RADIATION CONDITION AND INSTABILITY PHENOMENON AT A CORNER INTERFACE BETWEEN A DIELECTRIC AND A NEGATIVE MATERIAL

Augmented Singular Days 2013
in honour of Martin Costabel's 65th birthday

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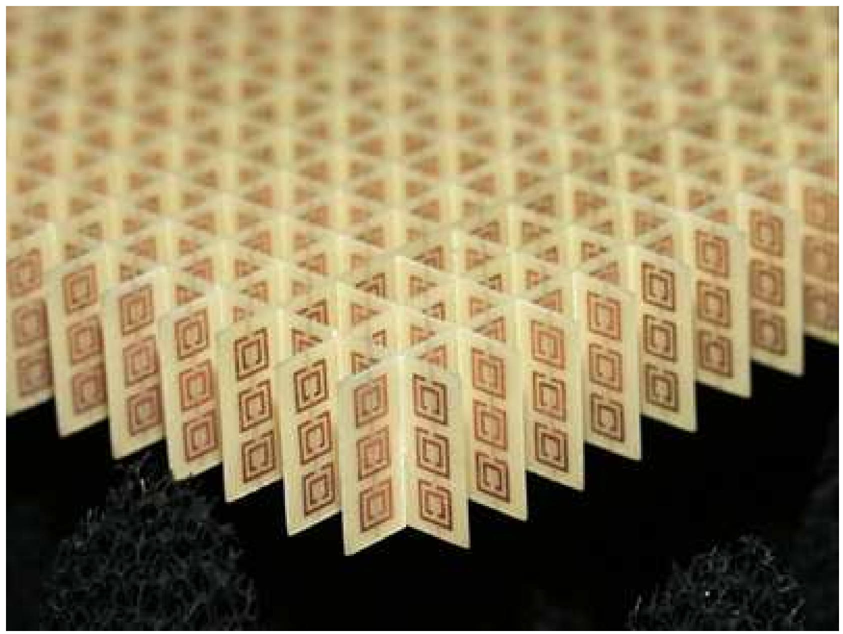
[*] Russian Academy of Sciences

Metamaterials

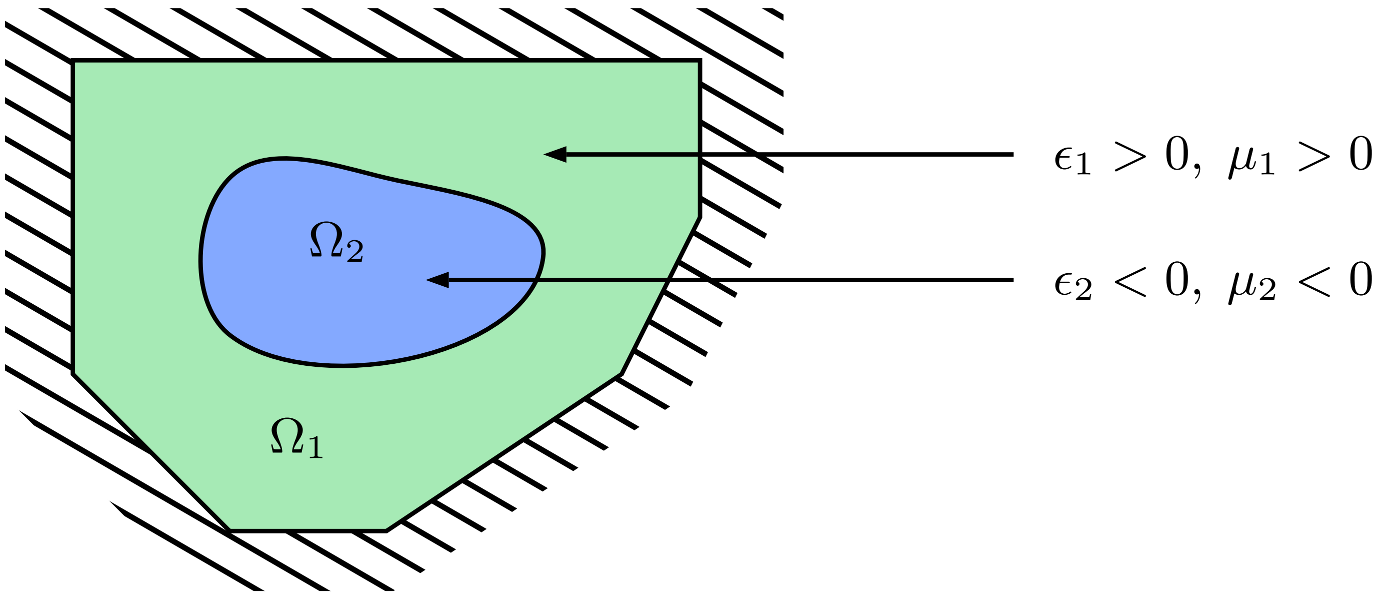
In the context of electromagnetic wave propagation, metamaterials are periodic assemblies of small resonators whose caracteristic size is much smaller than the average wavelength.



The periodicity cell (at the "micro scale") can be chosen so as to provide particular effective caracteristics at the macroscopic level.

Modelling: negative caracteristics

For particular choices of the periodicity cells, metamaterials can be modelled by homogeneous materials admitting negative effective permittivity/permeabilitty at some frequency: $\epsilon(\omega) < 0, \mu(\omega) < 0$.



Interesting applications rely on interfaces metamaterial/standard materials. The mathematical modelling is necessarily non-standard due to the sign shift of ϵ, μ through the interface.

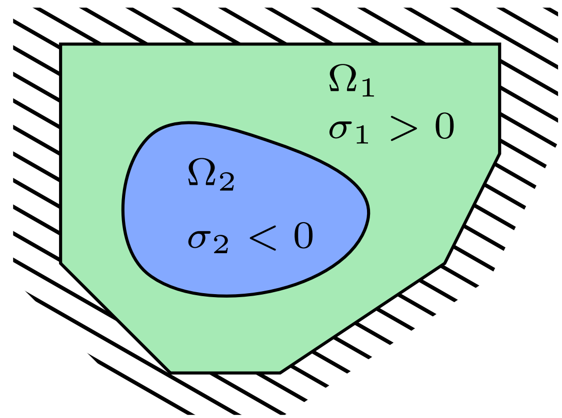
Refs: [Bouchitté-Bourel-Felbacq, 2009], [Bouchitté-Schweizer, 2010]

Model problem

Interesting mathematical difficulties are already contained in a 2-D "diffusion-like" model problem. Let $\mathrm{H}^1_0(\Omega) = \{v \in \mathrm{L}^2(\Omega) \mid \nabla v \in \mathrm{L}^2(\Omega),\ v|_{\Omega} = 0\ \}$. Given some $f \in \mathrm{H}^{-1}(\Omega) = \mathrm{H}^1_0(\Omega)^*$,

Find $u \in \mathrm{H}^1_0(\Omega)$ such that

$$-\mathrm{div}(\sigma \nabla u) = f \quad \text{in } \Omega.$$



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$\sigma = \sigma_j$ in Ω_j

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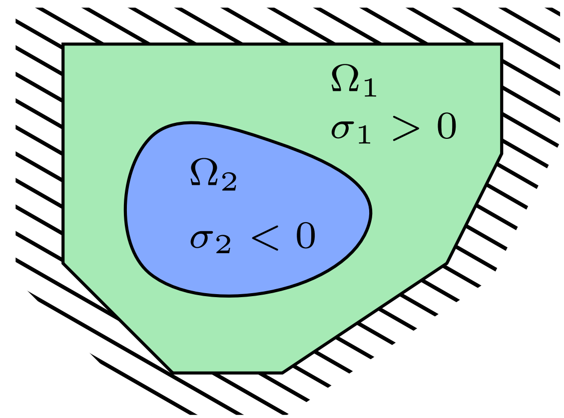

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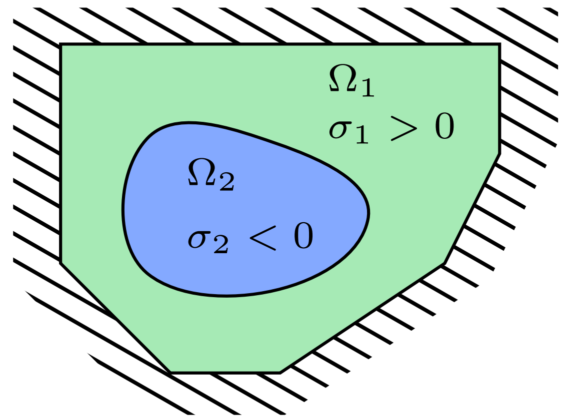

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Is this problem well posed (= existence + uniqueness of the solution)?

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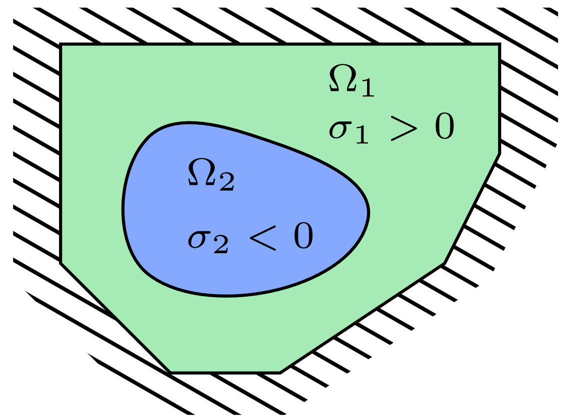

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$$\int_{\Omega} \sigma |\nabla u|^2 \, d\mathbf{x} \ \geq \ \min(\sigma) \ \|u\|^2_{\mathrm{H}^1_0(\Omega)}$$

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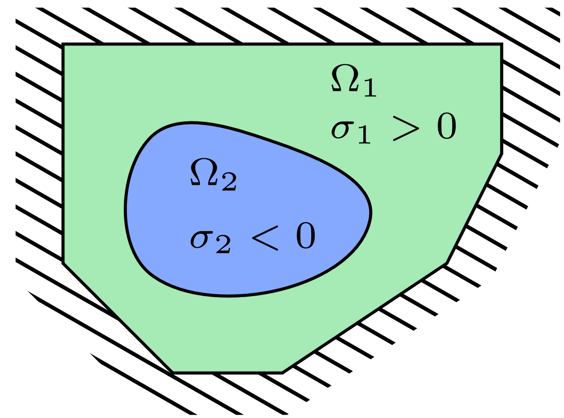

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no coercivity as σ changes sign
$\Rightarrow$ Lax-Milgram not available...

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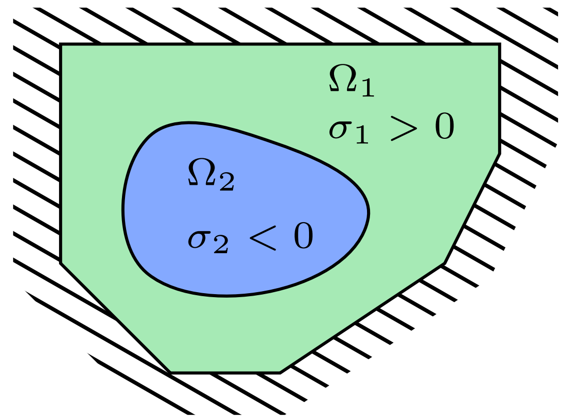

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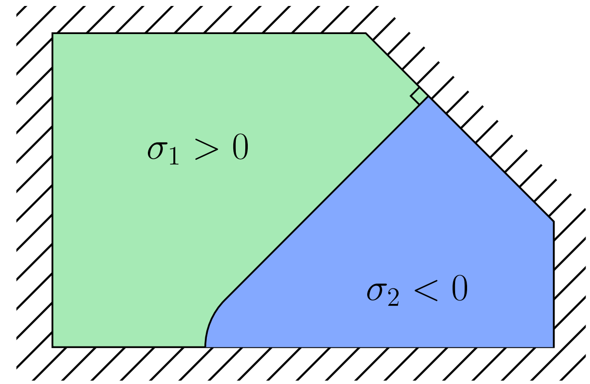
Refs: **[Costabel & Stephan, 1985]**,

[Bonnet-Ben Dhia, Ciarlet Jr., Zwölf, 2010],
[Bonnet-Ben Dhia, Chesnel, Ciarlet Jr., 2012],
[Chesnel, 2012].

Case 1: smooth interface

Case 1 corresponds to:

- smooth interface $\Sigma := \partial\Omega_1 \cap \partial\Omega_2$,
- if Σ meets $\partial\Omega$, it does with perpendicular angle.



