} = -\partial_j q_j^{(m)} + X_4^{(m)}.$

$c_{ijkl}^{(m)} = c_{jikl}^{(m)} = c_{klij}^{(m)}, \ \gamma_{ij}^{(m)} = \gamma_{ji}^{(m)}, \ \varkappa_{ij}^{(m)} = \varkappa_{ji}^{(m)},$

$c_{ijkl}^{(m)} \xi_{ij} \xi_{kl} \geq c_0 \xi_{ij} \xi_{ij}$ for all $\xi_{ij} = \xi_{ji} \in \mathbb{R},$

$\varkappa_{ij}^{(m)} \xi_i \xi_j \geq c_1 \xi_i \xi_i$ for all $\xi_j \in \mathbb{R}, \ i, j, k, l = 1, 2, 3.$

The system of thermo-elastic pseudo-oscillations $(\tau = \sigma + i\omega)$

$$c^{(m)}_{ijlk}\,\partial_i\partial_l\,u^{(m)}_k - \varrho^{(m)}\tau^2\,u^{(m)}_j - \gamma^{(m)}_{ij}\,\partial_i\vartheta^{(m)} + X^{(m)}_j = 0,\ j = 1, 2, 3,$$
$$-\tau\,T^{(m)}_0\,\gamma^{(m)}_{il}\,\partial_l u^{(m)}_i + \varkappa^{(m)}_{il}\,\partial_i\partial_l\vartheta^{(m)} - \tau\,\alpha^{(m)}\,\vartheta^{(m)} + X^{(m)}_4 = 0.$$

Matrix form

$$A^{(m)}(\partial,\tau)\,U^{(m)}(x) + \tilde{X}^{(m)}(x) = 0 \ \text{ in } \ \Omega^{(m)}, \qquad (1)$$

$U^{(m)} := (u^{(m)}, \vartheta^{(m)})^\top$ is the unknown vector,
$\tilde{X}^{(m)} = (X^{(m)}_1, X^{(m)}_2, X^{(m)}_3, X^{(m)}_4)^\top$ is a given vector,
$A^{(m)}(\partial,\tau)$ **- nonselfadjoint strongly elliptic** 4×4 **matrix operator**

$$A^{(m)}(\partial,\tau) = [\,A^{(m)}_{jk}(\partial,\tau)\,]_{4\times 4},$$

$$A^{(m)}_{jk}(\partial,\tau) = c^{(m)}_{ijlk}\,\partial_i\,\partial_l - \varrho^{(m)}\,\tau^2\,\delta_{jk},$$

$$A^{(m)}_{4k}(\partial,\tau) = -\tau\,T^{(m)}_0\,\gamma^{(m)}_{kl}\,\partial_l, \quad A^{(m)}_{j4}(\partial,\tau) = -\gamma^{(m)}_{ij}\,\partial_i,$$

$$A^{(m)}_{44}(\partial,\tau) = \varkappa^{(m)}_{il}\,\partial_i\,\partial_l - \alpha^{(m)}\,\tau$$

Generalized thermo-stress operator

$$\mathcal{T}^{(m)}(\partial,\nu) = [\, \mathcal{T}_{jk}^{(m)}(\partial,\nu)\,]_{4\times 4},$$

$$\mathcal{T}_{jk}^{(m)}(\partial,\nu) = c_{ijlk}^{(m)}\, \nu_i\, \partial_l, \quad \mathcal{T}_{j4}^{(m)}(\partial,\nu) = -\gamma_{ij}^{(m)}\, \nu_i,$$

$$\mathcal{T}_{4k}^{(m)}(\partial,\nu) = 0, \quad \mathcal{T}_{44}^{(m)}(\partial,\nu) = \varkappa_{il}^{(m)}\, \nu_i\, \partial_l,$$

$\nu = (\nu_1, \nu_2, \nu_3)$ – outward unit normal vector on $\partial\Omega^{(m)}$.

For a four–vector $U^{(m)} = (u^{(m)}, \vartheta^{(m)})^\top$:

$$\mathcal{T}^{(m)} U^{(m)} = (\, \sigma_{i1}^{(m)}\, \nu_i,\ \sigma_{i2}^{(m)}\, \nu_i,\ \sigma_{i3}^{(m)}\, \nu_i,\ -q_i^{(m)}\, \nu_i\,)^\top.$$

The first three components correspond to the mechanical stress vector in the theory of thermoelasticity, while the forth one is the normal component of the heat flux vector (with opposite sign).

