About some operators on the unit disc related to the Laplace equation

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Abstract

We introduce four integral operators closely related to the Laplace equation in three-dimensions on the circular unit disc. Two of them are closed to the simple layer on the disc and the other two are related to the hyper singular operator. We establish their variational formulations and the coercivity properties in some unknows Sobolev spaces. They are also linked to the Laplace operator on the disc.
These results are a tentative extension to R^3 of previous results in R^2, contains in a common work with Carlos Jerez-Hanckes that we present in the beginning of the talk.

Log-Kernel

Consider first the isotropic space $\mathbb{R}^2$ divided into two half-planes:

$$\pi_{\pm} := \left\{ \mathbf{x} \in \mathbb{R}^2 \; : \; x_2 \lessgtr 0 \right\} \tag{1}$$

with interface Γ given by the line $x_2 = 0$. The interface is further divided into the open disjoint segments $\Gamma_m := (-1, 1) \times \{0\}$ and $\Gamma_f := \Gamma \setminus \bar{\Gamma}_m$. Consequently, we have defined the domain $\Omega := \mathbb{R}^2 \setminus \bar{\Gamma}_m$. We seek u such that

$$\begin{cases} -\Delta u = 0 & \text{for} \quad \mathbf{x} \in \Omega \\ \quad u = g & \text{for} \quad \mathbf{x} \in \Gamma_m; \quad \text{with} \quad g \in H^{1/2}(\Gamma_m). \end{cases} \tag{2}$$

Then, the potential u can be represented as a single layer potential:

$$u(\mathbf{x}) = L_1\varphi = \frac{1}{\pi} \int_{\Gamma_m} \log \frac{1}{|\mathbf{x}-\mathbf{y}|} \varphi(\mathbf{y}) \mathrm{d}\mathbf{y} , \qquad \text{for} \quad \mathbf{x} \in \Omega , \tag{3}$$

Then φ is the solution of the logarithmic integral equation:

$$g(\mathbf{x}) = \frac{1}{\pi} \int_{\Gamma_m} \log \frac{1}{|\mathbf{x}-\mathbf{y}|} \varphi(\mathbf{y}) \mathrm{d}\mathbf{y} \qquad \text{for} \quad \mathbf{x} \in \Gamma . \tag{4}$$

The equation (4) has a variational formulation in the space $\widetilde{H}_0^{-1/2}(\Gamma_m)$ which is:

$$\frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m}\log\frac{1}{|\tau-t|}\varphi(t)\varphi^t(\tau)dtd\tau=\int_{\Gamma_m}g(\tau)\varphi^t(\tau)d\tau,\forall\varphi^t\in\widetilde{H}_0^{-1/2}(\Gamma_m) \tag{5}$$

This operator is a bijection between $\widetilde{H}_0^{-1/2}(\Gamma_m)$ and the $H_*^{1/2}(\Gamma_m)$ of functions in $H^{1/2}(\Gamma_m)$ satisfying $\int_{\Gamma_m}\frac{1}{\sqrt{1-t^2}}\,g(t)\,dt \;=\; 0$. and moreover we have

$$\frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m}\log\frac{1}{|\tau-t|}\varphi(t)\varphi(\tau)dtd\tau\geq C\,\|\varphi\|^2_{\widetilde{H}_0^{-1/2}(\Gamma_m)},\forall\varphi\in\widetilde{H}_0^{-1/2}(\Gamma_m). \tag{6}$$

The inverse operator is a bijection of $H_*^{1/2}(\Gamma_m)$ onto $\widetilde{H}_0^{-1/2}(\Gamma_m)$. This operator N_1 is symmetric and coercive in the space $H_*^{1/2}(\Gamma_m)$. It admits two variational formulations. Let $M(x,y)$ be the function

$$M(x,y)=\frac{1}{2}\left((y-x)^2+\left(\sqrt{1-x^2}+\sqrt{1-y^2}\right)^2\right) \tag{7}$$

$$L_2g=\frac{1}{\pi}\int_{\Gamma_m}\log\left\{\frac{M(x,y)}{|x-y|}\right\}g(y)dy \tag{8}$$

The first one is:

$$(N_1 g, g^t) = \frac{1}{\pi} \int_{\Gamma_m} \int_{\Gamma_m} \log \left\{ \frac{M(x,y)}{|x-y|} \right\} g'(x) (g^t(y))' \, dy dx = \int_{\Gamma_m} \varphi(x) g^t(x) dx \qquad (9)$$

for all $g^t \in H_*^{1/2}(\Gamma_m)$, which gives a first norm on the space $H_*^{1/2}(\Gamma_m)$:

$$\frac{1}{\pi} \int_{\Gamma_m} \int_{\Gamma_m} \log \left[\frac{M(x,y)}{|x-y|} \right] g'(x) \, g'(y) \, dy \, dx \ \geq \ C \, \|g\|^2_{H_*^{1/2}(\Gamma_m)} \, ; \forall g \in H_*^{1/2}(\Gamma_m) \qquad (10)$$

The second one is

$$\frac{1}{2\pi} \int_{\Gamma_m} \int_{\Gamma_m} \frac{d^2}{dxdy} \log \left[\frac{M(x,y)}{|x-y|} \right] (g(x) - g(y)) (g^t(x) - g^t(y)) \, dy dx = \int_{\Gamma_m} \varphi(x) g^t(x) dx \qquad (11)$$

for all $g^t \in H_*^{1/2}(\Gamma_m)$,

So we have a second norm on the space $H_*^{1/2}(\Gamma_m)$ which is:

$$\frac{1}{2\pi} \int_{\Gamma_m} \int_{\Gamma_m} \left\{ \frac{1 - xy}{w(x) w(y)} \right\} \frac{(g(x) - g(y))^2}{(x-y)^2} dy dx \geq C \, \|g\|^2_{H_*^{1/2}(\Gamma_m)}, \forall g \in H_*^{1/2}(\Gamma_m) \qquad (12)$$

where the weight function w is given by

$$w(x) := \sqrt{1 - x^2} \qquad \text{for } x \in (-1, 1). \qquad (13)$$

We can also consider the Neumann problem

$$\begin{cases} -\Delta u = 0 & \text{for} \quad \mathbf{x} \in \Omega \\ \gamma_m^+ \partial_n u = \gamma_m^- \partial_n u = \varphi & \text{for} \quad \mathbf{x} \in \Gamma_m, \quad \varphi \in H^{-1/2}(\Gamma_m) \end{cases} \tag{14}$$

which can be represent as a double layer potential of harmonic solution in the domain Ω of the form .

$$u(\mathbf{x}) = \frac{1}{\pi} \int_{\Gamma_m} \frac{x_2}{|\mathbf{x} - \mathbf{y}|^2} \alpha(\mathbf{y}) \mathrm{d}\mathbf{y} \,, \qquad \text{for} \quad \mathbf{x} \in \Omega \,, \tag{15}$$

Then the unknown α is the solution of the hyper singular integral equation:

$$\varphi(x) = N_2\, \alpha = \frac{1}{\pi} \int\!\!\!\!-_{\Gamma_m} \frac{1}{|x - y|^2} \alpha(y) dy \qquad \text{for} \quad x \in \Gamma \,. \tag{16}$$

α is also the jump of the Dirichlet trace of the solution of problem (14).