Then **T-coercivity techniques** show the following.

Theorem
If the geometry belongs to case 1, and $\kappa_\sigma := \sigma_2/\sigma_1 \neq -1$, then the operator $\mathrm{A} : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ defined by

$$\langle \mathrm{A}u, v\rangle \ := \int_\Omega \sigma \nabla u \, \nabla v \; d\mathbf{x} \qquad \forall u, v \in \mathrm{H}^1_0(\Omega)$$

is of Fredholm type with index 0.

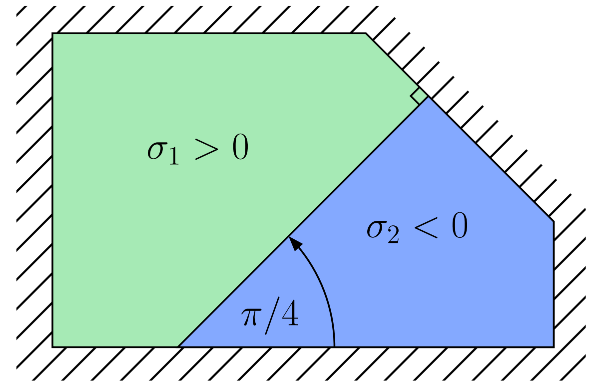
Main idea: A Fredholm $\iff$ $\mathrm{A}\,\mathrm{T}$ Fredholm

where $\mathrm{T} : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^1_0(\Omega)$ is a "well chosen" isomorphism

Case 2: corner interface

Case 2 is the same as case 1 except that:

- the interface Σ may admit corners,
- Σ may meet $\partial\Omega$ with an angle $\neq \pi/2$.



$\mathscr{I} = [-1, -\frac{1}{3}]$

Again $\langle \mathrm{A}u, v\rangle := \int_\Omega \sigma \nabla u \nabla v \, d\mathbf{x}$ and $\kappa_\sigma = \sigma_2/\sigma_1$.

Theorem
In case 2, there exists a closed interval $\mathscr{I} \subset \mathbb{R}_-$ depending on the corner angles of Σ, with $-1 \in \mathscr{I}$ and such that:

- if $\kappa_\sigma \in \mathbb{C} \setminus \mathscr{I}$, the operator $\mathrm{A} : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ is (index 0) - Fredholm,
- if $\kappa_\sigma \in \mathscr{I}$, the operator $\mathrm{A} : \mathrm{H}^1_0(\Omega) \to \mathrm{H}^{-1}(\Omega)$ is **NOT** of Fredholm type.

Questions: What exactly happens for $\kappa_\sigma \in \mathscr{I}$? Is it possible to recover Fredholmness by changing the functional setting?

Reduction of the problem

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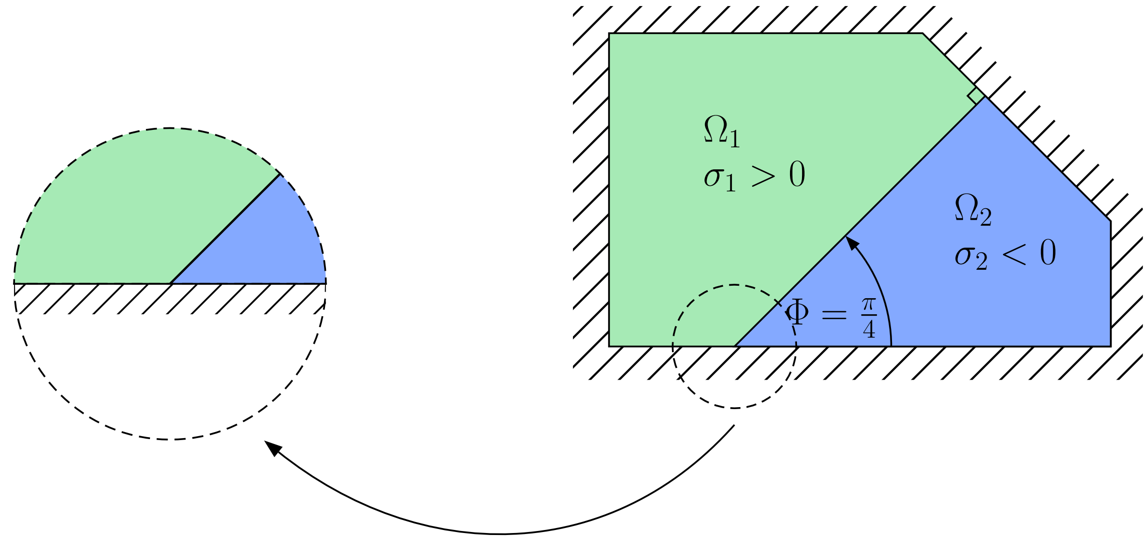
Relevant features of our problem are inherited from the metamaterial corner at the boundary.



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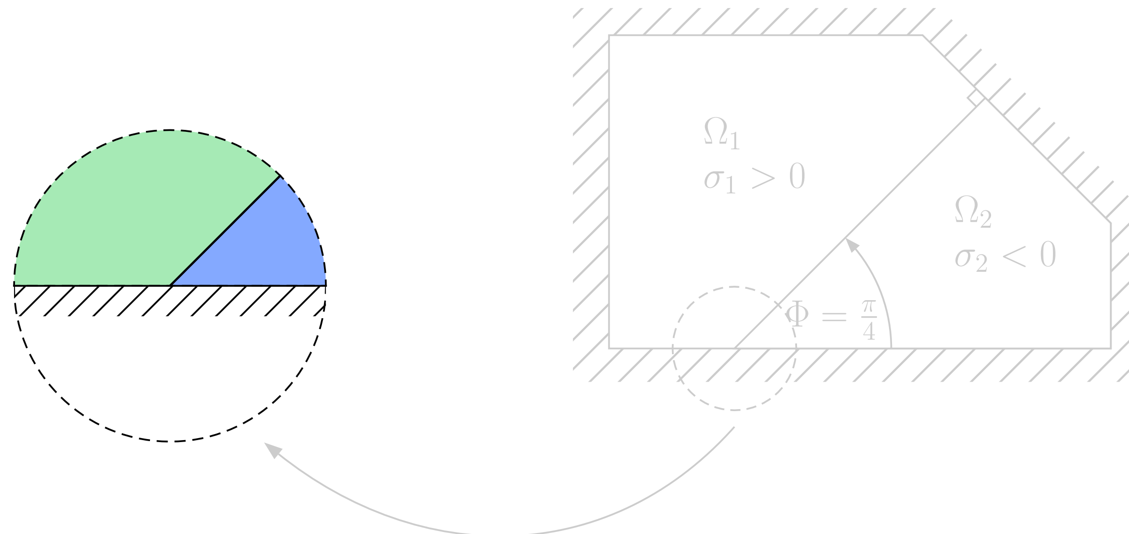

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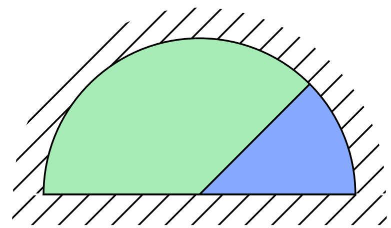
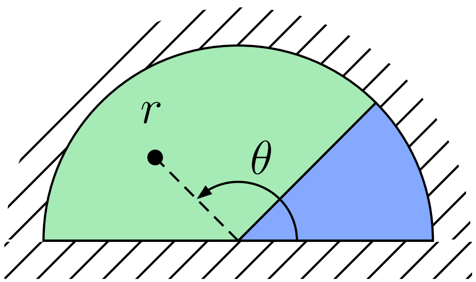


Reduction of the problem

Relevant features of our problem are inherited from the metamaterial corner at the boundary. Behaviour of solutions at the corner?



$\omega = \mathrm{D}(0,1) \cap \mathbb{R}^2_+$

Singular exponents:

$$\begin{cases} -\mathrm{div}(\sigma \nabla r^{\lambda} \varphi(\theta)) = 0 \\ \varphi(0) = \varphi(\pi) = 0 \end{cases}$$

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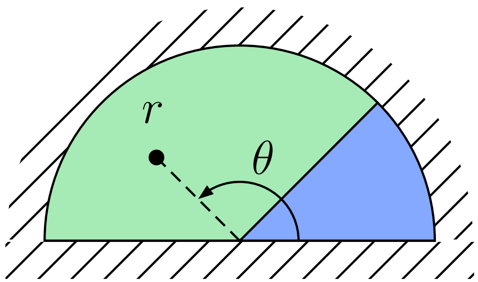

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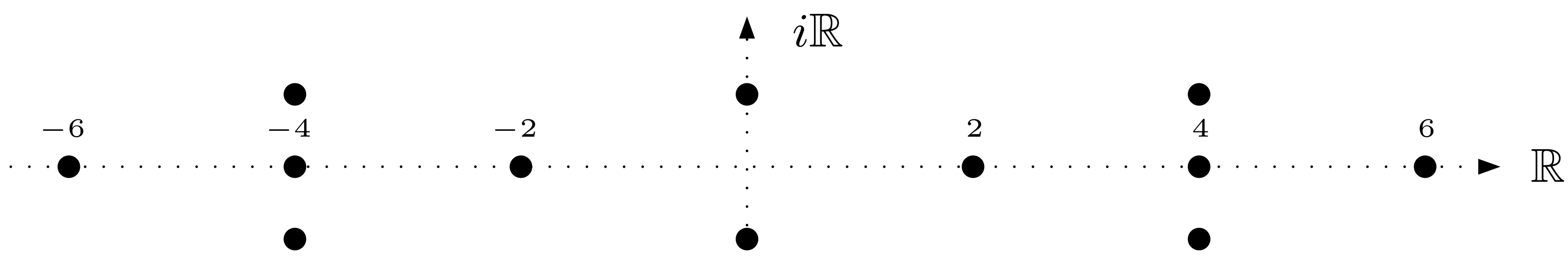
Eigenvalue pb:
Find $(\lambda, \varphi) \in \mathbb{C} \times \mathrm{H}^1_0(0,\pi)$ such that

$$\frac{\partial}{\partial\theta}\left(\sigma(\theta)\frac{\partial\varphi}{\partial\theta}\right) + \lambda^2\sigma(\theta)\varphi(\theta) = 0 \quad \text{on} \quad (0,\pi)$$

For $\kappa_\sigma := \sigma_2/\sigma_1 \in \mathscr{I}$, the strip $|\Re e\{\lambda\}| < 2$ contains exactly two purely imaginary roots: $\lambda = \pm i\mu$ associated to the behaviour $r^{\pm i\mu}\varphi_p(\theta) \notin \mathrm{H}^1(\omega)$.