Thermo-electro-elastic field equations in Ω :

Constitutive relations:

$$\begin{aligned} \sigma_{ij} &= c_{ijkl}\, s_{kl} - e_{lij} E_l - \gamma_{ij}\,\vartheta = c_{ijkl}\,\partial_l u_k + e_{lij}\,\partial_l \varphi - \gamma_{ij}\,\vartheta, \\ \mathcal{S} &= \gamma_{ij}\, s_{ij} + g_l E_l + \alpha\,[\,T_0\,]^{-1}\,\vartheta, \quad s_{kj} = 2^{-1}(\partial_k\, u_j + \partial_j\, u_k), \\ D_j &= e_{jkl}\, s_{kl} + \varepsilon_{jl}\, E_l + g_j\,\vartheta = e_{jkl}\,\partial_l u_k - \varepsilon_{jl}\,\partial_l \varphi + g_j\,\vartheta, \\ i, j &= 1, 2, 3. \end{aligned}$$

Fourier Law: $q_i = -\varkappa_{il}\,\partial_l T, \quad i = 1, 2, 3.$

Equations of motion: $\partial_i \sigma_{ij} + X_j = \varrho\,\partial_t^2 u_j, \quad j = 1, 2, 3.$

Equation of the entropy balance: $T\,\partial_t \mathcal{S} = -\partial_j q_j + X_4.$

Equation of static electric field: $\partial_i D_i - X_5 = 0.$

$c_{ijkl} = c_{jikl} = c_{klij},\ e_{ijk} = e_{ikj},\ \varepsilon_{ij} = \varepsilon_{ji},\ \gamma_{ij} = \gamma_{ji},\ \ \varkappa_{ij} = \varkappa_{ji},$

$c_{ijkl}\,\xi_{ij}\,\xi_{kl} \geq c_0\,\xi_{ij}\,\xi_{ij}, \ \ \varepsilon_{ij}\,\eta_i\,\eta_j \geq c_1\,\eta_i\,\eta_i\,, \ \ \varkappa_{ij}\,\eta_i\,\eta_j \geq c_2\,\eta_i\,\eta_i$

for all $\xi_{ij} = \xi_{ji} \in \mathbb{R}$ and for all $\eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3.$

Pseudo-oscillation equations of the thermo-electro-elasticity theory:

$$c_{ijlk}\,\partial_i\,\partial_l\,u_k-\varrho\,\tau^2\,u_j-\gamma_{ij}\,\partial_i\vartheta+e_{lij}\,\partial_l\partial_i\varphi+X_j=0,\ \ j=1,2,3,$$
$$-\tau\,T_0\,\gamma_{il}\,\partial_l u_i+\varkappa_{il}\,\partial_i\,\partial_l\vartheta-\tau\,\alpha\,\vartheta+\tau\,T_0\,g_i\,\partial_i\varphi+X_4=0,$$
$$-e_{ikl}\,\partial_i\partial_l u_k-g_i\,\partial_i\vartheta+\varepsilon_{il}\,\partial_i\partial_l\varphi+X_5=0,$$

Matrix form

$$A(\partial,\tau)\,U(x)+\widetilde{X}(x)=0 \ \text{ in } \ \Omega, \qquad (2)$$

$$U=(u,\vartheta,\varphi)^\top\ ,\ \widetilde{X}=(X_1,X_2,X_3,X_4,X_5)^\top,$$

$A(\partial,\tau)=[A_{jk}(\partial,\tau)]_{5\times 5}$ is a strongly elliptic nonselfadjoint matrix differential operator

$$A_{jk}=c_{ijlk}\,\partial_i\,\partial_l-\varrho\,\tau^2\,\delta_{jk},\ \ A_{j4}=-\gamma_{ij}\,\partial_i,\ \ A_{j5}=e_{lij}\partial_l\partial_i,$$
$$A_{4k}=-\tau\,T_0\,\gamma_{kl}\,\partial_l,\ \ A_{44}=\varkappa_{il}\,\partial_i\,\partial_l-\alpha\,\tau,\ \ A_{45}=\tau\,T_0\,g_i\,\partial_i,$$
$$A_{5k}=-e_{ikl}\,\partial_i\partial_l,\ \ A_{54}=-g_i\partial_i,\ \ A_{55}=\varepsilon_{il}\,\partial_i\partial_l,$$
$$j,k=1,2,3.$$

Generalized thermo-electro-elastic stress operator

$$\mathcal{T}(\partial, n) = \big[\mathcal{T}_{pq}(\partial, n)\big]_{5\times 5}$$

$$\begin{array}{lll}
\mathcal{T}_{jk} = c_{ijlk}\, n_i\, \partial_l, & \mathcal{T}_{j4} = -\gamma_{ij}\, n_i, & \mathcal{T}_{j5} = e_{lij}\, n_i\, \partial_l, \\
\mathcal{T}_{4k} = 0, & \mathcal{T}_{44} = \varkappa_{il}\, n_i\, \partial_l, & \mathcal{T}_{45} = 0, \\
\mathcal{T}_{5k} = -e_{ikl}\, n_i\, \partial_l, & \mathcal{T}_{54} = -g_i\, n_i, & \mathcal{T}_{55} = \varepsilon_{il}\, n_i\, \partial_l \\
 & j, k = 1, 2, 3. &
\end{array}$$

$n = (n_1, n_2, n_3)$ – outward unit normal vector on $\partial\Omega$.

