RADIATION CONDITION AND INSTABILITY PHENOMENON AT A CORNER INTERFACE BETWEEN A DIELECTRIC AND A NEGATIVE MATERIAL

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in honour of Martin Costabel’s 65th birthday

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Metamaterials

In the context of electromagnetic wave propagation, metamaterials are periodic assemblies of small resonators whose characteristic size is much smaller than the average wavelength.

The periodicity cell (at the "micro scale") can be chosen so as to provide particular effective characteristics at the macroscopic level.
Modelling: negative characteristics

For particular choices of the periodicity cells, metamaterials can be modelled by homogeneous materials admitting negative effective permittivity/permeability at some frequency: $\epsilon(\omega) < 0, \mu(\omega) < 0$.

Interesting applications rely on interfaces metamaterial/standard materials. The mathematical modelling is necessarily non-standard due to the sign shift of $\epsilon, \mu$ through the interface.

Refs: [Bouchitté-Bourel-Felbacq, 2009], [Bouchitté-Schweizer, 2010]
Interesting mathematical difficulties are already contained in a 2-D "diffusion-like" model problem. Let $H^1_0(\Omega) = \{ v \in L^2(\Omega) \mid \nabla v \in L^2(\Omega), \ v|_\Omega = 0 \}$. Given some $f \in H^{-1}(\Omega) = H^1_0(\Omega)^*$,

Find $u \in H^1_0(\Omega)$ such that

$$-\text{div}(\sigma \nabla u) = f \quad \text{in} \quad \Omega.$$
Interesting mathematical difficulties are already contained in a 2-D "diffusion-like" model problem. Let $H^1_0(\Omega) = \{ v \in L^2(\Omega) \mid \nabla v \in L^2(\Omega), \; v|_{\Omega} = 0 \}$. Given some $f \in H^{-1}(\Omega) = H^1_0(\Omega)^*$, find $u \in H^1_0(\Omega)$ such that

$$
\int_{\Omega} \sigma \nabla u \nabla v \, dx = \langle f, v \rangle \quad \forall v \in H^1_0(\Omega).
$$

$\Omega = \overline{\Omega}_1 \cup \overline{\Omega}_2$

$\sigma = \sigma_j$ in $\Omega_j$
Interesting mathematical difficulties are already contained in a 2-D "diffusion-like" model problem. Let $H^1_0(\Omega) = \{ v \in L^2(\Omega) \mid \nabla v \in L^2(\Omega), \ v|_{\Omega} = 0 \}$. Given some $f \in H^{-1}(\Omega) = H^1_0(\Omega)^*$, find $u \in H^1_0(\Omega)$ such that

$$
\int_{\Omega} \sigma \nabla u \nabla v \, dx = \langle f, v \rangle \quad \forall v \in H^1_0(\Omega).
$$

**Question:** Is this problem well posed (= existence + uniqueness of the solution)?
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$$\int_{\Omega} \sigma \nabla u \nabla v \, dx = \langle f, v \rangle \quad \forall v \in H^1_0(\Omega).$$

Question:
Is this problem well posed (= existence + uniqueness of the solution)?

$$\int_{\Omega} \sigma |\nabla u|^2 \, dx \geq \min(\sigma) \| u \|^2_{H^1_0(\Omega)}$$
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Find $u \in H^1_0(\Omega)$ such that

$$\int_\Omega \sigma \nabla u \nabla v \, dx = \langle f, v \rangle \quad \forall v \in H^1_0(\Omega).$$

**Question:**

Is this problem well posed (= existence + uniqueness of the solution)?

$$\int_\Omega \sigma |\nabla u|^2 \, dx \geq \min(\sigma) \| u \|_{H^1_0(\Omega)}^2$$

no coercivity as $\sigma$ changes sign

$\Rightarrow$ Lax-Milgram not available...
Interesting mathematical difficulties are already contained in a 2-D "diffusion-like" model problem. Let $H^1_0(\Omega) = \{ v \in L^2(\Omega) \mid \nabla v \in L^2(\Omega), \ v|_\Omega = 0 \}$. Given some $f \in H^{-1}(\Omega) = H^1_0(\Omega)^*$, find $u \in H^1_0(\Omega)$ such that

$$\int_{\Omega} \sigma \nabla u \nabla v \, dx = \langle f, v \rangle \quad \forall v \in H^1_0(\Omega).$$

**Question:**
Is this problem well posed (= existence + uniqueness of the solution)?

**Refs:**
[Costabel & Stephan, 1985],
[Bonnet-Ben Dhia, Ciarlet Jr., Zwölf, 2010],
[Bonnet-Ben Dhia, Chesnel, Ciarlet Jr., 2012],
[Chesnel, 2012].
Case 1: smooth interface

Case 1 corresponds to:

- **smooth interface** $\Sigma := \partial \Omega_1 \cap \partial \Omega_2$,
- if $\Sigma$ meets $\partial \Omega$, it does with **perpendicular angle**.