Refs: [Kozlov, Mazya & Rossmann, 97],
[Dauge & Texier, 97],
[Bonnet-Ben Dhia, Chesnel & Claeys, 2013]

Kondratiev's analysis

Weighted Sobolev space: $\beta \in (0,2)$,

$$\mathrm{V}^1_{\pm\beta}(\omega) := \{\, r^{-(\pm\beta)} v(r,\theta)\ ,\ v \in \mathrm{H}^1_0(\omega)\ \}$$

Operators:

$$\begin{array}{l} \mathrm{A}_{+\beta} : \mathrm{V}^1_{+\beta}(\omega) \to \mathrm{V}^1_{-\beta}(\omega)^* \\ \qquad\quad\ \bigcup \qquad\qquad\quad \bigcup \\ \mathrm{A}_{-\beta} : \mathrm{V}^1_{-\beta}(\omega) \to \mathrm{V}^1_{+\beta}(\omega)^* \end{array}$$

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functions possibly blowing up as $r \to 0$ like $r^{-\beta+1}$

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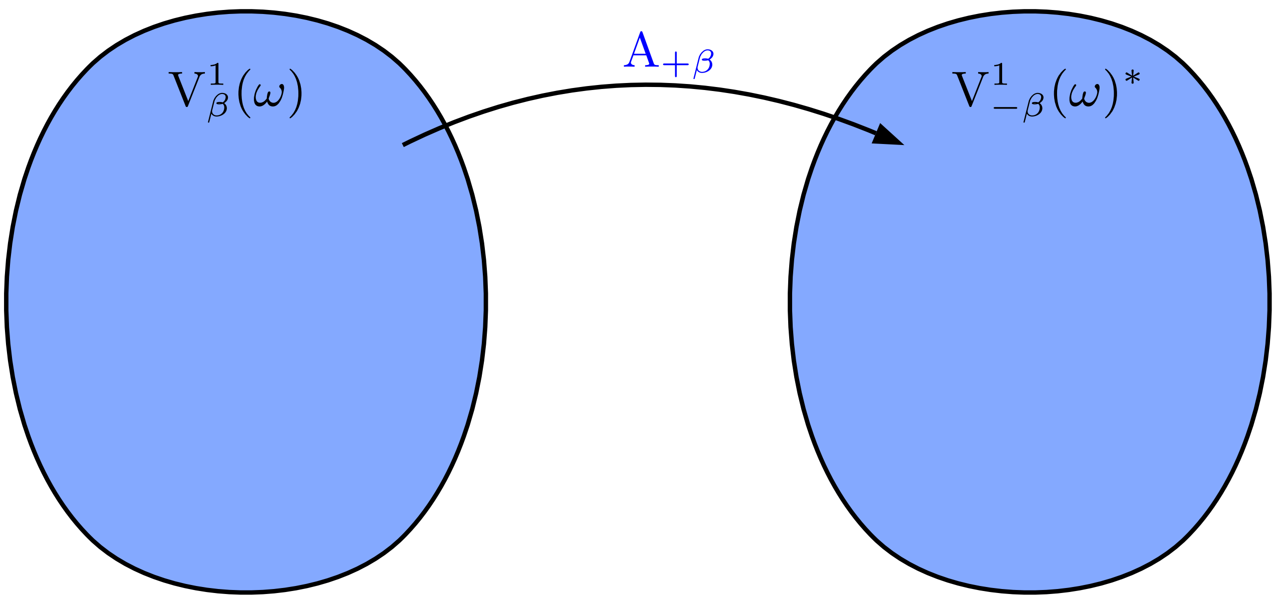

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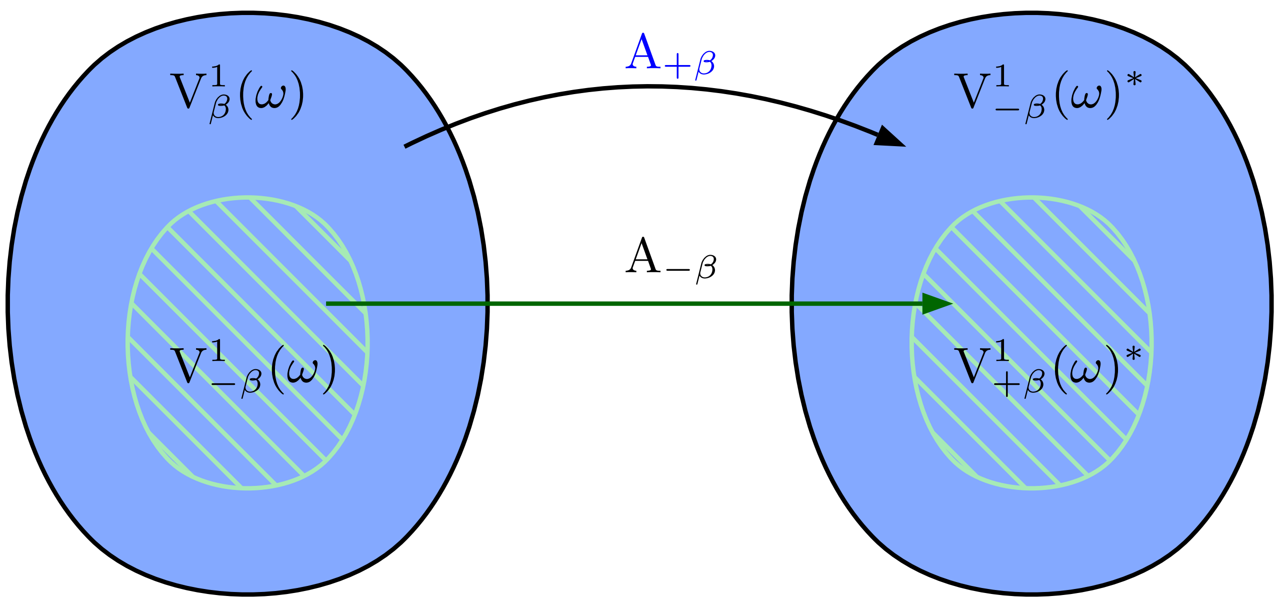

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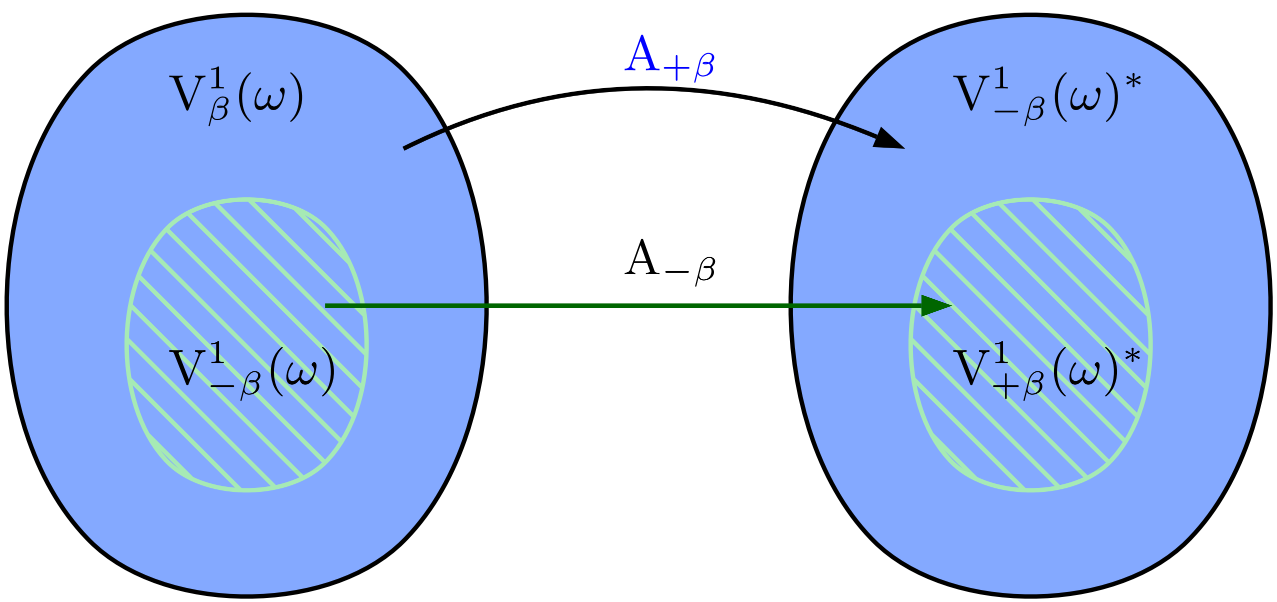

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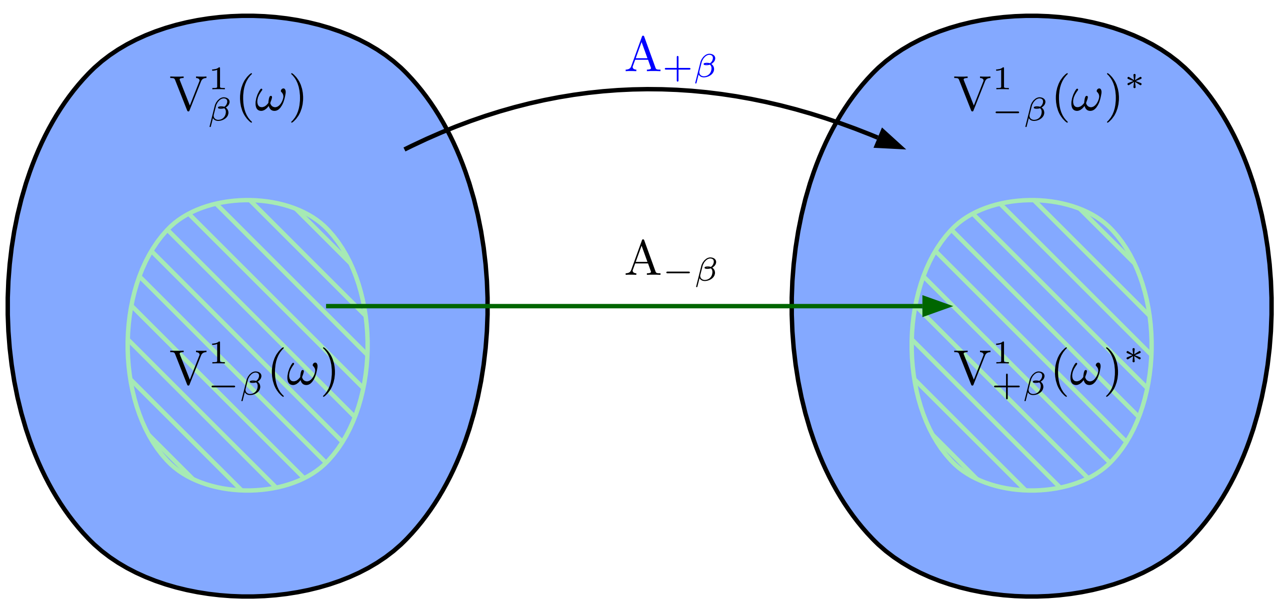

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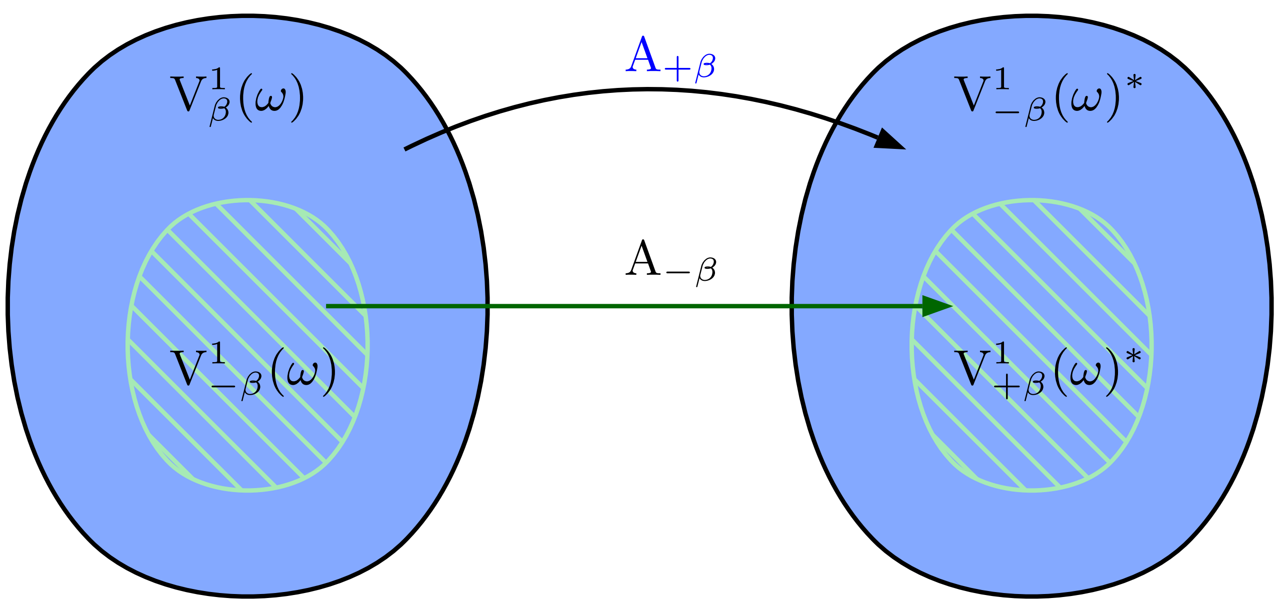

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$\mathrm{V}^{\mathrm{out}}_{\beta}(\omega) = \mathrm{span}\{r^{\pm i\mu}\varphi_p(\theta)\} \oplus \mathrm{V}^1_{-\beta}(\omega)$