A variational formulation of the integral equation (16) in the space $\widetilde{H}^{1/2}(\Gamma_m)$ is

$$\frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m}\log\frac{1}{|\tau-t|}\alpha'(t)(\alpha^t(\tau))'dtd\tau=\int_{\Gamma_m}\varphi(\tau)\alpha^t(\tau)d\tau,\forall\alpha^t\in\widetilde{H}^{1/2}(\Gamma_m) \quad (17)$$

The associated operator $\widetilde{\mathcal{D}}$ is a bijection from $\widetilde{H}^{1/2}(\Gamma_m)$ to $H^{-1/2}(\Gamma_m)$. Moreover, this bilinear form is coercive, i.e.,

$$\frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m}\log\frac{1}{|\tau-t|}\alpha'(t)\alpha(\tau)'dtd\tau\geq C\,\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_m)},\forall\alpha\in\widetilde{H}^{1/2}(\Gamma_m). \quad (18)$$

This operator admits a second variational formulation which is

$$\frac{1}{2\pi}\int_{\Gamma_m}\int_{\Gamma_m}\frac{(\alpha(x)-\alpha(y))\,(\alpha^t(x)-\alpha^t(y))}{|x-y|^2}dxdy+\frac{1}{\pi}\int_{\Gamma_m}\frac{\alpha(x)\alpha^t(x)}{1-x^2}dx=\int_{\Gamma_m}\varphi(x)\alpha^t(x)dx \quad (19)$$

for all $\alpha^t\in\widetilde{H}^{1/2}(\Gamma_m)$, and the next expression is a norm on $\widetilde{H}^{1/2}(\Gamma_m)$

$$\frac{1}{2\pi}\int_{\Gamma_m}\int_{\Gamma_m}\frac{(\alpha(x)-\alpha(y))^2}{|x-y|^2}dxdy+\frac{1}{\pi}\int_{\Gamma_m}\frac{\alpha(x)^2}{1-x^2}dx\geq C\,\|\alpha\|^2_{\widetilde{H}^{1/2}(\Gamma_m)},\forall\alpha\in\widetilde{H}^{1/2}(\Gamma_m) \quad (20)$$

The inverse operator is a bijection of $H^{-1/2}(\Gamma_m)$ onto $\widetilde{H}^{1/2}(\Gamma_m)$. The associated operator is symmetric and coercive in the space $H^{-1/2}(\Gamma_m)$.
It admits the following variational formulation:

$$\frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m}\log\left[\frac{M(x,y)}{|x-y|}\right]\varphi(x)\varphi^t(y)dydx=\int_{\Gamma_m}\alpha(x)\varphi^t(x)dx,\ \forall\varphi\in H^{-1/2}(\Gamma_m) \quad (21)$$

and thus the following expression is a norm on the space $H^{-1/2}(\Gamma_m)$

$$\frac{1}{\pi}\int_{\Gamma_m}\int_{\Gamma_m}\log\left[\frac{M(x,y)}{|x-y|}\right]\varphi(x)\varphi(y)dydx\geq C\,\|\varphi\|^2_{H^{-1/2}(\Gamma_m)},\ \forall\varphi\in H^{-1/2}(\Gamma_m) \quad (22)$$

The operators L_1, L_2, N_1, N_2, D, D^* are linked by the identities

$$L_2 \circ N_2 = -L_2 \circ D^* \circ L_1 \circ D = I\,, \qquad I \in \widetilde{H}^{1/2}(\Gamma_m)$$

$$L_1 \circ N_1 = -L_1 \circ D \circ L_2 \circ D^* = I\,, \qquad I \in H_*^{1/2}(\Gamma_m)$$

$$N_1 \circ L_1 = -D \circ L_2 \circ D^* \circ L_1 = I\,, \qquad I \in \widetilde{H}_0^{-1/2}(\Gamma_m)$$

$$N_2 \circ L_2 = -D^* \circ L_2 \circ D \circ L_1 = I\,, \qquad I \in H^{-1/2}(\Gamma_m)$$

$L_1 \circ D$ is continuous and invertible from $\widetilde{H}^{1/2}(\Gamma_m)$ into $H_*^{1/2}(\Gamma_m)$.
$L_2 \circ D^*$ is continuous and invertible from $H_*^{1/2}(\Gamma_m)$ into $\widetilde{H}^{1/2}(\Gamma_m)$.
$D^* \circ L_1$ is continuous and invertible from $\widetilde{H}_0^{-1/2}(\Gamma_m)$ into $H^{-1/2}(\Gamma_m)$.
$D \circ L_2$ is continuous and invertible from $H^{-1/2}(\Gamma_m)$ into $\widetilde{H}_0^{-1/2}(\Gamma_m)$.

The Dirichlet and Neumann Laplacian Δ_D, Δ_N are linked to L_1, L_2 and N_1, N_2:

$$L_1 = (-\Delta_D)^{-\frac{1}{2}}\,; \qquad -N_1 = (-\Delta_D)^{\frac{1}{2}}\,;$$

$$L_2 = (-\Delta_N)^{-\frac{1}{2}}\,; \qquad -N_2 = (-\Delta_N)^{\frac{1}{2}}\,.$$

The disc in R^3

We try now to extend these results to the unit disc in $\mathbb{R}^3$.

We Introduce the splitting of the space $\mathbb{R}^3$ into two half-spaces $\pi_\pm := \left\{\mathbf{x} \in \mathbb{R}^3 \,:\, x_3 \gtrless 0\right\}$, by the plane $x_3 = 0$ that will be denote as Γ.
Let $\mathfrak{c}$ be the circle of center at the origin and of radius 1 in the plane Γ.
Let $\mathbb{D}$ be the plane disc delimitated by the circle $\mathfrak{c}$ and $\mathbb{D}$ the associated flat domain in $\mathbb{R}^3$.
Now its complement in $\mathbb{R}^2$, is $\Gamma_f := \Gamma \setminus \bar{\mathbb{D}}$.
Henceforth, the problem domain is denoted by $\Omega := \mathbb{R}^3 \setminus \bar{\mathbb{D}}$.
We also consider the sphere $\mathbb{S}$ of radius 1 and center at the origin in R^3.
The disc $\mathbb{D}$ divide this sphere into two half-sphere that we denote respectively $\mathbb{S}^+$ and $\mathbb{S}^-$.
For any $s > 0$, $\widetilde{H}^s(\mathbb{D})$ is the space of functions whose extension by zero to Γ belongs to $H^s(\Gamma)$. We identify

$$\widetilde{H}^{-1/2}(\mathbb{D}) \equiv \left(H^{1/2}(\mathbb{D})\right)' \quad \text{and} \quad H^{-1/2}(\mathbb{D}) \equiv \left(\widetilde{H}^{1/2}(\mathbb{D})\right)', \tag{25}$$

The unit sphere in $\mathbb{R}^3$

We consider the unit sphere $\mathbb{S}$ in $\mathbb{R}^3$ (Fig. 1) and the spherical coordinates: (r, θ, φ), where r is the radius and θ, φ the two Euler angles.

$$\begin{cases} x_1 = r \sin\theta \cos\varphi, \\ x_2 = r \sin\theta \sin\varphi, \\ x_3 = r \cos\theta. \end{cases} \tag{26}$$

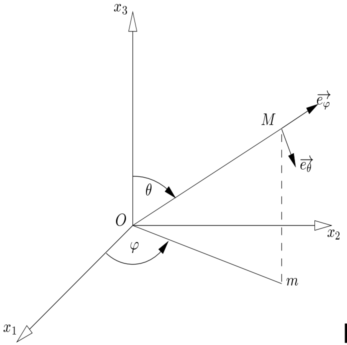

Fig. 1: Spherical coordinates

The vectors $\mathbf{e}_\theta$ and $\mathbf{e}_\varphi$ are unitary. The vector $\mathbf{e}_\rho$ directed along Om is unitary.

some geometry

–A point **x** on the circular domain $\mathbb{D}$ will be defined using its coordinates (x_1, x_2) or in circular coordinates by $(0 \leq \rho \leq 1, 0 \leq \varphi \leq 2\pi)$.

–A point $\mathbf{x}^+$ (resp. $\mathbf{x}^-$) on the half sphere $\mathbb{S}^+$ (resp. $\mathbb{S}^-$) will be defined using $(0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \varphi \leq 2\pi)$ (resp. $(\frac{\pi}{2} \leq \theta \leq \pi, 0 \leq \varphi \leq 2\pi)$).

–The projection **x** of a point $\mathbf{x}^+$ situed on the half sphere $\mathbb{S}^+$ onto the domain $\mathbb{D}$ has for circular coordinates **x** : $(\rho = \sin(\theta), \varphi)$.