For a vector $U = (u, \vartheta, \varphi)^\top$

$$\mathcal{T}(\partial, n)\, U = (\, \sigma_{i1}\, n_i,\; \sigma_{i2}\, n_i,\; \sigma_{i3}\, n_i,\; -q_i\, n_i,\; -D_i\, n_i\,)^\top.$$

The first three components correspond to the mechanical stress vector in the theory of thermo-electro-elasticity, the forth and fifth ones are the normal components of the heat flux vector and the electric displacement vector (with opposite sign), respectively.

Physical problem under consideration are described mathematically by essentially mixed boundary, transmission and crack conditions. Solutions to this kind of crack and mixed type boundary value problems and related mechanical, thermal and electric characteristics usually have **singularities** in a neighbourhood of **exceptional curves** , $\partial\Gamma_C^{(m)}$, ∂S_D, $\partial\Gamma^{(m)}$.

Our goal is to study the solvability of the mixed transmission problems in appropriate function spaces and analyse regularity properties of solutions. In particular, we describe dependence of the **stress singularity exponents** on the material parameters.

L_p, W_p^r, H_p^s, $B_{p,q}^s$ ($r \geq 0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$) stand for the well-known *Lebesgue, Sobolev-Slobodetski, Bessel potential, and Besov* function spaces.

Problem (ICP-A) - the crack gap is thermally insulated dielectric:

Find vector-functions

$U^{(m)} = (u_1^{(m)}, u_2^{(m)}, u_3^{(m)}, u_4^{(m)})^\top \in [W_p^1(\Omega^{(m)})]^4$

$U = (u_1, u_2, u_3, u_4, u_5)^\top \in [W_p^1(\Omega)]^5$

(i) the systems of partial differential equations:

$$\left[A^{(m)}(\partial_x, \tau)\, U^{(m)} \right]_j = 0 \quad \text{in} \quad \Omega^{(m)}, \quad j = \overline{1,4},$$
$$[\, A(\partial_x, \tau)\, U \,]_k = 0 \quad \text{in} \quad \Omega, \quad k = \overline{1,5},$$

(ii) the boundary conditions:

$$r_{S_N^{(m)}} \{ [\, \mathcal{T}^{(m)}(\partial, \nu)\, U^{(m)} \,]_j \}^+ = Q_j^{(m)} \quad \text{on} \quad S_N^{(m)}, \quad j = \overline{1,4},$$
$$r_{S_N} \{ [\, \mathcal{T}(\partial, n)\, U \,]_k \}^+ = Q_k \quad \text{on} \quad S_N, \quad k = \overline{1,5},$$
$$r_{S_D} \{ u_k \}^+ = f_k \quad \text{on} \quad S_D, \quad k = \overline{1,5},$$
$$r_{\Gamma_T^{(m)}} \{ u_5 \}^+ = f_5^{(m)} \quad \text{on} \quad \Gamma_T^{(m)},$$

(iii) the transmission conditions on $\Gamma_T^{(m)}$ $(j=\overline{1,4})$:

$$r_{\Gamma_T^{(m)}}\{u_j\}^+ - r_{\Gamma_T^{(m)}}\{u_j^{(m)}\}^+ = f_j^{(m)},$$

$$r_{\Gamma_T^{(m)}}\{[\mathcal{T}(\partial,n)U]_j\}^+ + r_{\Gamma_T^{(m)}}\{[\mathcal{T}^{(m)}(\partial,\nu)U^{(m)}]_j\}^+ = F_j^{(m)},$$

(iv) the interface crack conditions:

$$r_{\Gamma_C^{(m)}}\{[\mathcal{T}^{(m)}(\partial,\nu)\,U^{(m)}]_j\}^+ = \widetilde{Q}_j^{(m)} \quad \text{on} \quad \Gamma_C^{(m)}, \;\; j=\overline{1,4},$$

$$r_{\Gamma_C^{(m)}}\{[\mathcal{T}(\partial,n)\,U]_k\}^+ = \widetilde{Q}_k \quad \text{on} \quad \Gamma_C^{(m)}, \;\; k=\overline{1,5},$$

where $n=-\nu$ on $\Gamma^{(m)}$,

$$Q_k\in B_{p,p}^{-1/p}(S_N),\; Q_j^{(m)}\in B_{p,p}^{-1/p}(S_N^{(m)}),\; f_k\in B_{p,p}^{1/p'}(S_D),$$

$$f_k^{(m)}\in B_{p,p}^{1/p'}(\Gamma_T^{(m)}), F_j^{(m)}\in B_{p,p}^{-1/p}(\Gamma_T^{(m)}),\; \widetilde{Q}_j^{(m)}\in B_{p,p}^{-1/p}(\Gamma_C^{(m)}),$$

$$\widetilde{Q}_k\in B_{p,p}^{-1/p}(\Gamma_C^{(m)}),\; \frac{1}{p'}+\frac{1}{p}=1, \quad k=\overline{1,5}, \quad j=\overline{1,4}.$$