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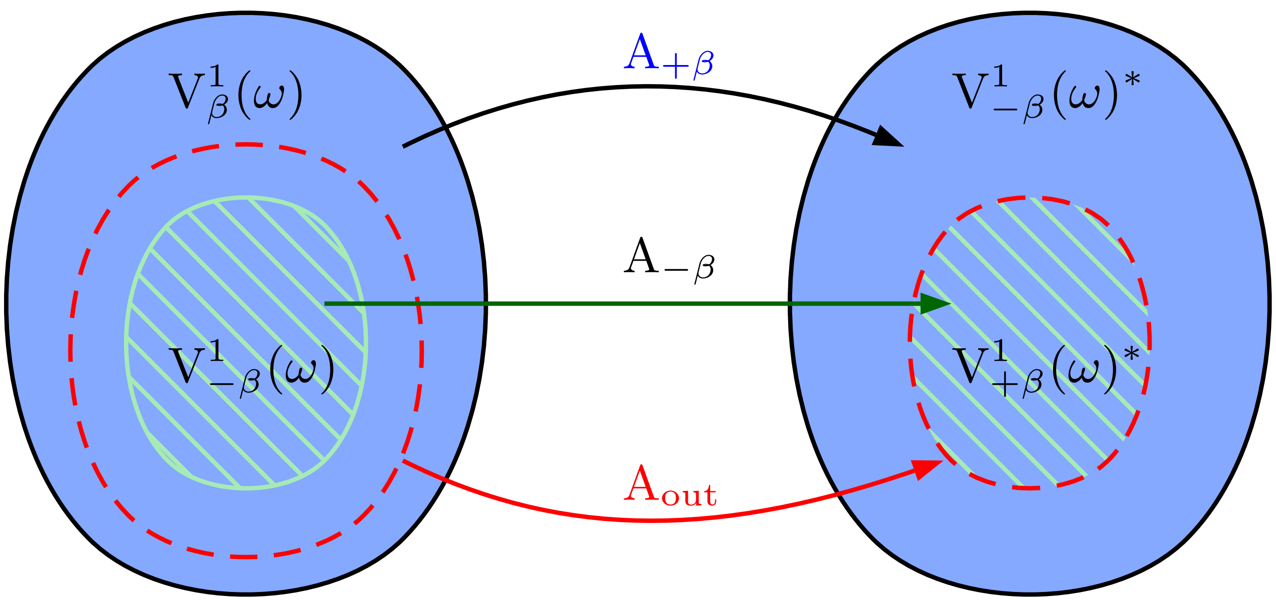

Kondratiev's analysis

Theorem

For $\beta \in (0,2)$, define $\mathrm{V}^{\mathrm{out}}_{\beta}(\omega) := \mathrm{span}\{\ r^{+i\mu}\varphi_{\mathrm{p}}(\theta)\chi(r)\ \} \oplus \mathrm{V}^1_{-\beta}(\omega)$. Then the operator $\mathrm{A}_{\mathrm{out}} : \mathrm{V}^{\mathrm{out}}_{\beta}(\omega) \to \mathrm{V}^1_{+\beta}(\omega)^*$ defined by

$$\langle \mathrm{A}_{\mathrm{out}} u, v \rangle \ := \ \int_{\omega} \sigma \nabla u \nabla v \, d\mathbf{x} \qquad \forall u \in \mathrm{V}^{\mathrm{out}}_{\beta}(\omega), \ \ \forall v \in \mathrm{V}^1_{+\beta}(\omega),$$

is an isomorphism.

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is an isomorphism.

Theorem

If $\kappa_{\sigma} \in \mathscr{I}$, suppose that $f \in \mathrm{V}^{1}_{+\beta}(\Omega)^{*}$ for some $\beta \in (0,2)$. Then the following problem is of Fredholm type with index 0:

Find $u \in \mathrm{V}^{\mathrm{out}}_{\beta}(\Omega)$ such that

$$\int_{\Omega} \sigma \nabla u \nabla v \, d\mathbf{x} \;=\langle f, v\rangle \quad \forall v \in \mathrm{V}^{1}_{+\beta}(\Omega).$$

Ref: [Bakharev & Nazarov 2009], [Nazarov & Taskinen 2011], [Bonnet-BenDhia, Chesnel, Claeys, 2013]

Rounded corner problem

Sobolev spaces not adapted to our corner problem.
$\Rightarrow$ uncomfortable for numerical simulation.

Is it possible to regularize this problem so as to make it fit the standard Sobolev framework?

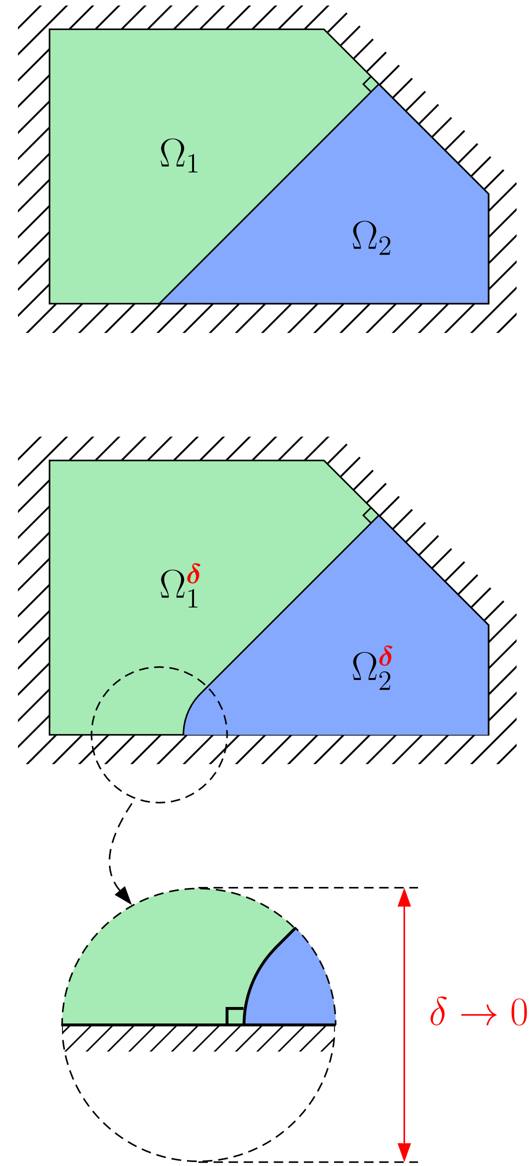
Question: What about rounding the corner?
Is this an admissible regularization process?

Set $\sigma^\delta := \sigma_j$ in Ω_j^δ, and take $f \in \mathrm{H}^{-1}(\Omega)$, with $f = 0$ next to $r = 0$ (for simplicity. . .).

$$\left\{\begin{array}{l}\text{Find } u^\delta \in \mathrm{H}^1_0(\Omega) \text{ such that}\\ -\mathrm{div}(\sigma^\delta \nabla u^\delta) = f \text{ in } \Omega\,.\end{array}\right. \qquad (3)$$

Question: Assuming (3) well posed, $u^\delta \underset{\delta \to 0}{\sim}$?

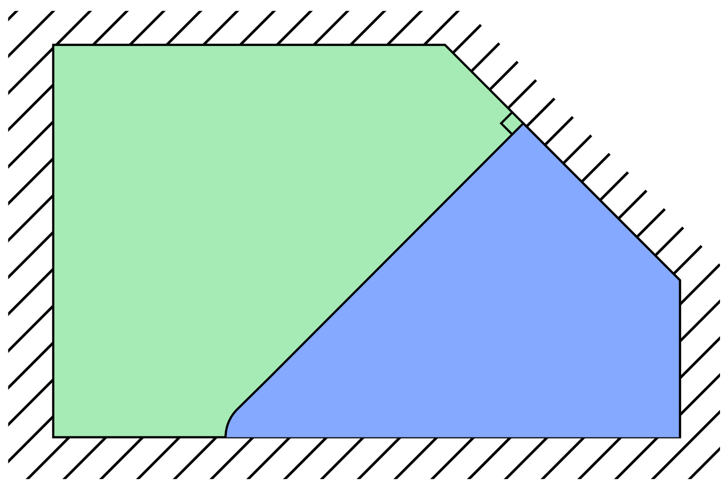
We present formal matched asymptotics. . .



Construction of the asymptotics

The method of matched asymptotics consist in approximating the exact solution u^δ with a "more explicit" function defined by

$$\widetilde{u}^\delta(r,\theta) :=$$



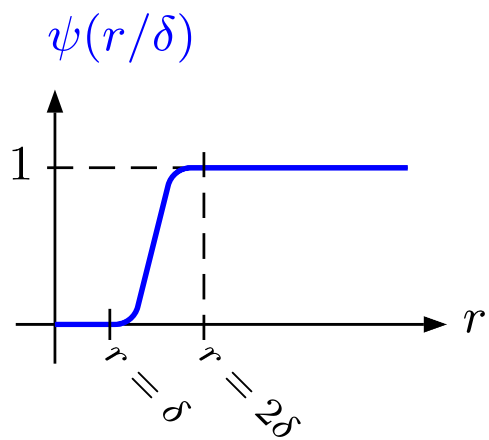
Construction of the asymptotics

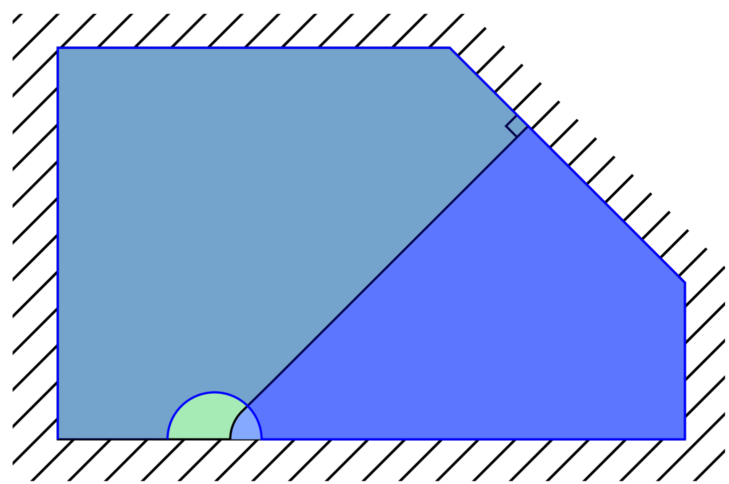
The method of matched asymptotics consist in approximating the exact solution u^δ with a "more explicit" function defined by

$$\widetilde{u}^\delta(r,\theta) := \psi(r/\delta) \times \text{far field expansion}$$

Far field

$$u^\delta(r,\theta) = u^0(r,\theta) + a(\delta)\zeta(r,\theta) + \dots$$





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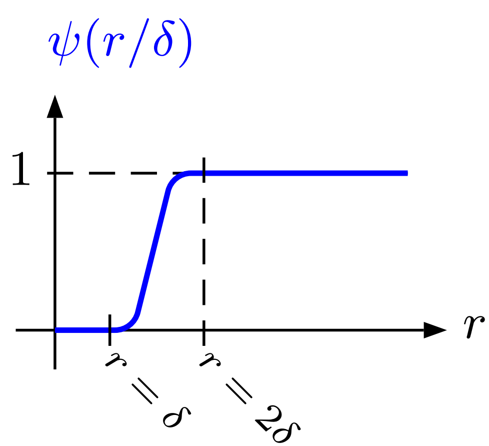
$$\widetilde{u}^\delta(r,\theta) := \psi(r/\delta) \times \text{far field expansion} + \chi(r) \times \text{near field expansion}$$

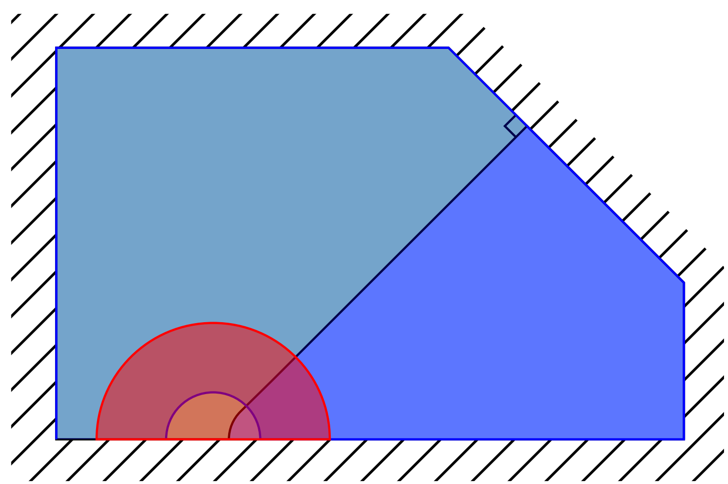
Far field

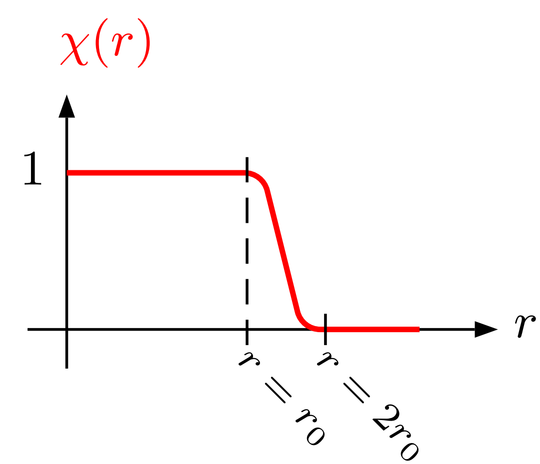
$$u^\delta(r,\theta) = u^0(r,\theta) + a(\delta)\zeta(r,\theta) + \dots$$