– The projection **x** of a point $\mathbf{x}^-$ situed on the half sphere $\mathbb{S}^-$ onto the domain $\mathbb{D}$ has for circular coordinates **x** : $(\rho = \sin(\theta), \varphi)$.

–To a point **x**, we associate the points $\mathbf{x}^+$ and $\mathbf{x}^-$ which projections are **x**.

We introduce the **simple layer potential** on the sphere $\mathbb{S}$ defined, for $\mathbf{x} \in \mathbb{S}$ and $\mathbf{y} \in \mathbb{S}$, as

$$(\mathbb{S}I)u(\mathbf{y}) := \frac{1}{4\pi} \int_{\mathbb{S}} \frac{1}{|\mathbf{x}-\mathbf{y}|} u(\mathbf{x}) \sin(\theta) d\theta d\varphi \tag{27}$$

We consider also **the hyper singular potential** on the sphere $\mathbb{S}$ given by

$$(\mathbb{N})u(\mathbf{y}) := \frac{1}{4\pi} \oint_{\mathbb{S}} \frac{\partial^2}{\partial n_{\mathbf{x}} \partial n_{\mathbf{y}}} \Big(\frac{1}{|\mathbf{x}-\mathbf{y}|} \Big) u(\mathbf{x}) \sin(\theta) d\theta d\varphi \tag{28}$$

In the case of the sphere $\mathbb{S}$ **the double layer potential** is in fact equal to $-\frac{1}{2}\mathbb{S}$

$$(\mathbb{D}I)u = \frac{1}{4\pi} \int_{\mathbb{S}} \frac{\partial}{\partial n_x} \frac{1}{|\mathbf{x}-\mathbf{y}|} u(\mathbf{x}) d\gamma(\mathbf{x}), \tag{29}$$

The kernel of the hyper singular potential has a symmetric expression which is

$$\frac{\partial^2}{\partial n_{\mathbf{x}} \partial n_{\mathbf{y}}} \Big(\frac{1}{|\mathbf{x}-\mathbf{y}|} \Big) = \frac{(n_{\mathbf{x}} . n_{\mathbf{y}})}{|\mathbf{x}-\mathbf{y}|^3} + \frac{3}{4|\mathbf{x}-\mathbf{y}|} = \frac{1}{|\mathbf{x}-\mathbf{y}|^3} + \frac{1}{4|\mathbf{x}-\mathbf{y}|} \tag{30}$$

So in that situation, the **Calderon relations** for **the operators** $(\mathbb{S}, \mathbb{N})$, are

$$\mathbb{N} \circ \mathbb{S}I = \mathbb{S}I \circ \mathbb{N}, \tag{31}$$

$$-(\mathbb{N} - \frac{1}{4}\mathbb{S}I) \circ \mathbb{S}I = -\mathbb{S}I \circ (\mathbb{N} - \frac{1}{4}\mathbb{S}I) = \frac{1}{4}\mathbb{I}, \tag{32}$$

– To a function $u(\mathbf{x})$ defined on the sphere $\mathbb{S}$, we associate its symmetric and its antisymmetric parts defined on $\mathbb{S}^+$ and $\mathbb{S}^-$ as

$$\begin{cases} u_s(\mathbf{x}^+) = u_s(\mathbf{x}^-) = \frac{1}{2}(u(\mathbf{x}^+) + u(\mathbf{x}^-)) \\ u_{as}(\mathbf{x}^+) = -u_{as}(\mathbf{x}^-) = \frac{1}{2}(u(\mathbf{x}^+) - u(\mathbf{x}^-)) \end{cases} \tag{33}$$

To a point $\mathbf{y}$ on the sphere $\mathbb{S}$, we associated the symmetric point $\mathbf{y}^s$ which is either $\mathbf{y}^-$ or $\mathbf{y}^+$ and we define the four following operators

$$\begin{cases}(\mathbb{S}_s)u(\mathbf{y})=(\mathbb{S}I)u_s(\mathbf{y})=\dfrac{1}{4\pi}\displaystyle\int_{\mathbb{S}^+}\Big(\frac{1}{|\mathbf{x}^+-\mathbf{y}^+|}+\frac{1}{|\mathbf{x}^+-\mathbf{y}^-|}\Big)u_s(\mathbf{x}^+)\sin(\theta)d\theta d\varphi\\(\mathbb{S}_{as})u(\mathbf{y})=(\mathbb{S}I)u_{as}(\mathbf{y})=\dfrac{1}{4\pi}\displaystyle\int_{\mathbb{S}^+}\Big(\frac{1}{|\mathbf{x}^+-\mathbf{y}^+|}-\frac{1}{|\mathbf{x}^+-\mathbf{y}^-|}\Big)u_{as}(\mathbf{x}^+)\sin(\theta)d\theta d\varphi\end{cases}\tag{34}$$

$$\begin{cases}(\mathbb{N}_s)u(\mathbf{y})=(\mathbb{N})u_s(\mathbf{y})=\dfrac{1}{4\pi}\displaystyle\oint_{\mathbb{S}^+}\Big(\frac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^+}}\Big(\frac{1}{|\mathbf{x}^+-\mathbf{y}^+|}\Big)\\\qquad+\dfrac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^-}}\Big(\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^-|}\Big)\Big)u_s(\mathbf{x}^+)\sin(\theta)d\theta d\varphi\\(\mathbb{N}_{as})u(\mathbf{y})=(\mathbb{N})u_{as}(\mathbf{y})=\dfrac{1}{4\pi}\displaystyle\oint_{\mathbb{S}^+}\Big(\frac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^+}}\Big(\frac{1}{|\mathbf{x}^+-\mathbf{y}^+|}\Big)\\\qquad-\dfrac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^-}}\Big(\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^-|}\Big)\Big)u_{as}(\mathbf{x}^+)\sin(\theta)d\theta d\varphi\end{cases}\tag{35}$$

$$(\mathbb{S}_s)u_{as}(\mathbf{y})=(\mathbb{N}_s)u_{as}(\mathbf{y})=0\tag{36}$$

$$(\mathbb{S}_{as})u_s(\mathbf{y})=(\mathbb{N}_{as})u_s(\mathbf{y})=0\tag{37}$$

We define the operator $\overrightarrow{\mathbf{curl}}_{\mathbb{S}}$ on the sphere $\mathbb{S}$ as

$$\overrightarrow{\mathbf{curl}}_{\mathbb{S}} u(\mathbf{x}) = \frac{\partial u}{\partial \theta}\mathbf{e}_{\varphi} - \frac{1}{\sin\theta}\frac{\partial u}{\partial \varphi}\mathbf{e}_{\theta} = \frac{\partial u}{\partial \theta}\mathbf{e}_{\varphi} - \frac{\cos\theta}{\sin\theta}\frac{\partial u}{\partial \varphi}\mathbf{e}_{\rho} + \frac{\partial u}{\partial \varphi}\mathbf{e}_{3} \tag{38}$$

The bilinear form associated to the surfacic Laplacian $\Delta_{\mathbb{S}}$ is

$$\langle -\Delta_{\mathbb{S}} u \, , \, \overline{v} \rangle_{\mathbb{S}} = \int_{\mathbb{S}} \left(\overrightarrow{\mathbf{curl}}_{\mathbb{S}} u(\mathbf{x}) \cdot \overrightarrow{\mathbf{curl}}_{\mathbb{S}} \overline{v}(\mathbf{x}) \right) \sin\theta \, d\theta d\varphi \tag{39}$$