UNIQUENESS THEOREM:

Let $\Omega^{(m)}$ and Ω be Lipschitz and either $\tau = \sigma + i\omega$, $(\sigma > 0)$, or $\tau = 0$. The above formulated interface crack problem (ICP-A) has at most one solution in the space $[W_2^1(\Omega^{(m)})]^4 \times [W_2^1(\Omega)]^5$, provided $\mathrm{mes}\, S_D > 0$.

Existence and regularity results for Problem (ICP-A)

Representation formulas of solutions:

Let $\Psi^{(m)}(\cdot\,,\tau) = \big[\,\Psi^{(m)}_{kj}(\cdot\,,\tau)\,\big]_{4\times 4}$ and $\Psi(\cdot\,,\tau) = \big[\,\Psi_{kj}(\cdot\,,\tau)\,\big]_{5\times 5}$ be the fundamental matrix-functions of the operators $A^{(m)}(\partial_x,\tau)$ and $A(\partial_x,\tau)$ and introduce the single layer potentials:

$$V^{(m)}_\tau(h^{(m)})(x) = \int_{\partial\Omega^{(m)}} \Psi^{(m)}(x-y,\tau)\, h^{(m)}(y)\, d_yS,$$

$$V_\tau(h)(x) = \int_{\partial\Omega} \Psi(x-y,\tau)\, h(y)\, d_yS,$$

$h^{(m)} = (h^{(m)}_1,\cdots,h^{(m)}_4)^\top$, $h = (h_1,\cdots,h_5)^\top$ - potential densities.

Jump relations:

$$\{V^{(m)}_\tau(h^{(m)})\}^+_{\partial\Omega^{(m)}} = \{V^{(m)}_\tau(h^{(m)})\}^-_{\partial\Omega^{(m)}} = \mathcal{H}^{(m)}_\tau\, h^{(m)},$$

$$\{\mathcal{T}^{(m)}\, V^{(m)}_\tau(h^{(m)})\}^\pm_{\partial\Omega^{(m)}} = [\mp\, 2^{-1} I_4 + \mathcal{K}^{(m)}_\tau]\, h^{(m)}.$$

The boundary integral (pseudodifferential) operators:

$$\mathcal{H}_\tau^{(m)} h^{(m)} := \int\limits_{\partial\Omega^{(m)}} \Psi^{(m)}(x-y,\tau)\, h^{(m)}(y)\, d_y S, \quad x \in \partial\Omega^{(m)},$$

$$\mathcal{K}_\tau^{(m)} h^{(m)} := \int\limits_{\partial\Omega^{(m)}} \big[\mathcal{T}^{(m)}(\partial_x, \nu(x))\Psi^{(m)}(x-y,\tau)\big] h^{(m)}(y)\, d_y S,$$

$$\mathcal{H}_\tau h := \int\limits_{\partial\Omega} \Psi(x-y,\tau)\, h(y)\, d_y S, \qquad x \in \partial\Omega,$$

$$\mathcal{K}_\tau h := \int\limits_{\partial\Omega} \big[\,\mathcal{T}(\partial_x, n(x))\Psi(x-y,\tau)\,\big]\, h(y)\, d_y S,$$

The layer boundary operators $\mathcal{H}_\tau^{(m)}$, $\mathcal{H}_\tau$ are pseudodifferential operators of order -1, while the operators $\mathcal{K}_\tau^{(m)}$ and $\mathcal{K}_\tau$ are singular integral operators – pseudodifferential operators of order 0.

Representation Lemma I: Let $\mathrm{Re}\,\tau = \sigma > 0$. An arbitrary solution vector $U^{(m)} \in [\,W_p^1(\Omega^{(m)})\,]^4$ to the homogeneous equation

$$A^{(m)}(\partial,\tau)\,U^{(m)} = 0$$

can be uniquely represented by the single layer potential

$$U^{(m)}(x) = V_\tau^{(m)}\big(\,\big[\mathcal{P}_\tau^{(m)}\big]^{-1}\chi^{(m)}\,\big)(x), \quad x \in \Omega^{(m)}, \qquad (3)$$

where $\chi^{(m)} = \{\,\mathcal{T}^{(m)}U^{(m)}\,\}^+ \in \big[\,B_{p,p}^{-\frac{1}{p}}(\partial\Omega^{(m)})\,\big]^4$, $1 < p < \infty$,