Then **T-coercivity techniques** show the following.

---

**Theorem**

If the geometry belongs to case 1, and $\kappa_\sigma := \sigma_2 / \sigma_1 \neq -1$, then the operator $A : H^1_0(\Omega) \to H^{-1}(\Omega)$ defined by

$$\langle Au, v \rangle := \int_{\Omega} \sigma \nabla u \cdot \nabla v \, dx \quad \forall u, v \in H^1_0(\Omega)$$

is of Fredholm type with index 0.

**Main idea:** A Fredholm $\iff$ A $T$ Fredholm

where $T : H^1_0(\Omega) \to H^1_0(\Omega)$ is a "well chosen" isomorphism
Case 2: corner interface

Case 2 is the same as case 1 except that:

- the interface $\Sigma$ may admit corners,
- $\Sigma$ may meet $\partial \Omega$ with an angle $\neq \pi/2$.

Again $\langle Au, v \rangle := \int_{\Omega} \sigma \nabla u \nabla v \, dx$ and $\kappa_\sigma = \sigma_2 / \sigma_1$.

Theorem
In case 2, there exists a closed interval $I \subset \mathbb{R}_-$ depending on the corner angles of $\Sigma$, with $-1 \in I$ and such that:

- if $\kappa_\sigma \in \mathbb{C} \setminus I$, the operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ is (index 0) - Fredholm,
- if $\kappa_\sigma \in I$, the operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$ is NOT of Fredholm type.

Questions: What exactly happens for $\kappa_\sigma \in I$? Is it possible to recover Fredholmness by changing the functional setting?
Reduction of the problem
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Relevant features of our problem are inherited from the metamaterial corner at the boundary.
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Relevant features of our problem are inherited from the metamaterial corner at the boundary. **Behaviour of solutions at the corner?**

\[ \omega = D(0, 1) \cap \mathbb{R}^2_+ \]

**Singular exponents:**

\[
\begin{cases}
- \div(\sigma \nabla r^\lambda \varphi(\theta)) = 0 \\
\varphi(0) = \varphi(\pi) = 0
\end{cases}
\]
Reduction of the problem

Relevant features of our problem are inherited from the metamaterial corner at the boundary. Behaviour of solutions at the corner?

Eigenvalue pb:
Find $(\lambda, \varphi) \in \mathbb{C} \times H^1_0(0, \pi)$ such that
\[
\frac{\partial}{\partial \theta} \left( \sigma(\theta) \frac{\partial \varphi}{\partial \theta} \right) + \lambda^2 \sigma(\theta) \varphi(\theta) = 0 \quad \text{on} \quad (0, \pi)
\]

For $\kappa_\sigma := \sigma_2/\sigma_1 \in \mathcal{I}$, the strip $|\Re\{\lambda\}| < 2$ contains exactly two purely imaginary roots: $\lambda = \pm i\mu$ associated to the behaviour $r^{\pm i\mu} \varphi_p(\theta) \notin H^1(\omega)$.

Refs: [Kozlov, Mazya & Rossmann, 97],
[Dauge & Texier, 97],
[Bonnet-Ben Dhia, Chesnel & Claeys, 2013]
Kondratiev’s analysis

Weighted Sobolev space: $\beta \in (0, 2)$,

$$V^{1}_{\pm\beta} (\omega) := \{ r^{-\beta} v(r, \theta) \mid v \in H^{1}_{0}(\omega) \}$$

Operators:

$$A_{\pm\beta} : V^{1}_{\pm\beta} (\omega) \rightarrow V^{1}_{\mp\beta} (\omega)^{*}$$

with

$$\langle A_{\pm\beta} u, v \rangle := \int_{\omega} \sigma \nabla u \nabla v \, dx$$
**Kondratiev’s analysis**

**Weighted Sobolev space:** $\beta \in (0, 2)$,

$$V_{\pm \beta}(\omega) := \{ r^{-(\pm \beta)} v(r, \theta) \ , \ v \in H^{1}_{0}(\omega) \}$$