The hermitian product of two vectors of this form is

$$\begin{cases} \left(\overrightarrow{\mathbf{curl}}_{\mathbb{S}} u(\mathbf{x}) \cdot \overrightarrow{\mathbf{curl}}_{\mathbb{S}} \overline{v}(\mathbf{y}) \right) = \dfrac{\partial u(\mathbf{x})}{\partial \varphi}\dfrac{\partial \overline{v}(\mathbf{y})}{\partial \varphi} \\ +\left(\dfrac{\partial u(\mathbf{x})}{\partial \theta}\dfrac{\partial \overline{v}(\mathbf{y})}{\partial \theta} + \dfrac{\cos\theta(\mathbf{x})}{\sin\theta(\mathbf{x})}\dfrac{\cos\theta(\mathbf{y})}{\sin\theta(\mathbf{y})}\dfrac{\partial u(\mathbf{x})}{\partial \varphi}\dfrac{\partial \overline{v}(\mathbf{y})}{\partial \varphi} \right) \cos(\varphi(\mathbf{y}) - \varphi(\mathbf{x})) \\ -\left(\dfrac{\partial u(\mathbf{x})}{\partial \theta}\dfrac{\cos\theta(\mathbf{y})}{\sin\theta(\mathbf{y})}\dfrac{\partial \overline{v}(\mathbf{y})}{\partial \varphi} - \dfrac{\partial \overline{v}(\mathbf{y})}{\partial \theta}\dfrac{\cos\theta(\mathbf{x})}{\sin\theta(\mathbf{x})}\dfrac{\partial u(\mathbf{x})}{\partial \varphi} \right) \sin(\varphi(\mathbf{y}) - \varphi(\mathbf{x})) \end{cases} \tag{40}$$

The operator $\mathbb{N}$ defined by (28) is an isomorphism from $H^{1/2}(\mathbb{S})/\mathbb{R}$ onto the space $H^{-1/2}(\mathbb{S})$ with $\langle u_n, 1\rangle = 0$. It admits two variational formulations

$$\begin{cases} \langle -\mathbb{N}u\,,\,\overline{v}\rangle_{\mathbb{S}} = -\dfrac{1}{8\pi}\displaystyle\int_{\mathbb{S}}\int_{\mathbb{S}}\frac{\partial^2}{\partial n_{\mathbf{x}}\partial n_{\mathbf{y}}}\Big(\frac{1}{|\mathbf{x}-\mathbf{y}|}\Big)(u(\mathbf{x})-u(\mathbf{y}))(\overline{v}(\mathbf{y})-\overline{v}(\mathbf{x}))d\gamma(\mathbf{x})d\gamma(\mathbf{y}) \\ = \dfrac{1}{4\pi}\displaystyle\int_{\mathbb{S}}\int_{\mathbb{S}}\frac{\Big(\overrightarrow{\mathbf{curl}}_{\mathbb{S}}u(\mathbf{x})\cdot\overrightarrow{\mathbf{curl}}_{\mathbb{S}}\overline{v}(\mathbf{y})\Big)}{|\mathbf{x}-\mathbf{y}|}d\gamma(\mathbf{x})d\gamma(\mathbf{y});\ \forall\, u,v\in H^{1/2}(\mathbb{S})/\mathbb{R}. \end{cases} \tag{41}$$

Using (41), the two operators N_s and N_{as} admits the variational formulation

$$\begin{cases} \langle -\mathbb{N}_s u_s(\mathbf{x})\,,\,\overline{v}_s(\mathbf{y})\rangle_{\mathbb{S}} = \langle -\mathbb{N}u_s(\mathbf{x})\,,\,\overline{v}_s(\mathbf{y})\rangle_{\mathbb{S}} \\ = \dfrac{1}{8\pi}\displaystyle\int_{\mathbb{S}^+}\int_{\mathbb{S}}\Big(\frac{\Big(\overrightarrow{\mathbf{curl}}_{\mathbb{S}}u_s(\mathbf{x})\cdot\overrightarrow{\mathbf{curl}}_{\mathbb{S}}\overline{v}_s(\mathbf{y})\Big)}{|\mathbf{x}-\mathbf{y}|}\Big)d\gamma(\mathbf{x})d\gamma(\mathbf{y}) \end{cases} \tag{42}$$

$$\begin{cases} \langle -\mathbb{N}_{as} u_{as}(\mathbf{x})\,,\,\overline{v}_{as}(\mathbf{y})\rangle_{\mathbb{S}} = \langle -\mathbb{N}u_{as}(\mathbf{x})\,,\,\overline{v}_{as}(\mathbf{y})\rangle_{\mathbb{S}} \\ = \dfrac{1}{8\pi}\displaystyle\int_{\mathbb{S}^+}\int_{\mathbb{S}}\Big(\frac{\Big(\overrightarrow{\mathbf{curl}}_{\mathbb{S}}u_{as}(\mathbf{x})\cdot\overrightarrow{\mathbf{curl}}_{\mathbb{S}}\overline{v}_{as}(\mathbf{y})\Big)}{|\mathbf{x}-\mathbf{y}|}\Big)d\gamma(\mathbf{x})d\gamma(\mathbf{y}) \end{cases} \tag{43}$$

The identity (41) is related to the following identities ($\Delta_{\mathbb{S}}$ is the Laplace Beltrami operator on the sphere $\mathbb{S}$):

$$L_3 u = \frac{1}{i}\frac{\partial}{\partial\varphi}u. \tag{44}$$

$$L_+ u = e^{i\varphi}\left(\frac{\partial}{\partial\theta}u + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}u\right). \tag{45}$$

$$L_- u = e^{-i\varphi}\left(-\frac{\partial}{\partial\theta}u + i\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\varphi}u\right). \tag{46}$$

The operators L_+ and L_- are associated to the representation of a two dimension vector in $\mathbb{R}^2$ by a complex number.

$$-\Delta_{\mathbb{S}} = \frac{1}{2}\left(L_- L_+ + L_+ L_-\right) + L_3 L_3 \tag{47}$$

$$\mathbb{S}I^{-1} \circ \mathbb{N} = \Delta_{\mathbb{S}}, \tag{48}$$

$$\mathbb{S}I = \frac{1}{2}\left(-\Delta_{\mathbb{S}} + \frac{I}{4}\right)^{-1/2}, \tag{49}$$

$$\mathbb{N} - \frac{1}{4}\mathbb{S}I = -\frac{1}{2}\left(-\Delta_{\mathbb{S}} + \frac{I}{4}\right)^{1/2}. \tag{50}$$

Operators on the disc

We associate to the functions $U_s(\mathbf{x}^+)$ and $U_{as}(\mathbf{x}^+))$, both defined on the sphere $\mathbb{S}^+$ (variables: θ, φ), the functions $u_s(\mathbf{x})$ and $u_{as}(\mathbf{x})$ defined on the disc $\mathbb{D}$ (variables: $\rho = \sin(\theta)$, φ, $0 \leq \theta \leq \frac{\pi}{2}$), where $\mathbf{x}$ is the projection on the disc of the vector $\mathbf{x}^+$. Let $w(\mathbf{x}) = \sqrt{1-\rho(\mathbf{x})^2}$

We define the operators $\mathcal{L}_+$, $\mathcal{L}_-$, $\mathcal{L}_3$ of derivation on the disc , analogue to the operators L_+, L_-, L_3.

In this representation, the vector $\mathbf{e}_1$ is the number 1 and the vector $\mathbf{e}_2$ is the number i. Thus $\mathbf{e}_\rho$ is $\mathbf{e}_\rho := e^{i\varphi}$ and $\mathbf{e}_\varphi$ is $\mathbf{e}_\varphi := ie^{i\varphi}$. We have

$$\begin{cases} \mathcal{L}_+ u = e^{i\varphi}\Big(\dfrac{\partial u}{\partial \rho} + i\dfrac{1}{\rho}\dfrac{\partial u}{\partial \varphi}\Big) \\ \mathcal{L}_- u = e^{-i\varphi}\Big(-\dfrac{\partial u}{\partial \rho} + i\dfrac{1}{\rho}\dfrac{\partial u}{\partial \varphi}\Big) \\ \mathcal{L}_3 u = \dfrac{1}{i}\dfrac{\partial u}{\partial \varphi} \end{cases} \tag{51}$$

They trivially satisfy

$$\overline{\mathcal{L}_+ u} = -\mathcal{L}_- \overline{u}; \qquad \overline{\mathcal{L}_- u} = -\mathcal{L}_+ \overline{u}; \qquad \overline{\mathcal{L}_3 u} = -\mathcal{L}_3 \overline{u} \tag{52}$$