$\mathcal{P}_\tau^{(m)} \equiv -2^{-1}\,I_4 + \mathcal{K}_\tau^{(m)} \;:\; \big[\,B_{p,q}^s(\partial\Omega^{(m)})\,\big]^4 \to \big[\,B_{p,q}^s(\partial\Omega^{(m)})\,\big]^4.$

Auxiliary problem I:

$$A^{(m)}(\partial,\tau)\,U^{(m)} = 0 \quad \text{in} \quad \Omega^{(m)},$$

$$\big\{\,\mathcal{T}^{(m)}\,U^{(m)}\,\big\}^+ = \chi^{(m)} \quad \text{on} \quad \partial\Omega^{(m)},$$

$$\chi^{(m)} = (\,\chi_1^{(m)},\,\chi_2^{(m)},\,\chi_3^{(m)},\,\chi_4^{(m)})^\top \in \big[\,H_2^{-\frac{1}{2}}(\partial\Omega^{(m)})\,\big]^4.$$

Representation Lemma II: Let $\mathrm{Re}\,\tau = \sigma > 0$ and $1 < p < \infty$. A solution $U \in [\,W_p^1(\Omega)\,]^5$ to the homogeneous equation $A(\partial, \tau)U = 0$ can be uniquely represented by the single layer potential

$$U = V_\tau\left(\mathcal{P}_\tau^{-1}\chi\right) \quad \text{in} \quad \Omega, \tag{4}$$

where $\chi = \{\mathcal{T}U\}^+ + \beta\ \{U\}^+ \in \left[\,B_{p,p}^{-\frac{1}{p}}(\partial\Omega)\,\right]^5$ with a smooth real valued scalar function β which does not vanish identically, $\beta \geq 0$, $\mathrm{supp}\ \beta \subset S_D$, and the operator

$$\mathcal{P}_\tau \equiv -2^{-1} I_5 + \mathcal{K}_\tau + \beta\,\mathcal{H}_\tau \ :\ \left[\,B_{p,q}^s(\partial\Omega)\,\right]^5 \to \left[\,B_{p,q}^s(\partial\Omega)\,\right]^5, \tag{5}$$

is invertible for all $1 < p < \infty$, $1 \leq q \leq \infty$, and $s \in \mathbb{R}$.

Auxiliary problem II:

$$A(\partial, \tau)U = 0 \quad \text{in} \quad \Omega, \quad \mathrm{Re}\,\tau = \sigma > 0,$$

$$\{\,\mathcal{T}U\,\}^+ + \beta\,\{\,U\,\}^+ = \chi \quad \text{on} \quad \partial\Omega,$$

$$\chi := (\chi_1, \cdots, \chi_5)^\top \in \left[\,H_2^{-\frac{1}{2}}(\partial\Omega)\,\right]^5.$$

Reduction to boundary pseudodifferential equations:

We look for solutions in the form

$$U^{(m)} = V_{\tau}^{(m)}\Big(\big[\mathcal{P}_{\tau}^{(m)} \big]^{-1} \big[G_0^{(m)} + h^{(m)} \big] \Big) \quad \text{in} \quad \Omega^{(m)}, \tag{6}$$

$$U = V_{\tau} \left(\mathcal{P}_{\tau}^{-1} \left[G_0 + \psi + h \right] \right) \quad \text{in} \quad \Omega, \tag{7}$$

where $G_0^{(m)} \in [B_{p,p}^{-1/p}(\partial\Omega^{(m)})]^4$ and $G_0 \in [B_{p,p}^{-1/p}(\partial\Omega)]^5$, are known vectors, and

$$\psi := (\psi_1, \cdots, \psi_5)^\top \in \big[\widetilde{B}_{p,p}^{-1/p}(S_D) \big]^5,$$

$$h := (h_1, \cdots, h_5)^\top \in \big[\widetilde{B}_{p,p}^{-1/p}(\Gamma_T^{(m)}) \big]^5,$$

$$h^{(m)} := (h_1^{(m)}, \cdots, h_4^{(m)})^\top \in \big[\widetilde{B}_{p,p}^{-1/p}(\Gamma_T^{(m)}) \big]^4$$

are sought vector functions.