Near field

$$u^\delta(\delta\rho,\theta) = b(\delta)\mathrm{Z}(\rho,\theta) + \dots$$







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Far field

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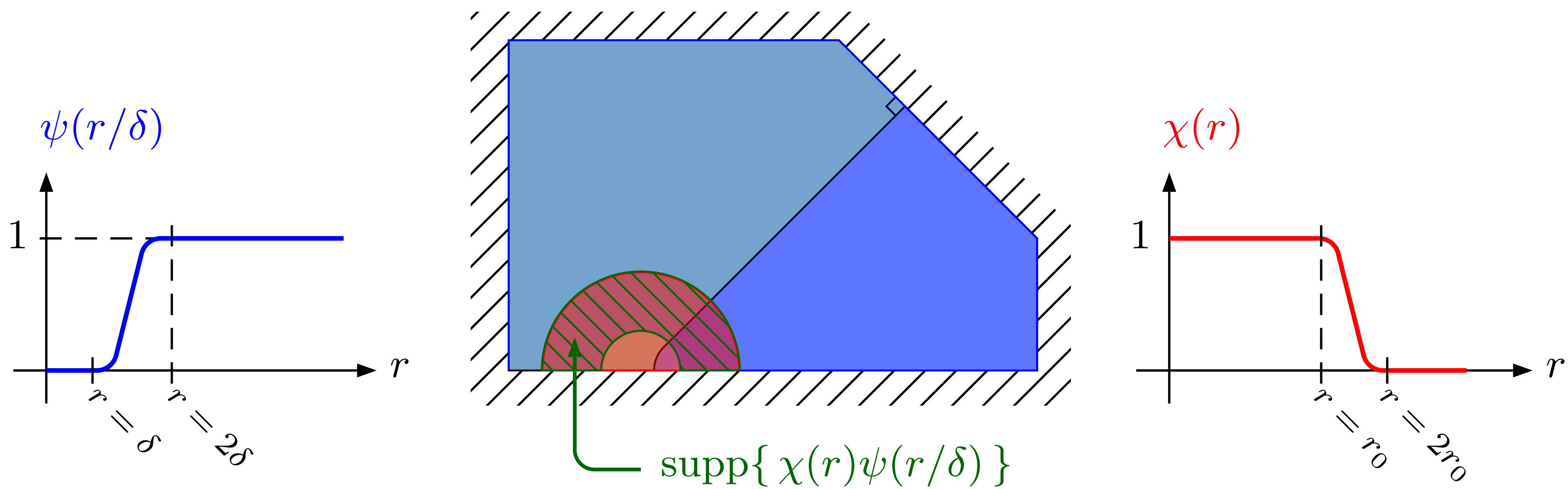
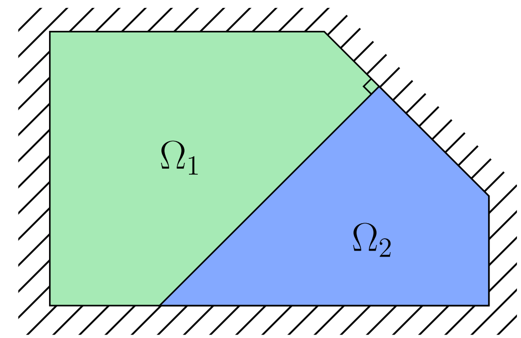


Far field expansion

Ansatz: $u^\delta(r,\theta) = u^0(r,\theta) + a(\delta)\,\zeta(r,\theta) + \ldots$



Far field expansion

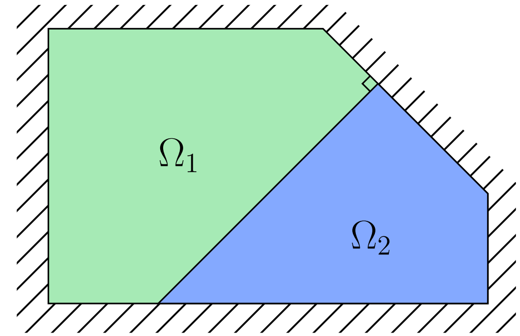
Ansatz: $u^\delta(r,\theta) = u^0(r,\theta) + a(\delta)\,\zeta(r,\theta) + \dots$

limit field

$u^0 \in \mathrm{V}^{\mathrm{out}}_\beta(\Omega)$ and

$-\mathrm{div}(\sigma^0 \nabla u^0) = f$ in Ω

corrector (=?)



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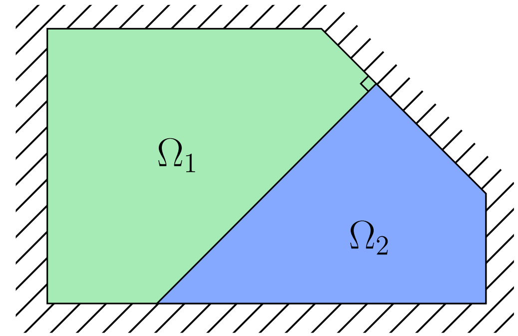

$$f = -\mathrm{div}(\sigma^\delta \nabla u^\delta) \approx -\mathrm{div}(\sigma^0 \nabla u^\delta)$$

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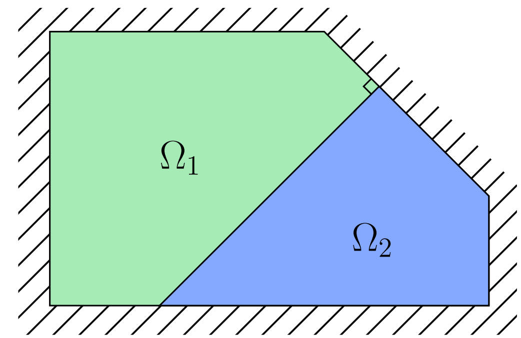

$$f \approx -\mathrm{div}\big(\,\sigma^0 \nabla (u^0 + a(\delta)\zeta + \dots)\,\big)$$

Far field expansion

Ansatz: $u^\delta(r,\theta) = u^0(r,\theta) + a(\delta)\,\zeta(r,\theta) + \dots$

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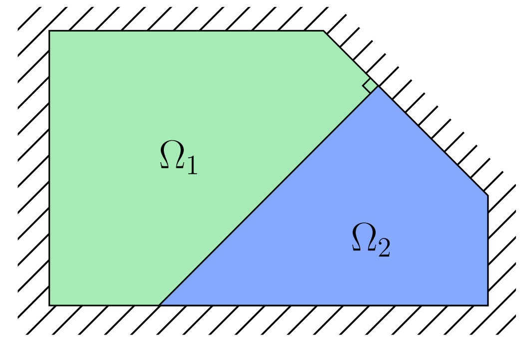


$f \approx -\mathrm{div}(\sigma^0\nabla u^0) - a(\delta)\mathrm{div}(\sigma^0\nabla\zeta) + \dots \quad \Longrightarrow \quad -\mathrm{div}(\sigma^0\nabla\zeta) = 0$ in Ω

$\zeta = 0$ on $\partial\Omega$, $\zeta \neq 0$

Recall: $\mathrm{V}^1_{-\beta}(\Omega) \subset \mathrm{V}^{\mathrm{out}}_\beta(\Omega) \subset \mathrm{V}^1_{+\beta}(\Omega)$

$\langle \mathrm{A}_\star u, v\rangle = \int_\Omega \sigma^0 \nabla u \,\nabla v \, d\mathbf{x}$, where $\star = \beta, \mathrm{out}$.

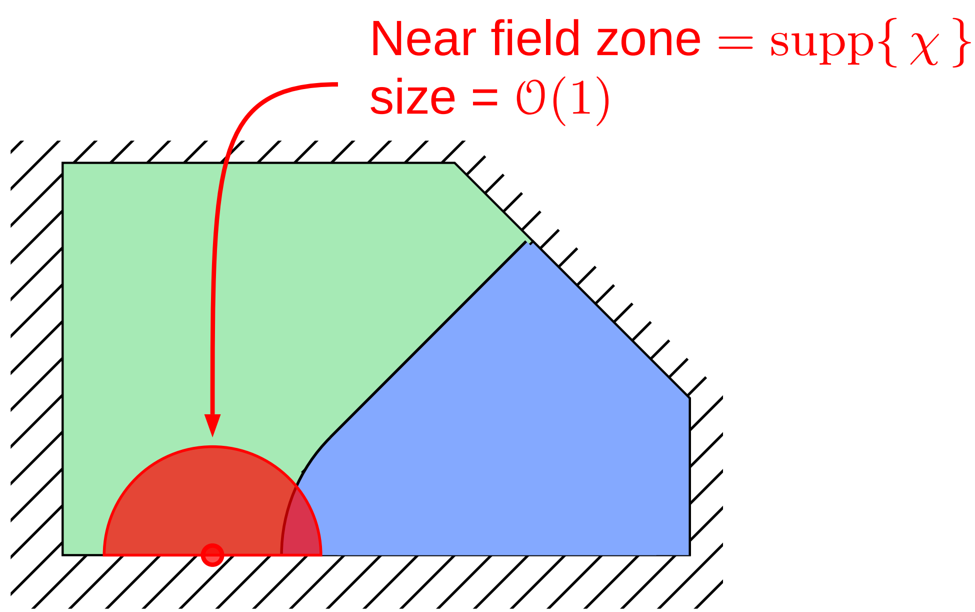
Proposition

$\mathrm{Ker}(\mathrm{A}_\beta) = \mathrm{span}\{\,\zeta\,\} \oplus \mathrm{Ker}(\mathrm{A}_{-\beta})$ with

$\zeta(r,\theta) \;=\; (r^{-i\mu} + c_\zeta r^{+i\mu})\varphi_p(\theta) + \dots$ remainder $\in \mathrm{V}^1_{-\beta}(\Omega)$

Near field expansion

Next to the rounded corner, we use the fast variable $\rho = r/\delta$ so as to normalize the geometry of the perturbation.



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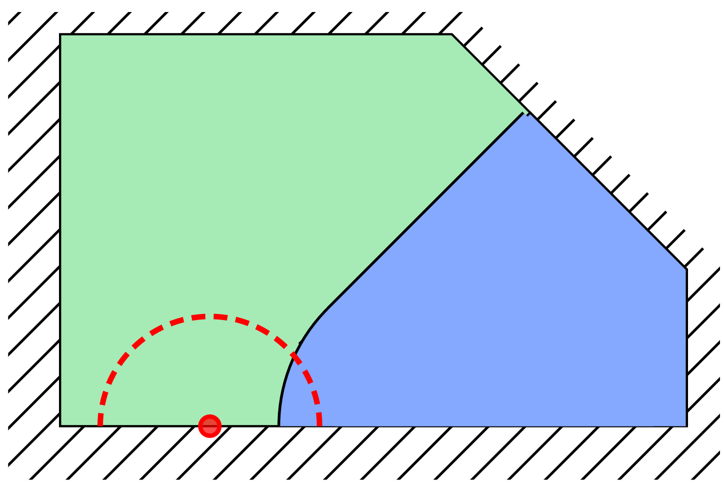