An integration by part give the following result

$$\int_{\mathbb{D}} e^{i\varphi}\Big(\frac{\partial u}{\partial \rho} + i\frac{1}{\rho}\frac{\partial u}{\partial \varphi}\Big) v\, \rho d\rho d\varphi = -\int_{\mathbb{D}} e^{i\varphi}\Big(\frac{\partial v}{\partial \rho} + i\frac{1}{\rho}\frac{\partial v}{\partial \varphi}\Big) u\, \rho d\rho d\varphi \tag{53}$$

which means that the operators $\mathcal{L}_+$, $\mathcal{L}_-$ and $\mathcal{L}_3$ are formally anti-adjoint with respect to the duality in $L^2(\mathbb{D})$.

$$\overrightarrow{\mathbf{grad}}_{\mathbb{D}} u(\mathbf{x}) := e^{i\varphi}\Big(\frac{\partial u}{\partial \rho} + i\frac{1}{\rho}\frac{\partial u}{\partial \varphi}\Big) = \mathcal{L}_+ u \tag{54}$$

$$\overrightarrow{\mathbf{curl}}_{\mathbb{D}} u(\mathbf{x}) := ie^{i\varphi}\Big(\frac{\partial u}{\partial \rho} + i\frac{1}{\rho}\frac{\partial u}{\partial \varphi}\Big) = i\,\mathcal{L}_+ u \tag{55}$$

$$\left\{\begin{array}{l} \Big(\overrightarrow{\mathbf{curl}}_{\mathbb{D}} u(\mathbf{x}).\overrightarrow{\mathbf{curl}}_{\mathbb{D}} v(\mathbf{y})\Big) = \Big(\overrightarrow{\mathbf{grad}}_{\mathbb{D}} u(\mathbf{x}).\overrightarrow{\mathbf{grad}}_{\mathbb{D}} v(\mathbf{y})\Big) \\ = -\frac{1}{2}\Big(\mathcal{L}_+ u(\mathbf{x})\,\mathcal{L}_- v(\mathbf{y}) + \mathcal{L}_- u(\mathbf{x})\,\mathcal{L}_+ v(\mathbf{y})\Big) \end{array}\right. \tag{56}$$

We introduce the four following integral operators defined for $\mathbf{x} \in \mathbb{D}$ and $\mathbf{y} \in \mathbb{D}$:

$$(\mathcal{S}_s)u_s(\mathbf{y}) := \frac{1}{4\pi}\int_{\mathbb{D}}\Big(\frac{1}{|\mathbf{x}^+ - \mathbf{y}^+|} + \frac{1}{|\mathbf{x}^+ - \mathbf{y}^-|}\Big)u_s(\mathbf{x})\,\rho d\rho d\varphi \qquad (57)$$

$$(\mathcal{S}_{as})u_{as}(\mathbf{y}) := \frac{1}{4\pi}\int_{\mathbb{D}}\Big(\frac{1}{|\mathbf{x}^+ - \mathbf{y}^+|} - \frac{1}{|\mathbf{x}^+ - \mathbf{y}^-|}\Big)u_{as}(\mathbf{x})\,\rho d\rho d\varphi \qquad (58)$$

$$\begin{cases} (\mathcal{N}_s)u_s(\mathbf{y}) := \frac{1}{4\pi}\oint_{\mathbb{D}}\Big(\frac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^+}}\Big(\frac{1}{|\mathbf{x}^+ - \mathbf{y}^+|}\Big) \\ \qquad + \frac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^-}}\Big(\frac{1}{|\mathbf{x}^+ - \mathbf{y}^-|}\Big)\Big)u_s(\mathbf{x})\rho d\rho d\varphi \end{cases} \qquad (59)$$

$$\begin{cases} (\mathcal{N}_{as})u_{as}(\mathbf{y}) := \frac{1}{4\pi}\oint_{\mathbb{D}}\Big(\frac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^+}}\Big(\frac{1}{|\mathbf{x}^+ - \mathbf{y}^+|}\Big) \\ \qquad - \frac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^-}}\Big(\frac{1}{|\mathbf{x}^+ - \mathbf{y}^-|}\Big)u_{as}(\mathbf{x})\rho d\rho d\varphi \end{cases} \qquad (60)$$

We will denoted by K_s, K_{as}, NK_s and NK_{as} the associated kernels

$$\begin{cases} K_s = \dfrac{1}{4\pi}\Big(\dfrac{1}{|\mathbf{x}^+ - \mathbf{y}^+|} + \dfrac{1}{|\mathbf{x}^+ - \mathbf{y}^-|}\Big) \\ K_{as} = \dfrac{1}{4\pi}\Big(\dfrac{1}{|\mathbf{x}^+ - \mathbf{y}^+|} - \dfrac{1}{|\mathbf{x}^+ - \mathbf{y}^-|}\Big) \\ NK_s = \dfrac{1}{4\pi}\Big(\dfrac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^+}}\Big(\dfrac{1}{|\mathbf{x}^+ - \mathbf{y}^+|}\Big) + \dfrac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^-}}\Big(\dfrac{1}{|\mathbf{x}^+ - \mathbf{y}^-|}\Big)\Big) \\ NK_{as} = \dfrac{1}{4\pi}\Big(\dfrac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^+}}\Big(\dfrac{1}{|\mathbf{x}^+ - \mathbf{y}^+|}\Big) - \dfrac{\partial^2}{\partial n_{\mathbf{x}^+}\partial n_{\mathbf{y}^-}}\Big(\dfrac{1}{|\mathbf{x}^+ - \mathbf{y}^-|}\Big)\Big) \end{cases} \tag{61}$$

These kernels are not the kernels of the Laplace operator on the disk. But they are closely related to them.

Consider the following function

$$E^2 = 1 - \rho(\mathbf{x})\rho(\mathbf{y})\cos^2\left(\frac{\varphi(\mathbf{x})-\varphi(\mathbf{y})}{2}\right) \quad (62)$$

We have the inequalities

$$\begin{cases} \dfrac{E^2}{4\pi\,|\mathbf{x}-\mathbf{y}|} \leq \dfrac{1}{4\pi}\left(\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^+|}+\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^-|}\right) \leq \dfrac{2}{4\pi\,|\mathbf{x}-\mathbf{y}|} \\ \dfrac{1}{2}\dfrac{w(\mathbf{x})w(\mathbf{y})}{4\pi\,|\mathbf{x}-\mathbf{y}|} \leq \dfrac{1}{4\pi}\left(\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^+|}-\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^-|}\right) \leq \dfrac{1}{E^3}\dfrac{w(\mathbf{x})w(\mathbf{y})}{4\pi\,|\mathbf{x}-\mathbf{y}|} \end{cases} \quad (63)$$

The function E^2 is regular with a positive value between zero and one. It is zero on the circle $\mathfrak{c}$, when $\rho(\mathbf{x}) = \rho(\mathbf{y}) = 1$ and $\varphi(\mathbf{x})-\varphi(\mathbf{y}) = 0$.
Its order closed to the zero ($1 - \rho(\mathbf{x})^2 \leq |\mathbf{x}-\mathbf{y}|$), expressed in term of $|\mathbf{x}-\mathbf{y}|$ is one or two, depending on the direction of the vector $\mathbf{x}-\mathbf{y}$.

$$
\begin{cases}
\dfrac{E}{4\pi\,|\mathbf{x}-\mathbf{y}|^3}\Big(\dfrac{3}{4}+\dfrac{w(\mathbf{x})^2 w(\mathbf{y})^2}{4}\Big) \\
\qquad \leq \dfrac{1}{4\pi}\Big(\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^+|^3}+\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^-|^3}\Big)+\dfrac{1}{16\pi}\Big(\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^+|}+\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^-|}\Big) \\
\qquad\qquad \leq \dfrac{1}{4\pi\,|\mathbf{x}-\mathbf{y}|^3}\Big(\dfrac{1}{2}+\dfrac{1}{E}+\dfrac{2w(\mathbf{x})^2 w(\mathbf{y})^2}{E^6}\Big) \\
\dfrac{1}{2}\dfrac{w(\mathbf{x})w(\mathbf{y})}{4\pi\,|\mathbf{x}-\mathbf{y}|^3}\Big(\dfrac{3}{4}+\dfrac{w(\mathbf{x})^2 w(\mathbf{y})^2}{4}\Big) \\
\qquad \leq \dfrac{1}{4\pi}\Big(\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^+|^3}-\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^-|^3}\Big)+\dfrac{1}{16\pi}\Big(\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^+|}-\dfrac{1}{|\mathbf{x}^+-\mathbf{y}^-|}\Big) \\
\qquad\qquad \leq \dfrac{1}{E^3}\dfrac{w(\mathbf{x})w(\mathbf{y})}{4\pi\,|\mathbf{x}-\mathbf{y}|\,3}\Big(\dfrac{1}{4}+\dfrac{1}{2E}+\dfrac{w(\mathbf{x})^2 w(\mathbf{y})^2}{E^6}\Big)
\end{cases}
\tag{64}
$$