The homogeneous differential equations, boundary conditions and crack conditions containing the Neumann type conditions are satisfied automatically. The remaining boundary and transmission conditions containing the Dirichlet type conditions lead to the pseudodifferential equations with respect to the unknown vector-functions ψ, h and $h^{(m)}$:

$$
\begin{aligned}
& r_{S_D} \big[\mathcal{H}_\tau \mathcal{P}_\tau^{-1} (\psi + h) \big]_k = \widetilde{f}_k \quad \text{on} \quad S_D, \quad k = \overline{1,5}, \\
& r_{\Gamma_T^{(m)}} \big[\mathcal{H}_\tau \mathcal{P}_\tau^{-1} (\psi + h) \big]_5 = \widetilde{f}_5^{(m)} \quad \text{on} \quad \Gamma_T^{(m)}, \\
& r_{\Gamma_T^{(m)}} \big[\mathcal{H}_\tau \mathcal{P}_\tau^{-1} (\psi + h) \big]_j - r_{\Gamma_T^{(m)}} \big[\mathcal{H}_\tau^{(m)} \big[\mathcal{P}_\tau^{(m)} \big]^{-1} h^{(m)} \big]_j \\
& \qquad\qquad\qquad = \widetilde{f}_j^{(m)} \quad \text{on} \quad \Gamma_T^{(m)}, \; j = \overline{1,4}, \\
& r_{\Gamma_T^{(m)}} h_j^{(m)} + r_{\Gamma_T^{(m)}} h_j = \widetilde{F}_j^{(m)} \; \text{on} \quad \Gamma_T^{(m)}, \quad j = \overline{1,4}.
\end{aligned}
$$

Introduce the Steklov-Poincaré type 5×5 pseudodifferential operators

$$
\mathcal{A}_\tau := \mathcal{H}_\tau \mathcal{P}_\tau^{-1}, \quad \mathcal{B}_\tau^{(m)} := \begin{bmatrix} [\mathcal{H}_\tau^{(m)} \big[\mathcal{P}_\tau^{(m)} \big]^{-1}]_{4\times 4} & [\,0\,]_{4\times 1} \\ [\,0\,]_{1\times 4} & [\,0\,]_{1\times 1} \end{bmatrix}_{5\times 5}
$$

Then the above obtained system can be rewritten as

$$\mathcal{N}_\tau^{(A)}\, \Phi = Y,$$

where $\Phi := (\psi,\ h,\ h^{(m)})^\top \in \mathbf{X}_p^s$ is the unknown vector,
$Y := (\widetilde{f},\ \widetilde{g}^{(m)},\ \widetilde{F}^{(m)})^\top \in \mathbf{Y}_p^s$ is a given vector,

$$\mathcal{N}_\tau^{(A)} := \begin{bmatrix} r_{S_D}\, \mathcal{A}_\tau & r_{S_D}\, \mathcal{A}_\tau & r_{S_D}\, [\,0\,]_{5\times 4} \\ r_{\Gamma_T^{(m)}}\, \mathcal{A}_\tau & r_{\Gamma_T^{(m)}} [\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}] & r_{\Gamma_T^{(m)}}\, [\,0\,]_{5\times 4} \\ r_{\Gamma_T^{(m)}}\, [\,0\,]_{4\times 5} & r_{\Gamma_T^{(m)}}\, I_{4\times 5} & r_{\Gamma_T^{(m)}}\, I_4 \end{bmatrix}_{14\times 14}$$

$$\mathcal{N}_\tau^{(A)}\ :\ \mathbf{X}_{p,q}^s \to \mathbf{Y}_{p,q}^s \quad \left[\,\mathbf{X}_p^s \to \mathbf{Y}_p^s\,\right], \tag{8}$$

$$\mathbf{X}_{p,q}^s := [\widetilde{B}_{p,q}^s(S_D)]^5 \times [\widetilde{B}_{p,q}^s(\Gamma_T^{(m)})]^5 \times [\widetilde{B}_{p,q}^s(\Gamma_T^{(m)})]^4, \tag{9}$$

$$\mathbf{Y}_{p,q}^s := [B_{p,q}^{s+1}(S_D)]^5 \times [B_{p,q}^{s+1}(\Gamma_T^{(m)})]^5 \times [\widetilde{B}_{p,q}^s(\Gamma_T^{(m)})]^4, \tag{10}$$

$$\mathbf{X}_p^s = \mathbf{X}_{p,p}^s, \qquad \mathbf{Y}_p^s = \mathbf{Y}_{p,p}^s\,. \tag{11}$$

$r_{\mathcal{M}}$ - restriction operator onto $\mathcal{M}$.

[Vishik, Eskin, Grubb, Schulze, Duduchava, Shargorodsky, etc]

Denote by $\mathfrak{S}_1(x,\xi_1,\xi_2) := \mathfrak{S}(\mathcal{A}_\tau)(x,\xi_1,\xi_2)$ the principal homogeneous symbol matrix of the operator $\mathcal{A}_\tau$ and let $\lambda_j^{(1)}(x)$ $(j=\overline{1,5})$ be the eigenvalues of the matrix

$$\mathcal{D}_1(x) := [\mathfrak{S}_1(x,0,+1)]^{-1}\,\mathfrak{S}_1(x,0,-1), \qquad x \in \partial S_D.$$