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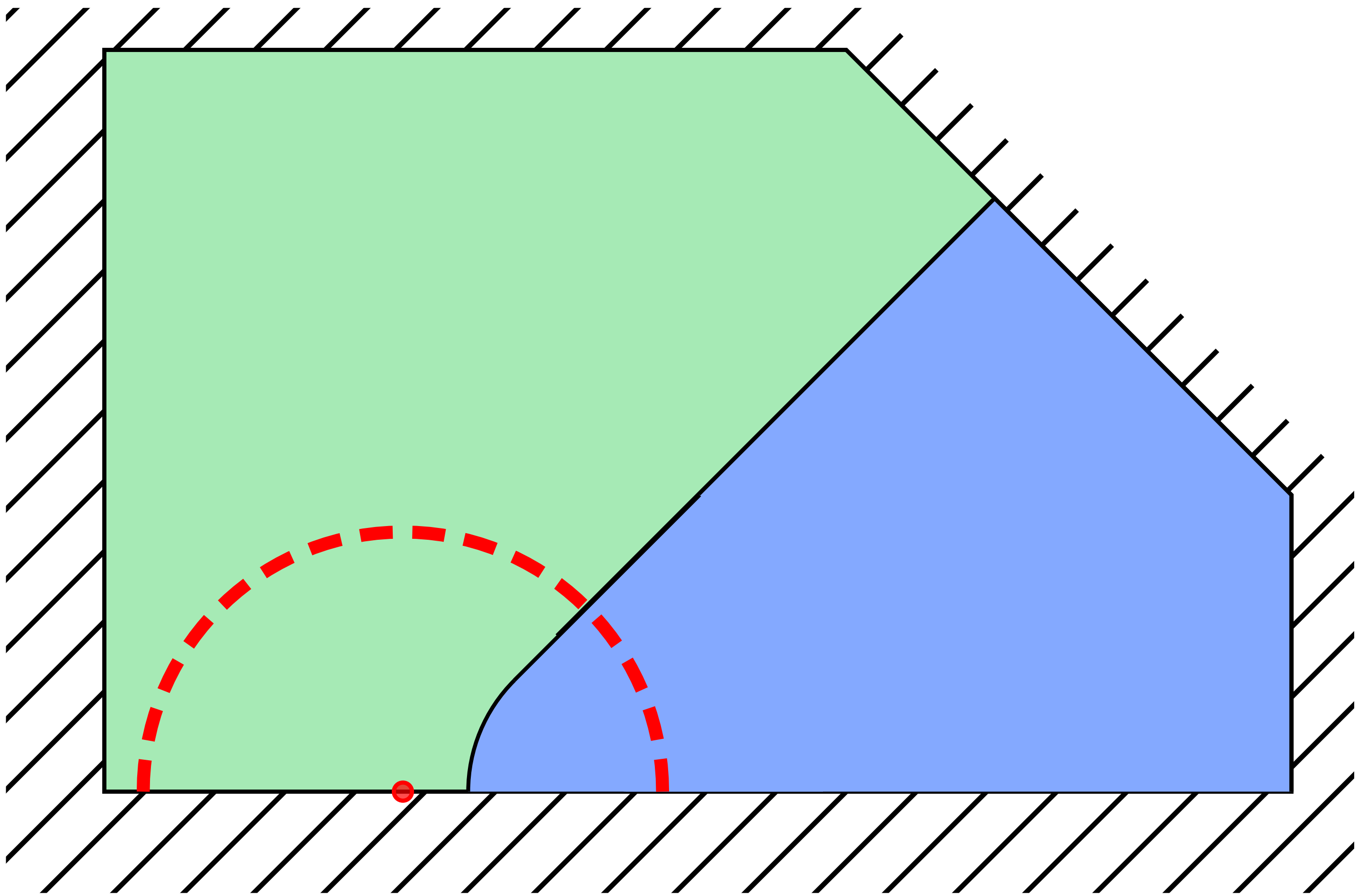
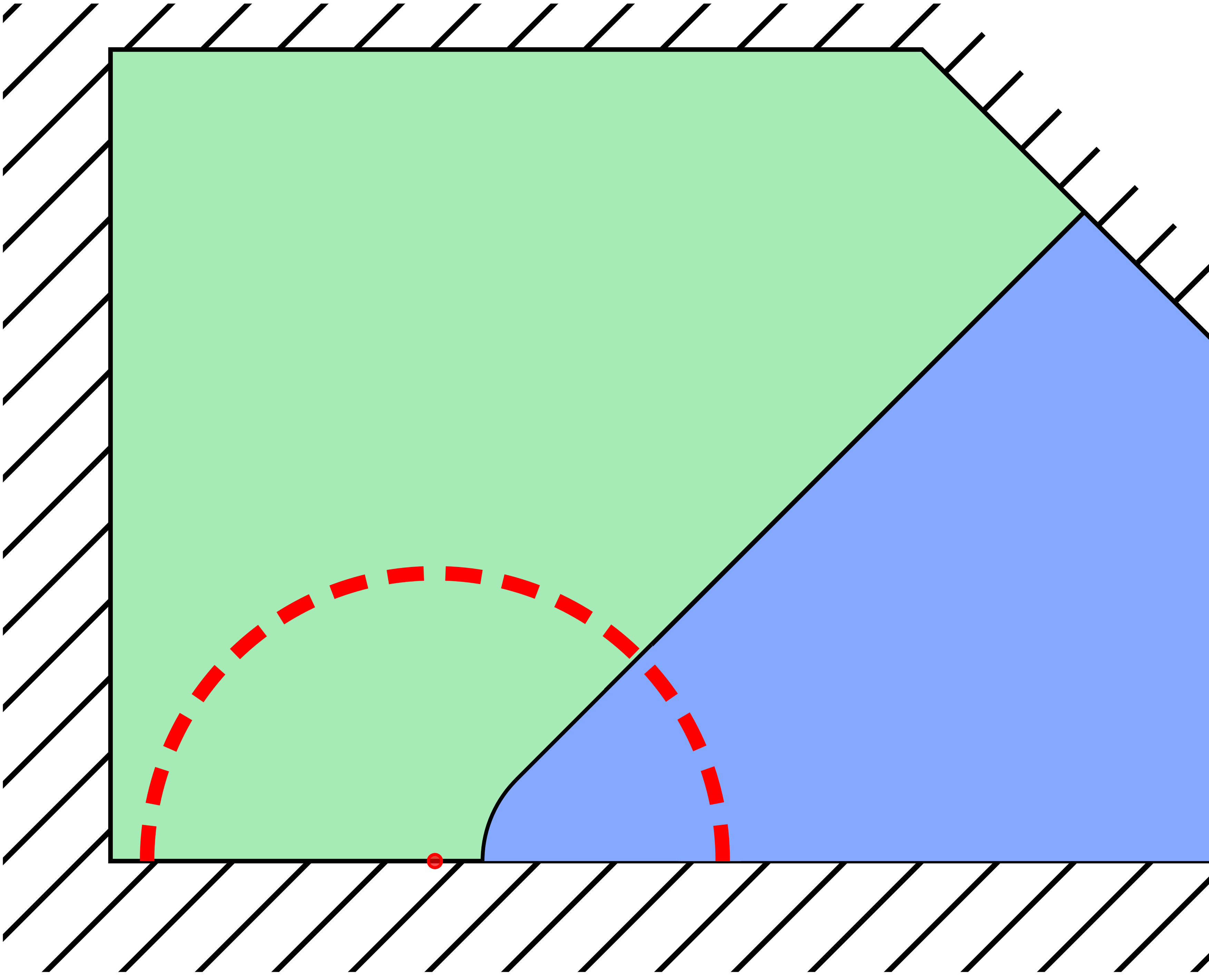
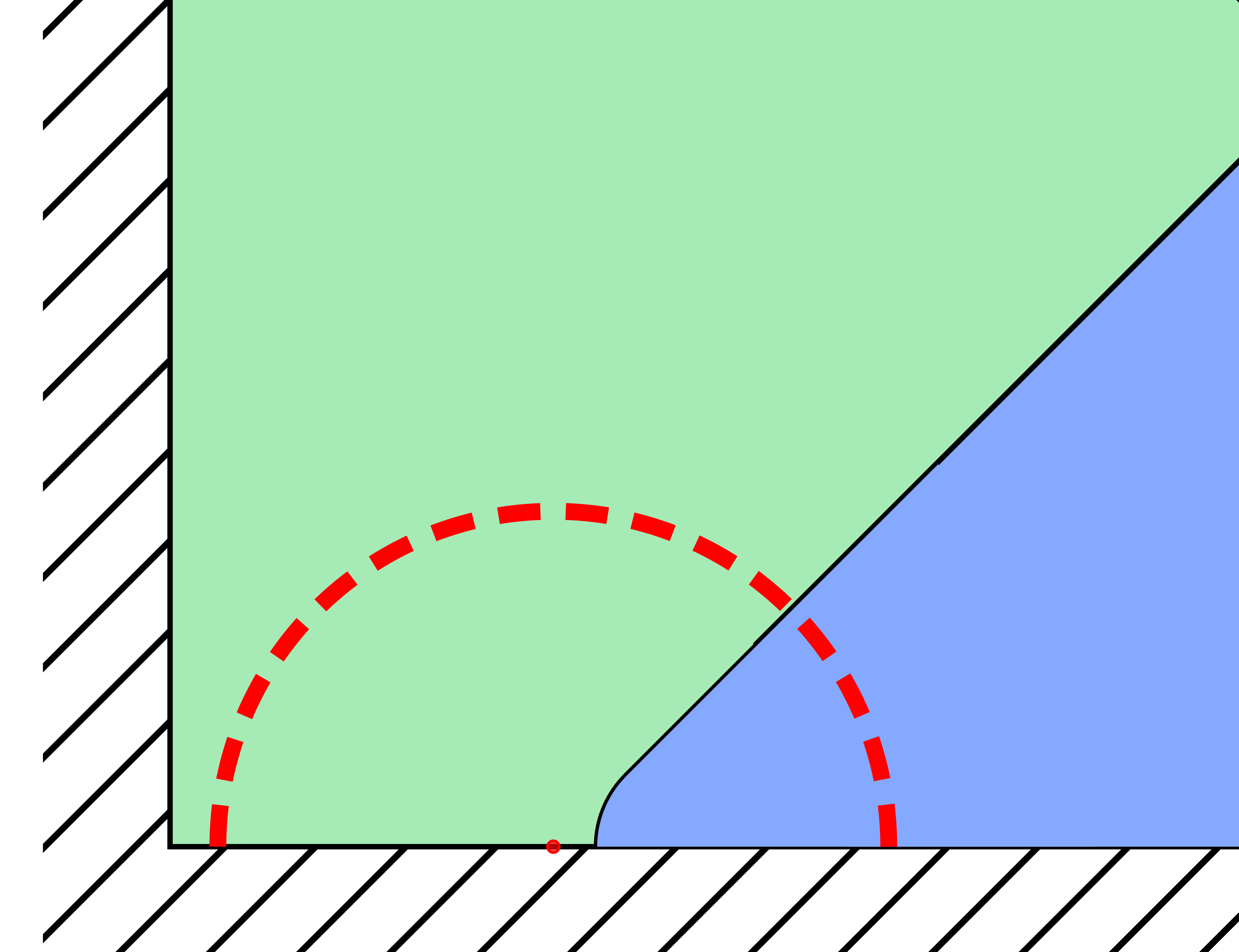
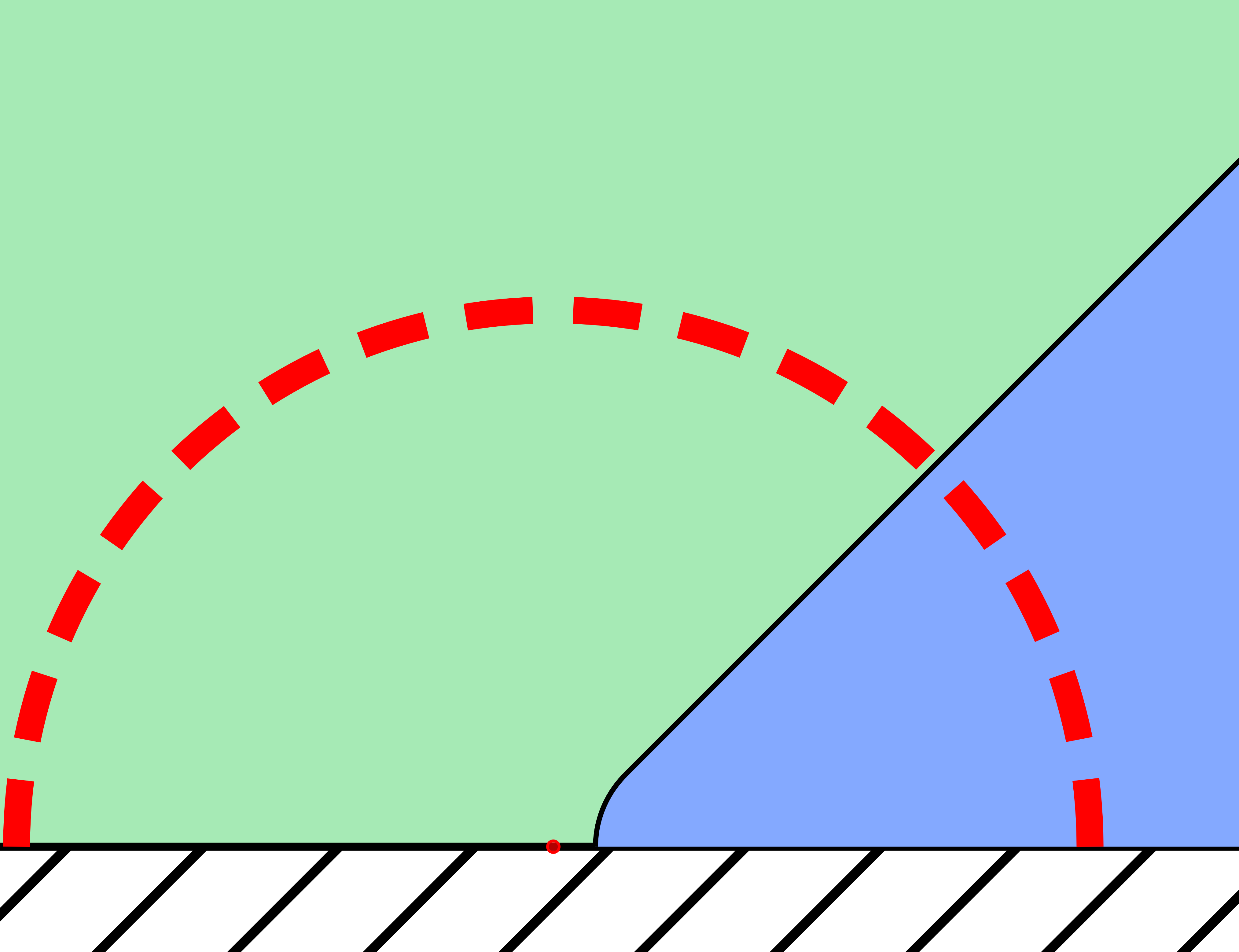
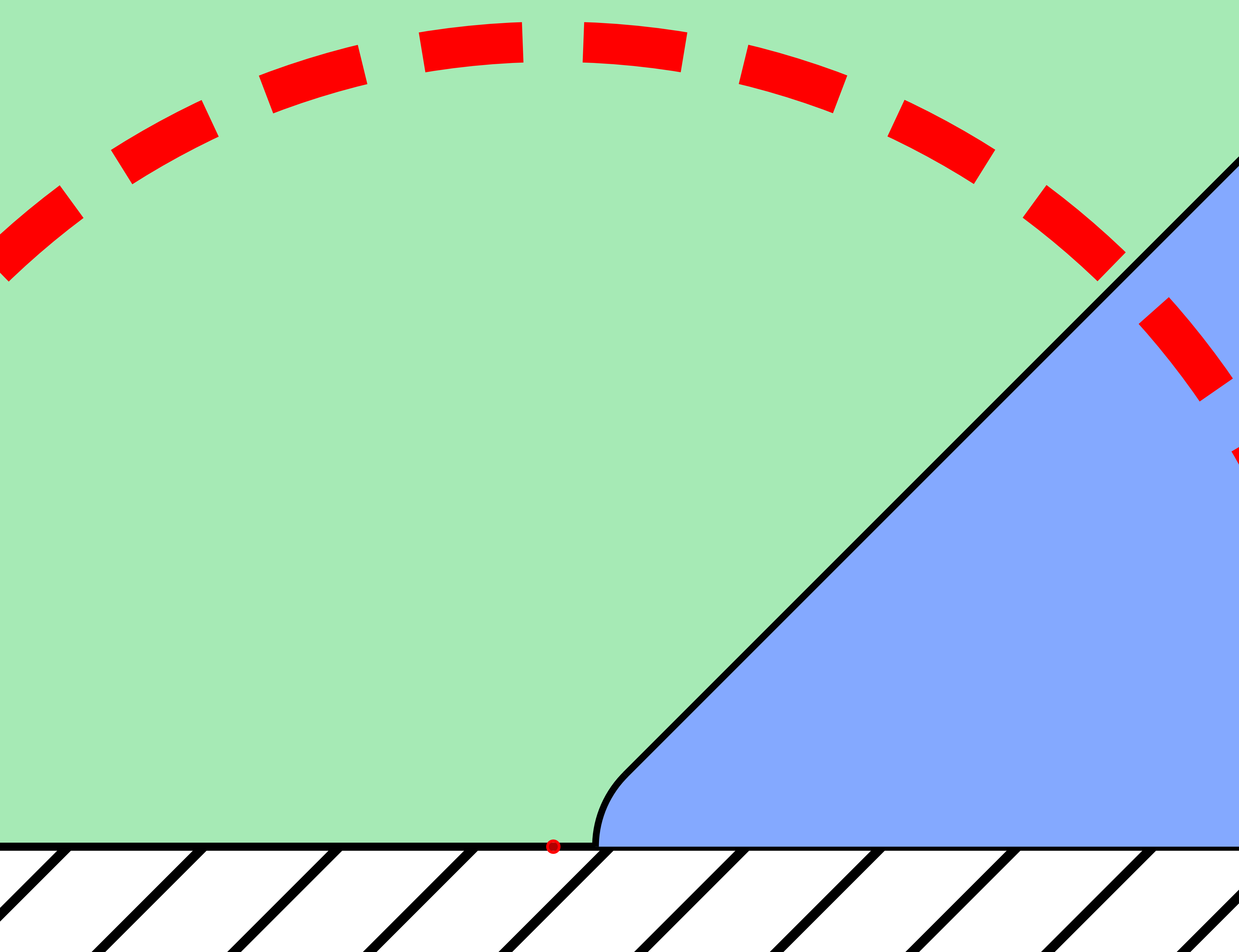