Let us give some links between these operators.
The identities (34) can be rewritted as

$$
\begin{cases}
(\mathbb{S}_s)u_s(\mathbf{y}^+) = \left(\mathcal{S}_s\left(\dfrac{u_s}{\sqrt{(1-\rho^2)}}\right)\right)(\mathbf{y}) \\[2ex]
= \dfrac{1}{4\pi}\displaystyle\int_{\mathbb{D}}\left(\dfrac{1}{|\mathbf{x}^+ - \mathbf{y}^+|} + \dfrac{1}{|\mathbf{x}^+ - \mathbf{y}^-|}\right)\dfrac{u_s(\mathbf{x})}{\sqrt{(1-\rho^2)}}\rho d\rho d\varphi \\[2ex]
(\mathbb{S}_{as})u_{as}(\mathbf{y}^+) = \left(\mathcal{S}_{as}\left(\dfrac{u_{as}}{\sqrt{(1-\rho^2)}}\right)\right)(\mathbf{y}) \\[2ex]
= \dfrac{1}{4\pi}\displaystyle\int_{\mathbb{D}}\left(\dfrac{1}{|\mathbf{x}^+ - \mathbf{y}^+|} - \dfrac{1}{|\mathbf{x}^+ - \mathbf{y}^-|}\right)\dfrac{u_{as}(\mathbf{x})}{\sqrt{(1-\rho^2)}}\rho d\rho d\varphi
\end{cases}
\tag{65}
$$

Using the identities (42) and (43) , we obtain the variational expressions

$$\begin{cases} \langle -\mathcal{N}_s\, u_s(\mathbf{x})\,,\, \overline{v}_s(\mathbf{y})\rangle_{\mathbb{D}} = \left\langle \mathcal{S}_{as}\, \overrightarrow{\mathbf{curl}}_{\mathbb{D}}\, u_s(\mathbf{x})\,,\, \overrightarrow{\mathbf{curl}}_{\mathbb{D}}\, \overline{v}_s(\mathbf{y}) \right\rangle_{\mathbb{D}} \\ \qquad + \left\langle \mathcal{S}_s\, \dfrac{\dfrac{\partial u_s(\mathbf{x})}{\partial \varphi}}{\sqrt{(1-\rho(\mathbf{x})^2)}}\,,\, \dfrac{\dfrac{\partial \overline{v}_s(\mathbf{y})}{\partial \varphi}}{\sqrt{(1-\rho(\mathbf{y})^2)}} \right\rangle_{\mathbb{D}} \end{cases} \tag{66}$$

$$\begin{cases} \langle -\mathcal{N}_s\, u_s(\mathbf{x})\,,\, \overline{v}_s(\mathbf{y})\rangle_{\mathbb{D}} = -\left\langle \mathcal{S}_s\, \dfrac{\mathcal{L}_3\, u_s(\mathbf{x})}{\sqrt{(1-\rho(\mathbf{x})^2)}}\,,\, \dfrac{\mathcal{L}_3\, \overline{v}_s(\mathbf{y})}{\sqrt{(1-\rho(\mathbf{y})^2)}} \right\rangle_{\mathbb{D}} \\ -\dfrac{1}{2}\, \langle \mathcal{S}_{as}\, \mathcal{L}_+ u_s(\mathbf{x})\,,\, \mathcal{L}_- \overline{v}_s(\mathbf{y})\rangle_{\mathbb{D}} - \dfrac{1}{2}\, \langle \mathcal{S}_{as}\, \mathcal{L}_- u_s(\mathbf{x})\,,\, \mathcal{L}_+ \overline{v}_s(\mathbf{y})\rangle_{\mathbb{D}} \end{cases} \tag{67}$$

$$\begin{cases}\langle -\mathcal{N}_{as}\, u_{as}(\mathbf{x})\,,\, \overline{v}_{as}(\mathbf{y})\rangle_{\mathbb{D}} = \left\langle \mathcal{S}_s\, \overrightarrow{\mathbf{curl}}_{\mathbb{D}}\, u_{as}(\mathbf{x})\,,\, \overrightarrow{\mathbf{curl}}_{\mathbb{D}}\, \overline{v}_{as}(\mathbf{y})\right\rangle_{\mathbb{D}} \\[2ex] \qquad + \left\langle \mathcal{S}_{as}\, \dfrac{\dfrac{\partial u_{as}(\mathbf{x})}{\partial\varphi}}{\sqrt{(1-\rho(\mathbf{x})^2)}}\,,\, \dfrac{\dfrac{\partial \overline{v}_{as}(\mathbf{y})}{\partial\varphi}}{\sqrt{(1-\rho(\mathbf{y})^2)}}\right\rangle_{\mathbb{D}}\end{cases} \tag{68}$$

$$\begin{cases}\langle -\mathcal{N}_{as}\, u_{as}(\mathbf{x})\,,\, \overline{v}_{as}(\mathbf{y})\rangle_{\mathbb{D}} = -\left\langle \mathcal{S}_{as}\, \dfrac{\mathcal{L}_3\, u_{as}(\mathbf{x})}{\sqrt{(1-\rho(\mathbf{x})^2)}}\,,\, \dfrac{\mathcal{L}_3\, \overline{v}_{as}(\mathbf{y})}{\sqrt{(1-\rho(\mathbf{y})^2)}}\right\rangle_{\mathbb{D}} \\[2ex] -\dfrac{1}{2}\langle \mathcal{S}_s\, \mathcal{L}_+ u_{as}(\mathbf{x})\,,\, \mathcal{L}_- \overline{v}_{as}(\mathbf{y})\rangle_{\mathbb{D}} - \dfrac{1}{2}\langle \mathcal{S}_s\, \mathcal{L}_- u_{as}(\mathbf{x})\,,\, \mathcal{L}_+ \overline{v}_{as}(\mathbf{y})\rangle_{\mathbb{D}}\end{cases} \tag{69}$$

Using the anti-duality, we can rewrite these identities as

$$\begin{cases} -\mathcal{N}_{as} = \dfrac{1}{2}\Big(\mathcal{L}_- \; \mathcal{S}_s \, \mathcal{L}_+ + \mathcal{L}_+ \; \mathcal{S}_s \, \mathcal{L}_-\Big) + \dfrac{1}{\sqrt{(1-\rho^2)}} \, \mathcal{L}_3 \, \mathcal{S}_{as} \, \dfrac{1}{\sqrt{(1-\rho^2)}} \, \mathcal{L}_3 \\ -\mathcal{N}_s = \dfrac{1}{2}\Big(\mathcal{L}_- \; \mathcal{S}_{as} \, \mathcal{L}_+ + \mathcal{L}_+ \; \mathcal{S}_{as} \, \mathcal{L}_-\Big) + \dfrac{1}{\sqrt{(1-\rho^2)}} \, \mathcal{L}_3 \, \mathcal{S}_s \, \dfrac{1}{\sqrt{(1-\rho^2)}} \, \mathcal{L}_3 \end{cases} \tag{70}$$