Similarly, let $\mathfrak{S}_2(x,\xi_1,\xi_2) = \mathfrak{S}(\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)})(x,\xi_1,\xi_2)$ be the principal homogeneous symbol matrix of the operator $\mathcal{A}_\tau + \mathcal{B}_\tau^{(m)}$ and $\lambda_j^{(2)}(x)$ $(j=\overline{1,5})$ be the eigenvalues of the matrix

$$\mathcal{D}_2(x) := [\mathfrak{S}_2(x,0,+1)]^{-1}\,\mathfrak{S}_2(x,0,-1), \qquad x \in \partial\Gamma_T^{(m)}.$$

Set

$$\gamma_1' := \inf_{x\in\partial S_D,\, 1\le j\le 5} \frac{1}{2\pi} \arg \lambda_j^{(1)}(x),$$
$$\gamma_1'' := \sup_{x\in\partial S_D,\, 1\le j\le 5} \frac{1}{2\pi} \arg \lambda_j^{(1)}(x),$$
$$\gamma_2' := \inf_{x\in\partial \Gamma_T^{(m)},\, 1\le j\le 5} \frac{1}{2\pi} \arg \lambda_j^{(2)}(x),$$
$$\gamma_2'' := \sup_{x\in\partial \Gamma_T^{(m)},\, 1\le j\le 5} \frac{1}{2\pi} \arg \lambda_j^{(2)}(x).$$

$$\gamma' := \min\{\gamma_1', \gamma_2'\}, \qquad \gamma'' := \max\{\gamma_1'', \gamma_2''\}. \tag{12}$$

$$-\frac{1}{2} < \gamma' \le 0 \le \gamma'' < \frac{1}{2}\,. \tag{13}$$

INVERTIBILITY THEOREM: Let the conditions

$$1 < p < \infty, \quad 1 \le q \le \infty, \quad \frac{1}{p} - 1 + \gamma'' < s + \frac{1}{2} < \frac{1}{p} + \gamma' \qquad (14)$$

be satisfied. Then the operators

$$\mathcal{N}_{\tau}^{(A)} : \mathbf{X}_p^s \to \mathbf{Y}_p^s \qquad \left[\mathbf{X}_{p,q}^s \to \mathbf{Y}_{p,q}^s\right],$$

are invertible.

BASIC EXISTENCE THEOREM: Let the inequality

$$\frac{4}{3 - 2\gamma''} < p < \frac{4}{1 - 2\gamma'} \qquad (15)$$

hold. Then the interface crack problem (ICP-A) has a unique solution

$$(U^{(m)}, U) \in [W_p^1(\Omega^{(m)})]^4 \times [W_p^1(\Omega)]^5,$$

which can be represented by the generalized single layer potentials.

REGULARITY THEOREM FOR PROBLEM (ICP-A):

Let $\alpha > 0$ be not integer and

$$Q_k \in B^{\alpha-1}_{\infty,\infty}(S_N),\ Q_j^{(m)} \in B^{\alpha-1}_{\infty,\infty}(S_N^{(m)}),\ f_k \in C^{\alpha}(\overline{S_D}),$$

$$f_k^{(m)} \in C^{\alpha}(\overline{\Gamma_T^{(m)}}), F_j^{(m)} \in B^{\alpha-1}_{\infty,\infty}(\Gamma_T^{(m)}),\ \widetilde{Q}_j^{(m)} \in B^{\alpha-1}_{\infty,\infty}(\Gamma_C^{(m)}),$$

$$\widetilde{Q}_k \in B^{\alpha-1}_{\infty,\infty}(\Gamma_C^{(m)}),\ k=\overline{1,5},\ j=\overline{1,4},$$

and the compatibility conditions

$$\widetilde{F}_j^{(m)} := F_j^{(m)} - r_{_{\Gamma_T^{(m)}}} G_{0j} - r_{_{\Gamma_T^{(m)}}} G_{0j}^{(m)} \in r_{_{\Gamma_T^{(m)}}} \widetilde{B}^{\alpha-1}_{\infty,\infty}(\Gamma_T^{(m)}),\ j=\overline{1,4},$$

be satisfied. Then

$$U^{(m)} \in \bigcap_{\alpha'<\kappa} [C^{\alpha'}(\overline{\Omega^{(m)}})]^4, \qquad U \in \bigcap_{\alpha'<\kappa} [C^{\alpha'}(\overline{\Omega})]^5,$$

where $\kappa = \min\{\alpha,\ \gamma' + \frac{1}{2}\} > 0$.

Asymptotic expansions:

[Eskin, Shamir, Duduchava, Chkadua, ...]