**Operators:**

$$A_{+\beta} : V^{1}_{+\beta}(\omega) \to V^{1}_{-\beta}(\omega)^*$$

$$A_{-\beta} : V^{1}_{-\beta}(\omega) \to V^{1}_{+\beta}(\omega)^*$$

with

$$\langle A_{\pm \beta} u, v \rangle := \int_{\omega} \sigma \nabla u \nabla v \, dx$$

- Functions possibly blowing up as $r \to 0$ like $r^{-\beta+1}$
- Functions decaying as $r \to 0$ as fast as $r^{+\beta+1}$
Kondratiev’s analysis

Weighted Sobolev space: \( \beta \in (0, 2) \),
\[ V^1_{\pm \beta}(\omega) := \{ r^{-(\pm \beta)}v(r, \theta) \ , \ v \in H^1_0(\omega) \} \]

Operators:
- \( A_{+\beta} : V^1_{+\beta}(\omega) \rightarrow V^1_{-\beta}(\omega)^* \)
- \( A_{-\beta} : V^1_{-\beta}(\omega) \rightarrow V^1_{+\beta}(\omega)^* \)

with \( \langle A_{\pm \beta}u, v \rangle := \int_{\omega} \sigma \nabla u \cdot \nabla v \ dx \)

functions possibly blowing up as \( r \rightarrow 0 \) like \( r^{-\beta + 1} \)
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Weighted Sobolev space: $\beta \in (0, 2)$,

$V^{1}_{\pm\beta}(\omega) := \{r^{-\pm\beta}v(r, \theta) \mid v \in H^{1}_{0}(\omega)\}$

Operators:

$A_{+\beta} : V^{1}_{+\beta}(\omega) \to V^{1}_{-\beta}(\omega)^{*}$

$A_{-\beta} : V^{1}_{-\beta}(\omega) \to V^{1}_{+\beta}(\omega)^{*}$

with $\langle A_{\pm\beta}u, v \rangle := \int_{\omega} \sigma \nabla u \nabla v \, dx$

functions possibly blowing up as $r \to 0$ like $r^{-\beta+1}$

functions decaying as $r \to 0$ as fast as $r^{+\beta+1}$
Kondratiev’s analysis

Weighted Sobolev space: $\beta \in (0, 2)$,

$$V_{\pm \beta}^1(\omega) := \{ r^{-\pm \beta} v(r, \theta), v \in H_0^1(\omega) \}$$

Operators:

$$A_{+\beta} : V_{+\beta}(\omega) \to V_{-\beta}(\omega)^*$$

$$A_{-\beta} : V_{-\beta}(\omega) \to V_{+\beta}(\omega)^*$$

with

$$\langle A_{\pm \beta} u, v \rangle := \int_{\omega} \sigma \nabla u \nabla v \, dx$$
Kondratiev’s analysis

Weighted Sobolev space: $\beta \in (0, 2)$,

$$V_{\pm \beta}^1(\omega) := \{ r^{-(\pm \beta)} v(r, \theta) , \; v \in H_0^1(\omega) \}$$

Operators:

$$A_{+\beta} : V_{+\beta}^1(\omega) \to V_{-\beta}^1(\omega)^* \quad \text{onto but not one-to-one}$$

$$A_{-\beta} : V_{-\beta}^1(\omega) \to V_{+\beta}^1(\omega)^* \quad \text{one-to-one but not onto}$$

with \[ \langle A_{\pm \beta} u, v \rangle := \int_\omega \sigma \nabla u \nabla v \, dx \]
Kondratiev’s analysis

Weighted Sobolev space: $\beta \in (0, 2)$,
$$V^1_{\pm \beta}(\omega) := \{ r^{-(\pm \beta)}v(r, \theta) \ , \ v \in H^1_0(\omega) \}$$
$$r^{\pm i \mu} \varphi_p(\theta) \in V^1_\beta(\omega) \setminus V^1_{-\beta}(\omega)$$

Operators:
$$A_{+\beta} : V^1_{+\beta}(\omega) \rightarrow V^1_{-\beta}(\omega)^* \ 	ext{onto but not one-to-one}$$
$$A_{-\beta} : V^1_{-\beta}(\omega) \rightarrow V^1_{+\beta}(\omega)^* \ 	ext{one-to-one but not onto}$$

with
$$\langle A_{\pm \beta} u, v \rangle := \int_\omega \sigma \nabla u \nabla v \, dx$$
Kondratiev’s analysis