Images of the Spherical Harmonics

The parity of the Spherical Harmonics Y_l^m with respect to the variable $x = \cos(\theta)$ is the parity of $l+m$. Thus the vectorial space $\mathbb{Y}$ generated by the Spherical Harmonics $Y_l^m; 0 \leq l; -l \leq m \leq l$, can be split into two **subspaces** $\mathbb{Y}_s$ **and** $\mathbb{Y}_{as}$ defined on $\mathbb{S}^+$ which are respectively :

$$\mathbb{Y}_s = \{Y_l^m; 0 \leq l; \quad -l \leq m \leq l; \quad l+m \quad \text{even}\}$$
$$\mathbb{Y}_{as} = \{Y_l^m; 1 \leq l; \quad -l \leq m \leq l; \quad l+m \quad \text{odd}\}$$

The orthogonality identities between the Spherical Harmonics functions $Y_{l_1}^{m_1}$ can be rewrite as

$$\int_{\mathbb{S}^+} \left(Y_{l_1}^{m_1}(\mathbf{x}) \overline{Y}_{l_1}^{m_1}(\mathbf{x}) \right) \sin(\theta) d\theta d\varphi = \frac{1}{2} \delta_{l_1}^{l_2} \delta_{m_1}^{m_2}, \tag{71}$$

$$\int_{\mathbb{S}^+} \left(\overrightarrow{\mathbf{curl}}_{\mathbb{S}} Y_{l_1}^{m_1}(\mathbf{x}) . \overrightarrow{\mathbf{curl}}_{\mathbb{S}} \overline{Y}_{l_1}^{m_1}(\mathbf{x}) \right) \sin(\theta) d\theta d\varphi = \frac{1}{2} l(l+1) \delta_{l_1}^{l_2} \delta_{m_1}^{m_2}, \tag{72}$$

$$\mathbb{S}l \; Y_l^m = \frac{1}{2l+1} Y_l^m, \tag{73}$$

$$\mathbb{N} \; Y_l^m = -\frac{l(l+1)}{2l+1} Y_l^m, \tag{74}$$

We introduce now the functions y_l^m defined on the disc $\mathbb{D}$, images of the Spherical Harmonics, which are

$$y_l^m(x,\varphi) = \gamma_l^m e^{im\varphi}\mathbb{P}_l^m(\sqrt{(1-\rho^2)}) \tag{75}$$

$$\begin{cases} y_0^0(x,\varphi) = \sqrt{\dfrac{1}{4\pi}}; \qquad y_1^1(x,\varphi) = -\sqrt{\dfrac{3}{8\pi}}e^{i\varphi}\rho \\ y_1^0(x,\varphi) = \sqrt{\dfrac{3}{4\pi}}\sqrt{(1-\rho^2)} \end{cases} \tag{76}$$

We associated to the two subspaces $\mathbb{Y}_s$ and $\mathbb{Y}_{as}$ defined on $\mathbb{S}^+$, the corresponding subspaces on the disc $\mathbb{D}$:

$$\mathcal{Y}_s = \{y_l^m(\mathbf{x}) = Y_l^m(\mathbf{x}^+); \quad Y_l^m \in \mathbb{Y}_s\}$$

$$\mathcal{Y}_{as} = \{y_l^m(\mathbf{x}) = Y_l^m(\mathbf{x}^+); \quad Y_l^m \in \mathbb{Y}_{as}\}$$

Using (71) and the identities in theorem (1), we obtain the identities

$$\mathcal{S}_s\Big(\frac{y_l^m}{\sqrt{(1-\rho^2)}}\Big)=\frac{1}{2l+1}y_l^m,\quad y_l^m\in\mathbb{Y}_s \tag{77}$$

$$\mathcal{S}_{as}\Big(\frac{y_l^m}{\sqrt{(1-\rho^2)}}\Big)=\frac{1}{2l+1}y_l^m,\quad y_l^m\in\mathbb{Y}_{as} \tag{78}$$

$$\int_{\mathbb{D}}\frac{y_{l_1}^{m_1}(\mathbf{x})\overline{y}_{l_2}^{m_2}(\mathbf{x})}{\sqrt{(1-\rho^2)}}\,\rho d\rho d\varphi=\frac{1}{2}\delta_{l_1}^{l_2}\delta_{m_1}^{m_2}, \tag{79}$$

$$\begin{cases}\mathcal{L}_+\,y_l^m=\sqrt{(l-m)(l+m+1)}\dfrac{y_l^{m+1}}{\sqrt{(1-\rho^2)}};\\[2ex] \mathcal{L}_-\,y_l^m=\sqrt{(l+m)(l-m+1)}\dfrac{y_l^{m-1}}{\sqrt{(1-\rho^2)}};\\[2ex] \mathcal{L}_3\,y_l^m=my_l^m\end{cases} \tag{80}$$

Using the identities (80) (77) and (78) , we remark that the functions $y_l^m \in \mathcal{Y}_s$ satisfy the identities

$$\begin{cases} \sqrt{(1-\rho^2)}\,\mathcal{L}_+\,\dfrac{\mathcal{S}_s}{\sqrt{(1-\rho^2)}} = \mathcal{S}_{as}\,\mathcal{L}_+; \quad \sqrt{(1-\rho^2)}\,\mathcal{L}_+\,\dfrac{\mathcal{S}_{as}}{\sqrt{(1-\rho^2)}} = \mathcal{S}_s\,\mathcal{L}_+ \\ \sqrt{(1-\rho^2)}\,\mathcal{L}_-\,\dfrac{\mathcal{S}_s}{\sqrt{(1-\rho^2)}} = \mathcal{S}_{as}\,\mathcal{L}_-; \quad \sqrt{(1-\rho^2)}\,\mathcal{L}_-\,\dfrac{\mathcal{S}_{as}}{\sqrt{(1-\rho^2)}} = \mathcal{S}_s\,\mathcal{L}_- \\ \mathcal{L}_3\,\dfrac{\mathcal{S}_s}{\sqrt{(1-\rho^2)}} = \dfrac{\mathcal{S}_s}{\sqrt{(1-\rho^2)}}\,\mathcal{L}_3; \quad \mathcal{L}_3\,\dfrac{\mathcal{S}_{as}}{\sqrt{(1-\rho^2)}} = \dfrac{\mathcal{S}_{as}}{\sqrt{(1-\rho^2)}}\,\mathcal{L}_3 \end{cases} \tag{81}$$

Using the identities (80) and (70), we obtain

$$\mathcal{N}_s\left(y_l^m\right) = -\frac{l(l+1)}{2l+1}\,\frac{y_l^m}{\sqrt{(1-\rho^2)}}; \quad y_l^m \in \mathbb{Y}_s \tag{82}$$

$$\mathcal{N}_{as}\left(y_l^m\right) = -\frac{l(l+1)}{2l+1}\,\frac{y_l^m}{\sqrt{(1-\rho^2)}}; \quad y_l^m \in \mathbb{Y}_{as} \tag{83}$$

The bilinear form of the Laplace operator $\Delta_{\mathbb{D}}^{D}$ (reps. $\Delta_{\mathbb{D}}^{N}$) for the Dirichet condition (resp.Neumann condition), defined in $H^1(\mathbb{D})$ is associated to the squared norm of **curl** u_{as} (resp. **curl** u_s). Using also the duality, we can rewrite these identities as

$$\begin{cases} -\Delta_{\mathbb{D}}^{N} u_s = \dfrac{1}{2}\Big(\mathcal{L}_- \ \mathcal{L}_+ + \mathcal{L}_+ \ \mathcal{L}_-\Big) u_s \\ -\Delta_{\mathbb{D}}^{D} u_{as} = \dfrac{1}{2}\Big(\mathcal{L}_- \ \mathcal{L}_+ + \mathcal{L}_+ \ \mathcal{L}_-\Big) u_{as} \end{cases} \tag{84}$$

Using the identities (80), we obtain also

$$\begin{cases} -\Delta_{\mathbb{D}}^{N} y_l^m = \Big(\dfrac{l(l+1) - m^2}{(1-\rho^2)}\Big) y_l^m; \quad l+m \quad \text{even} \\ -\Delta_{\mathbb{D}}^{D} y_l^m = \Big(\dfrac{l(l+1) - m^2}{(1-\rho^2)}\Big) y_l^m; \quad l+m \quad \text{odd} \end{cases} \tag{85}$$