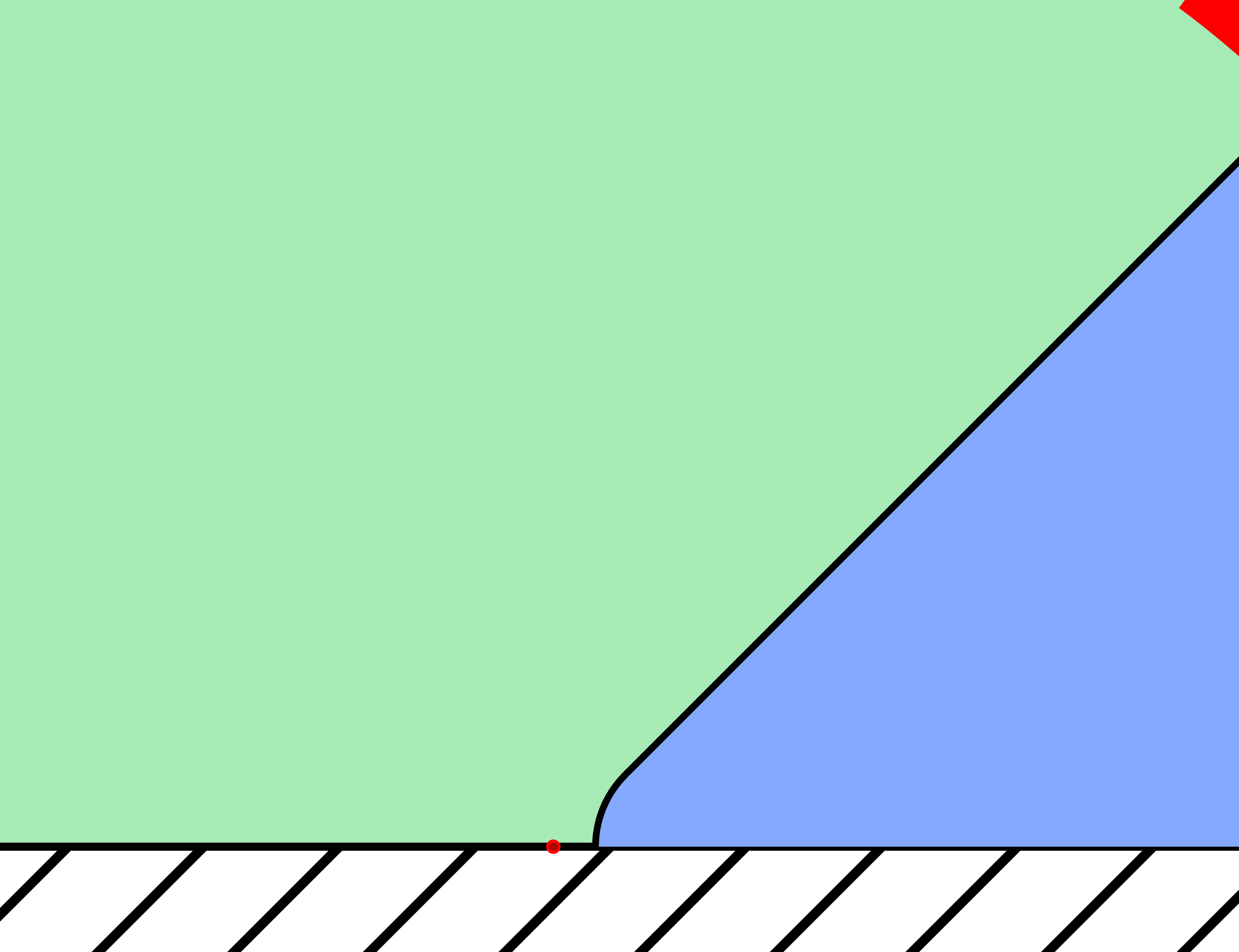
Near field expansion

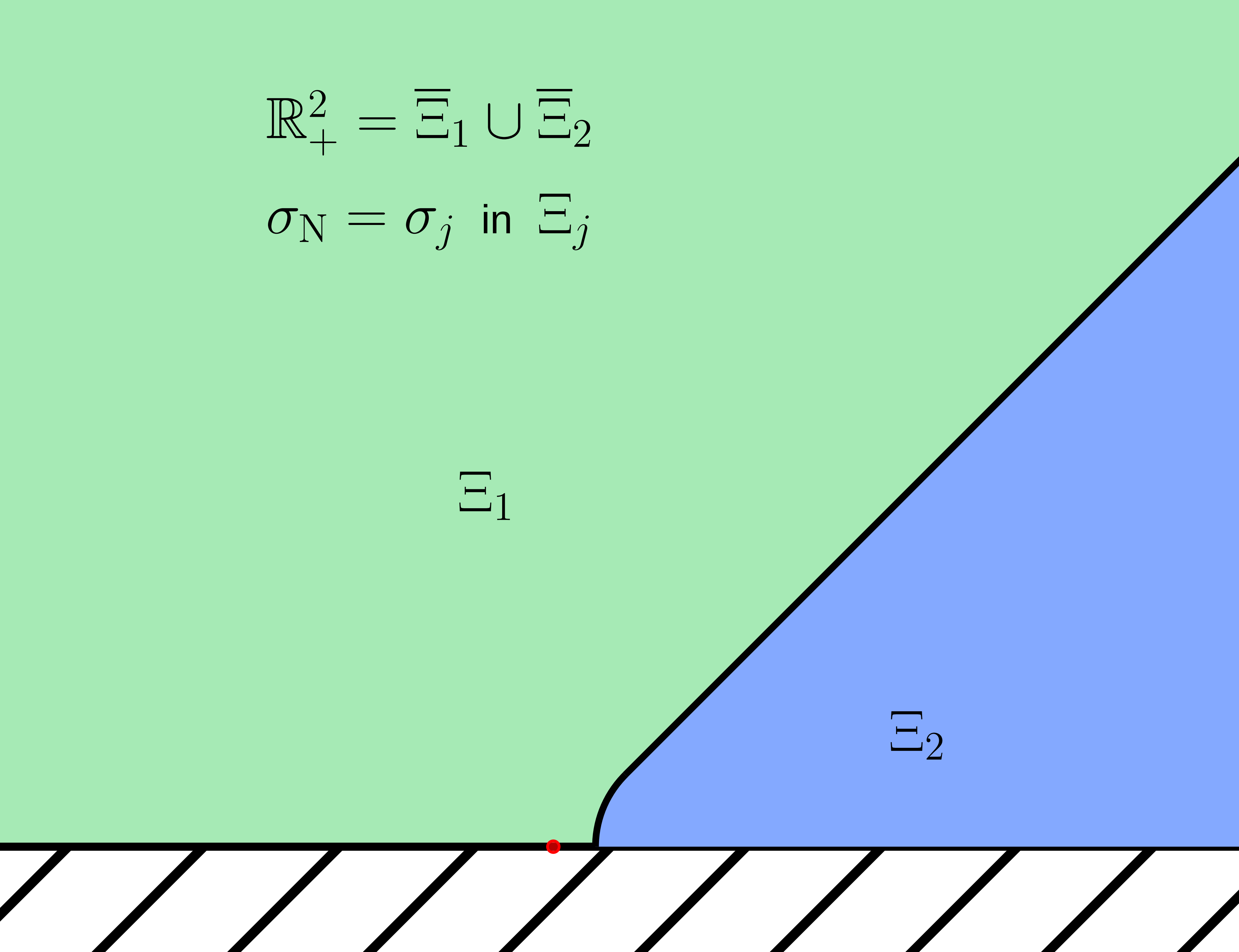








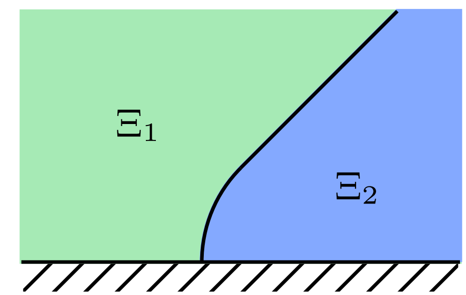



$\mathbb{R}^2_+ = \overline{\Xi}_1 \cup \overline{\Xi}_2$
$\sigma_{\mathrm{N}} = \sigma_j$ in Ξ_j
Ξ_1
Ξ_2