[Kondratiev, Maz'ya, Costabel, Dauge,...]

$$U(x) = \sum_{\mu=\pm 1} \sum_{j=0}^{n_s-1} c_{j\mu}(x', \alpha)\, r^{\gamma+i\delta} \big(\ln r \big)^{m_j-1}\, \widetilde{c}_{j\mu}(x', \alpha) + \dots,$$

$$U^{(m)}(x) = \sum_{\mu=\pm 1} \sum_{j=0}^{n_s^{(m)}-1} c_{j\mu}^{(m)}(x', \alpha)\, r^{\gamma+i\delta} \big(\ln r \big)^{m_j-1}\, \widetilde{c}_{j\mu}^{(m)}(x', \alpha) + \dots,$$

$$r^{\gamma+i\delta} = \mathrm{diag}\big\{ r^{\gamma_1+i\delta_1}, \dots, r^{\gamma_5+i\delta_5} \big\}$$

$$x' \in \{\partial\Gamma_C^{(m)}, \partial S_D, \partial\Gamma^{(m)}\}$$

$$\gamma_j = \frac{1}{2} + \frac{1}{2\pi} \arg \lambda_j^{(k)}(x'), \quad \delta_j = -\frac{1}{2\pi} \ln |\lambda_j^{(k)}(x')|;$$

Stress singularity exponents:
The above analysis based on the asymptotic expansions of solutions shows that for sufficiently smooth boundary data the principal **dominant singular terms** of solution vectors near the exceptional curves ∂S_D and $\partial \Gamma_T^{(m)}$ can be represented as a product of a "good" vector-function $G(x')$ of the variable $x' \in \partial S_D \cup \partial \Gamma_T^{(m)}$ and a singular factor

$$U^{(m)},\ U \quad \simeq \quad [\, r(x)\,]^{\gamma_j + i\,\delta_j}\, [\, \ln r(x)\,]^{m_j - 1} \times G(x') \tag{16}$$

$$\mathcal{T}^{(m)} U^{(m)},\ \mathcal{T} U \ \simeq\ [\, r(x)\,]^{-1+\gamma_j + i\,\delta_j} [\, \ln r(x)\,]^{m_j - 1} \times G(x') \tag{17}$$

$$\gamma_j = \frac{1}{2} + \frac{\arg\, \lambda_j}{2\pi}\,, \quad \delta_j = -\frac{\ln\, |\lambda_j|}{2\pi} \tag{18}$$

$r(x)$ is the distance from a reference point x to the exceptional curves ∂S_D, $\partial \Gamma^{(m)}$ and $\partial \Gamma_C^{(m)}$,
m_j denotes the multiplicity of the eigenvalue λ_j,
the numbers δ_j are different from zero, in general, and display the **oscillating character** of the stress singularities.

Numerical calculations:

$\Omega^{(m)}$ is occupied by the isotropic metallic material **silver-palladium alloy**,
Ω is occupied by one of the following piezoelectric materials:

$BaTiO_3$ (the crystal symmetry of the class **4mm**),
PZT-4 and **PZT-5A** (the crystal symmetry of the class **6mm**).

Calculations have shown that the parameters γ'_k and γ''_k essentially depend on the material parameters:

		$BaTiO_3$	PZT-4	PZT-5A
∂S_D :	$-0.5+\gamma'_1$	-0.62	-0.62	-0.63
$\partial \Gamma_T^{(m)}$:	$-0.5+\gamma'_2$	-0.56	-0.58	-0.59 .

For these particular cases, from the above table it follows that $\gamma'_1 < \gamma'_2$, which yields that the stress singularities at the curve ∂S_D are higher than the singularities near the curve $\partial \Gamma_T^{(m)}$.

NATROSHVILI, D., BUCHUKURI, T. AND CHKADUA, O.,(2006), *Mathematical modelling and analysis of interaction problems for piezoelectric composites.* Rendiconti Academia Nazionale delle Scienze detta dei XL, Memorie di Matematica e Applicazioni 124°, Vol. XXX, fasc. 1, 159–190.

BUCHUKURI, T., CHKADUA, O., NATROSHVILI, D. AND SÄNDIG, A.-M.,(2008), *Interaction problems of metallic and piezoelectric materials with regard to thermal stresses*. Memoirs on Differential Equations and Mathematical Physics, **45**, 7-74.

BUCHUKURI, T., CHKADUA, O., NATROSHVILI, D. AND SÄNDIG, A.-M.,(2009), *Solvability and regularity results to boundary-transmission problems for metallic and piezoelectric elastic materials.* Math. Nachr., **282**, No.8, 1079-1110.

BUCHUKURI, T., CHKADUA, O. AND NATROSHVILI, D.,(2009), *Mixed boundary value problems of thermopiezoelectricity for solids with interior cracks.* Integral Equations and Operator Theory, **64**, 4, 495-537.

T. BUCHUKURI, O. CHKADUA, R. DUDUCHAVA, D. NATROSHVILI, (2012). *Interface crack problems for metallic-piezoelectric composite structures*, Memoirs on Differential Equations and Math. Physics, **55**, 1-150.

THANK YOU !