Theorem

The operators $\mathcal{L}_3$, $\mathcal{S}_{as}$, $\mathcal{S}_s$, $\Delta_{\mathbb{D}}^N$ and $\Delta_{\mathbb{D}}^D$ are linked by the identities

$$\begin{cases} \frac{1}{4}\mathbb{I} = -\Big(\Delta_{\mathbb{D}}^D + \frac{1}{(1-\rho^2)}\Big(\frac{\partial^2}{\partial\varphi^2} - \frac{1}{4}\Big)\Big)\ \mathcal{S}_s\mathcal{S}_s \\ \qquad = -\mathcal{S}_s\mathcal{S}_s\Big(\Delta_{\mathbb{D}}^D + \frac{1}{(1-\rho^2)}\Big(\frac{\partial^2}{\partial\varphi^2} - \frac{1}{4}\Big)\Big) \\ \frac{1}{4}\mathbb{I} = -\Big(\Delta_{\mathbb{D}}^N + \frac{1}{(1-\rho^2)}\Big(\frac{\partial^2}{\partial\varphi^2} - \frac{1}{4}\Big)\Big)\ \mathcal{S}_{as}\mathcal{S}_{as} \\ \qquad = -\mathcal{S}_{as}\mathcal{S}_{as}\Big(\Delta_{\mathbb{D}}^N + \frac{1}{(1-\rho^2)}\Big(\frac{\partial^2}{\partial\varphi^2} - \frac{1}{4}\Big)\Big) \end{cases} \tag{86}$$

$$\begin{cases} \mathcal{N}_s\mathcal{S}_s^{-1} = \mathcal{S}_s^{-1}\mathcal{N}_s = \Big(\Delta_{\mathbb{D}}^D + \frac{1}{(1-\rho^2)}\frac{\partial^2}{\partial\varphi^2}\Big) \\ \mathcal{S}_{as}^{-1}\mathcal{N}_{as} = \Big(\Delta_{\mathbb{D}}^N + \frac{1}{(1-\rho^2)}\frac{\partial^2}{\partial\varphi^2}\Big) \end{cases} \tag{87}$$

decomposition on basis functions

Associated to the weight $w(\mathbf{x}) = \sqrt{1-\rho(\mathbf{x})^2}$ we introduce the weighted space:

$L^2_{\frac{1}{w}}(\mathbb{D}) = \{u(\mathbf{x}), \dfrac{u^2}{w} \in L^1(\mathbb{D})\}$.

Then both sets $\{y_l^m \in \mathcal{Y}_s\}$ and $\{y_l^m \in \mathcal{Y}_{as}\}$ are an orthogonal basis in the space $L^2_{\frac{1}{w}}(\mathbb{D})$.

Due to the properties of the associated Legendre functions, the functions in the space $\mathcal{Y}_s$ have a bounded non zero value and a bounded normal derivative closed to the circle $\mathfrak{c}$.

The functions in the space $\mathcal{Y}_{as}$ have closed to the circle $\mathfrak{c}$ a value which goes to zero as $\sqrt{(1-\rho^2}$ and a normal derivative which explodes as $\dfrac{1}{\sqrt{(1-\rho^2}}$.

A function u_{as} in the space $\mathcal{Y}_{as}$ can be extended on the basis $\{y_l^m\}$ which is an orthogonal basis in the weighted space $L^2_{\frac{1}{w}}(\mathbb{D})$ and a basis in the space $H^1_0(\mathbb{D})$.

$$u_{as} = \sum_{1\leq l}\sum_{m} u_l^m y_l^m; \quad -l \leq m \leq l; \quad l+m \quad \text{odd} \tag{88}$$

A function u_s in the space $\mathcal{Y}_s$ can be extended on the basis $\{y_l^m\}$ which is an orthogonal basis in the weighted space $L^2_{\frac{1}{w}}(\mathbb{D})$ and a basis in the space $H^1(\mathbb{D})$.

$$u_s = \sum_{1\leq l}\sum_{m} u_l^m y_l^m; \quad -l \leq m \leq l; \quad l+m \quad \text{even} \tag{89}$$

We consider the associated weighted space: $L^2_w(\mathbb{D}) = \{u(\mathbf{x}), wu^2 \in L^1(\mathbb{D})\}$.

Then both sets $\{\frac{y_l^m}{w}\}$ for $\{y_l^m \in \mathcal{Y}_s\}$ and $\{\frac{y_l^m}{w}\}$ for $\{y_l^m \in \mathcal{Y}_{as}\}$ are an orthogonal basis in the space $L^2_w(\mathbb{D})$. A function u can be extended on the basis $\{\frac{y_l^m}{w}\}$ for $\{y_l^m \in \mathcal{Y}_s\}$ which is an orthogonal basis in the weighted space $L^2_w(\mathbb{D})$

$$u = \sum_{0\leq l}\sum_{m} u_l^m \frac{y_l^m}{w}; \quad -l \leq m \leq l; \quad l+m \quad \text{even} \tag{90}$$

A function u can be also extended on the basis $\{\frac{y_l^m}{w}\}$ for $\{y_l^m \in \mathcal{Y}_{as}\}$ which is an orthogonal basis in the weighted space $L^2_w(\mathbb{D})$ and a basis in $L^2(\mathbb{D})$.

$$u = \sum_{1\leq l}\sum_{m} u_l^m \frac{y_l^m}{w}; \quad -l \leq m \leq l; \quad l+m \quad \text{odd} \tag{91}$$

Theorem

For all $\mathbf{x}, \mathbf{y} \in D \times D$, *(* $\mathbf{x} \neq \mathbf{y}$*),* K_s, K_{as}, NK_s, NK_{as} *admits the expansions:*

$$K_s(\mathbf{x}, \mathbf{y}) = 2 \sum_{0 \leq l} \sum_{m} \frac{y_l^m(\mathbf{x}) y_l^m(\mathbf{y})}{2l+1}; \quad -l \leq m \leq l; \quad l+m \quad \textit{even}. \tag{92}$$

$$K_{as}(\mathbf{x}, \mathbf{y}) = 2 \sum_{0 \leq l} \sum_{m} \frac{y_l^m(\mathbf{x}) y_l^m(\mathbf{y})}{2l+1}; \quad -l \leq m \leq l; \quad l+m \quad \textit{odd}. \tag{93}$$

$$NK_s(\mathbf{x}, \mathbf{y}) = -2 \sum_{0 \leq l} \sum_{m} \frac{l(l+1)}{2l+1} \frac{y_l^m(\mathbf{x})}{w(\mathbf{x})} \frac{y_l^m(\mathbf{y})}{w(\mathbf{y})}; \ -l \leq m \leq l; \ l+m \ \textit{even}. \tag{94}$$

$$NK_{as}(\mathbf{x}, \mathbf{y}) = -2 \sum_{0 \leq l} \sum_{m} \frac{l(l+1)}{2l+1} \frac{y_l^m(\mathbf{x})}{w(\mathbf{x})} \frac{y_l^m(\mathbf{y})}{w(\mathbf{y})}; \ -l \leq m \leq l; \ l+m \ \textit{odd}. \tag{95}$$

$$\int_{\mathbb{D}} K_s(\mathbf{x}, \mathbf{y}) d\mathbb{D}(\mathbf{y}) = \frac{1}{2}; \qquad \int_{\mathbb{D}} K_{as}(\mathbf{x}, \mathbf{y}) d\mathbb{D}(\mathbf{y}) = \frac{1}{3} w(\mathbf{x}) \tag{96}$$

$$\oint_{\mathbb{D}} NK_s(\mathbf{x}, \mathbf{y}) d\mathbb{D}(\mathbf{y}) = 0; \quad \oint_{\mathbb{D}} NK_{as}(\mathbf{x}, \mathbf{y}) d\mathbb{D}(\mathbf{y}) = -\frac{1}{3w(\mathbf{x})^3} \tag{97}$$

HAPPY BIRTHDAY MARTIN

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