Near field expansion

Next to the rounded corner, we use the fast variable $\rho = r/\delta$ so as to normalize the geometry of the perturbation. The normalized field $\mathrm{U}^\delta(\rho,\theta) := u^\delta(\delta\rho,\theta)$ satisfies

$$\begin{cases} -\mathrm{div}(\sigma_\mathrm{N} \nabla \mathrm{U}^\delta) = 0 \quad \text{in } \mathbb{R}^2_+, \\ \mathrm{U}^\delta = 0 \quad \text{on } \partial\mathbb{R}^2_+ . \end{cases}$$



Ansatz: $\mathrm{U}^\delta(\rho,\theta) = b(\delta)\,\mathrm{Z}(\rho,\delta) + \dots$

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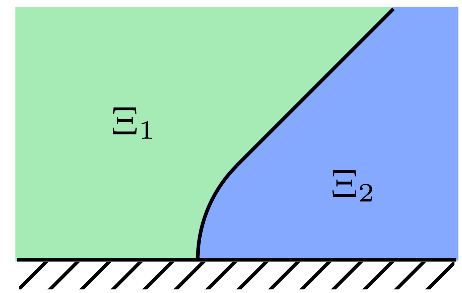

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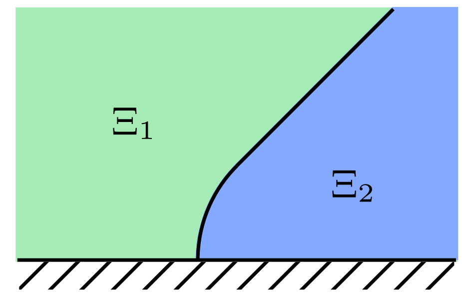

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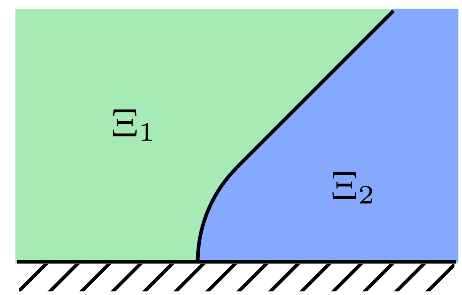

Weighted Sobolev framework:

$$\mathrm{W}^1_\beta(\mathbb{R}^2_+) = \{\ (1+\rho)^\beta v(\rho,\theta),\ v \in \mathrm{H}^1_0(\mathbb{R}^2_+)\ \}$$

$$\langle \mathscr{A}_\beta u, v\rangle = \int_{\mathbb{R}^2_+} \sigma_{\mathrm{N}} \nabla u \nabla v \, d\mathbf{x} \qquad u \in \mathrm{W}^1_\beta(\mathbb{R}^2_+),\ v \in \mathrm{W}^1_{-\beta}(\mathbb{R}^2_+)$$

Proposition

For $\beta \in (0,2)$, we have $\mathrm{Ker}(\mathscr{A}_{-\beta}) = \mathrm{span}\{\mathrm{Z}\} \oplus \mathrm{Ker}(\mathscr{A}_{+\beta})$ with

$$\mathrm{Z}(\rho,\theta) = (\rho^{+i\mu} + c_{\mathrm{Z}}\rho^{-i\mu})\varphi_{\mathrm{p}}(\theta) + \dots \qquad \text{remainder} \in \mathrm{W}^{\mathrm{out}}_\beta(\mathbb{R}^2_+).$$

Matching

We have defined $\zeta(r,\theta), \mathrm{Z}(\rho,\theta)$, but we still have to determine the jauge functions $a(\delta), b(\delta)$. We do so by matching radial expansions.

$$u^\delta(r,\theta) = u^0(r,\theta) + a(\delta)\,\zeta(r,\theta) + \dots$$

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$$= (\,c_0 + a(\delta)c_\zeta\,)\,r^{+i\mu}\varphi_{\mathrm{p}}(\theta) + a(\delta)r^{-i\mu}\varphi_{\mathrm{p}}(\theta) + \dots \qquad \text{for } r \to 0.$$

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$$u^\delta(r,\theta) = \mathrm{U}^\delta\left(\frac{r}{\delta},\theta\right) = b(\delta)\,\mathrm{Z}\left(\frac{r}{\delta},\theta\right) + \dots$$

$$= b(\delta)\left(\frac{r}{\delta}\right)^{+i\mu}\varphi_{\mathrm{p}}(\theta) + b(\delta)\,c_{\mathrm{Z}}\left(\frac{r}{\delta}\right)^{-i\mu}\varphi_{\mathrm{p}}(\theta) + \dots \qquad \text{for } \frac{r}{\delta} \to 0$$

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$$\left\{ \begin{array}{rcl} c_0 + a(\delta)c_\zeta & = & b(\delta)\,\delta^{-i\mu} \\ a(\delta)c_\zeta & = & b(\delta)c_{\mathrm{Z}}\,\delta^{+i\mu} \end{array} \right.$$

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oscillating jauge functions!

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Proposition
The coefficients $c_\zeta, c_\mathrm{Z} \in \mathbb{C}$ systematically verify $|c_\zeta| = |c_\mathrm{Z}| = 1$.

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oscillating jauge functions!

Proposition
The coefficients $c_\zeta, c_{\mathrm{Z}} \in \mathbb{C}$ systematically verify $|c_\zeta| = |c_{\mathrm{Z}}| = 1$.

Consequence: The matched asymptotic expansion is well defined only under the condition that

$$\delta \notin \mathcal{J} = \{\, \delta \in (0,1) \;\mid\; \delta^{-2i\mu} = c_\zeta c_{\mathrm{Z}} \,\}$$

Unfortunately $\mathcal{J}$ admits $\delta = 0$ as accumulation point.

(Simplified) convergence estimate

Theorem [Chesnel,Claeys,Nazarov]

Assume that $\kappa_\sigma \in \mathscr{I}$, and consider a datum $f \in \mathrm{H}^{-1}(\Omega)$ such that $f = 0$ near $r = 0$. Assume in addition:

- $\lim_{n\to\infty} \delta_n = 0$ with $\inf_{n\geq 0} |\delta_n^{-2i\mu} - c_\zeta c_\mathrm{Z}| > 0$,
- $\mathrm{Ker}(\mathrm{A}_{-\beta}) = \{0\}$,
- $\mathrm{Ker}(\mathscr{A}_\beta) = \{0\}$.

Then $\forall \epsilon \in (0,2)$, $\exists C_\epsilon > 0$ independent of δ such that

$$\|u^\delta - \widetilde{u}^\delta\|_{\mathrm{H}^1_0(\Omega)} \leq C_\epsilon\, \delta^{2-\epsilon}\, \|f\|_{\mathrm{H}^{-1}(\Omega)} \quad \forall \delta \in (0,1)$$

where $\widetilde{u}^\delta(r,\theta)$, the matched expansion of $u^\delta(r,\theta)$, is defined by:

$$\begin{aligned} \widetilde{u}^\delta(r,\theta) = \quad & \psi(r/\delta)\,\big(\,u^0(r,\theta) + a(\delta)\,\zeta(r,\theta)\,\big) \\ & + \chi(r)\; b(\delta)\,\mathrm{Z}\big(r/\delta,\theta\big) \\ & - \chi(r)\,\psi(r/\delta)\,\big(\,b(\delta)(r/\delta)^{+i\mu} + a(\delta)r^{-i\mu}\,\big)\,\varphi_\mathrm{P}(\theta) \end{aligned}$$

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- $\mathrm{Ker}(\mathrm{A}_{-\beta}) = \{0\}$,
- $\mathrm{Ker}(\mathscr{A}_\beta) = \{0\}$.

$\widetilde{u}^\delta$ oscillates as $\delta \to 0$
and $\lim_{\delta\to 0} \|\widetilde{u}^\delta\|_{\mathrm{H}_0^1(\Omega)} = +\infty$(!!!)

Then $\forall \epsilon \in (0,2)$, $\exists C_\epsilon > 0$ independent of δ such that

$$\|u^\delta - \widetilde{u}^\delta\|_{\mathrm{H}_0^1(\Omega)} \leq C_\epsilon\, \delta^{2-\epsilon}\, \|f\|_{\mathrm{H}^{-1}(\Omega)} \quad \forall \delta \in (0,1)$$

where $\widetilde{u}^\delta(r,\theta)$, the matched expansion of $u^\delta(r,\theta)$, is defined by:

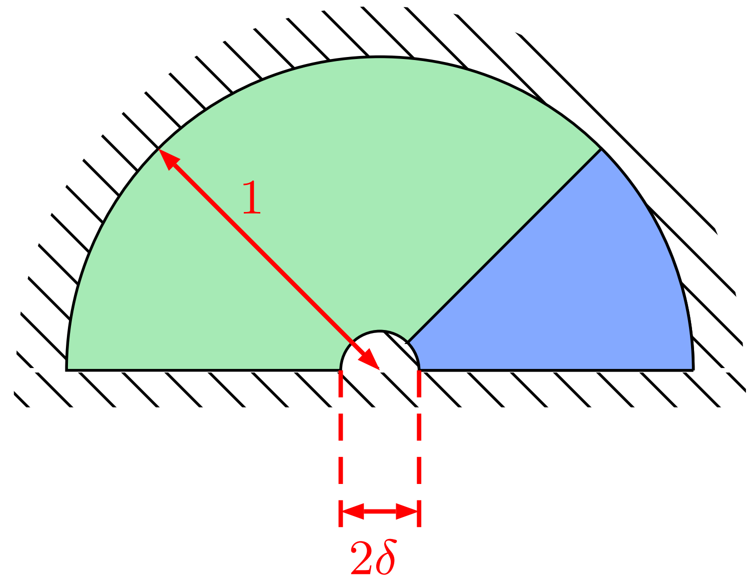
$$\begin{aligned}\widetilde{u}^\delta(r,\theta) = \quad & \psi(r/\delta)\left(u^0(r,\theta) + a(\delta)\,\zeta(r,\theta)\right)\\ & + \chi(r)\,b(\delta)\,\mathrm{Z}(r/\delta,\theta)\\ & - \chi(r)\,\psi(r/\delta)\left(b(\delta)(r/\delta)^{+i\mu} + a(\delta)r^{-i\mu}\right)\varphi_{\mathrm{P}}(\theta)\end{aligned}$$

Numerical illustration

$u^\delta \in \mathrm{H}^1_0(\Omega^\delta)$ satisfies

$-\mathrm{div}(\sigma \nabla u^\delta) = f$ in Ω^δ

with $f = \begin{cases} 1 & \text{for } x < 0 \\ 0 & \text{for } x \geq 0 \end{cases}$



We represent $\Re e\{u^\delta\}$ as $\delta \to 0$ for two values of $\kappa_\sigma = \sigma_2/\sigma_1$:

a) $\kappa_\sigma = -1.0001 \notin [-1, -1/3]$

b) $\kappa_\sigma = -0.9999 \in [-1, -1/3]$

Thank you

for your attention