Weighted Sobolev space: \( \beta \in (0, 2) \),
\[
V^1_{\pm \beta}(\omega) := \{ r^{-(\pm \beta)}v(r, \theta) : v \in H^1_0(\omega) \}
\]
\[
V^\text{out}_\beta(\omega) = \text{span}\{r^{\pm i \mu} \varphi_p(\theta)\} \oplus V^1_{-\beta}(\omega)
\]

Operators:
\[
A_{+\beta} : V^1_{+\beta}(\omega) \to V^1_{-\beta}(\omega)^* \quad \text{onto but not one-to-one}
\]
\[
A_{-\beta} : V^1_{-\beta}(\omega) \to V^1_{+\beta}(\omega)^* \quad \text{one-to-one but not onto}
\]

with \( \langle A_{\pm \beta} u, v \rangle := \int_{\omega} \sigma \nabla u \cdot \nabla v \, dx \)
Kondratiev’s analysis

**Theorem**

For $\beta \in (0, 2)$, define $V^\text{out}_\beta(\omega) := \text{span}\{ r^{+i\mu} \varphi_p(\theta) \chi(r) \} \oplus V^1_{-\beta}(\omega)$. Then the operator $A_{\text{out}} : V^\text{out}_\beta(\omega) \to V^1_{+\beta}(\omega)^*$ defined by

$$\langle A_{\text{out}} u, v \rangle := \int_\omega \sigma \nabla u \nabla v \, dx \quad \forall u \in V^\text{out}_\beta(\omega), \forall v \in V^1_{+\beta}(\omega),$$

is an isomorphism.
Kondratiev’s analysis

Theorem

For \( \beta \in (0, 2) \), define \( V_\beta^\text{out}(\omega) := \text{span}\{ r^{+i\mu} \varphi_p(\theta) \chi(r) \} \oplus V^1_{-\beta}(\omega) \). Then the operator \( A_{\text{out}} : V_\beta^\text{out}(\omega) \to V^1_{+\beta}(\omega)^* \) defined by

\[
\langle A_{\text{out}} u, v \rangle := \int_\omega \sigma \nabla u \nabla v \, dx \quad \forall u \in V_\beta^\text{out}(\omega), \quad \forall v \in V^1_{+\beta}(\omega),
\]

is an isomorphism.

Theorem

If \( \kappa_\sigma \in \mathcal{F} \), suppose that \( f \in V^1_{+\beta}(\Omega)^* \) for some \( \beta \in (0, 2) \). Then the following problem is of Fredholm type with index 0:

Find \( u \in V_\beta^\text{out}(\Omega) \) such that

\[
\int_\Omega \sigma \nabla u \nabla v \, dx = \langle f, v \rangle \quad \forall v \in V^1_{+\beta}(\Omega).
\]

Ref: [Bakharev & Nazarov 2009], [Nazarov & Taskinen 2011], [Bonnet-BenDhia, Chesnel, Claeys, 2013]
Rounded corner problem

Sobolev spaces not adapted to our corner problem. ⇒ uncomfortable for numerical simulation.

Is it possible to regularize this problem so as to make it fit the standard Sobolev framework?

**Question:** What about rounding the corner? Is this an admissible regularization process?

Set $\sigma^\delta := \sigma_j$ in $\Omega_j^\delta$, and take $f \in H^{-1}(\Omega)$, with $f = 0$ next to $r = 0$ (for simplicity...).

\[
\begin{align*}
\begin{cases}
\text{Find } u^\delta \in H^1_0(\Omega) \text{ such that } \\
-\text{div}(\sigma^\delta \nabla u^\delta) = f \text{ in } \Omega.
\end{cases}
\end{align*}
\]

**Question:** Assuming (3) well posed, $u^\delta \sim_{\delta \to 0}$?

We present formal matched asymptotics...
Construction of the asymptotics

The method of matched asymptotics consist in approximating the exact solution $u^\delta$ with a "more explicit" function defined by

$$
\tilde{u}^\delta (r, \theta) :=
$$
Construction of the asymptotics

The method of matched asymptotics consist in approximating the exact solution $u^\delta$ with a "more explicit" function defined by

$$\tilde{u}^\delta(r, \theta) := \psi(r/\delta) \times \text{far field expansion}$$

Far field

$$u^\delta(r, \theta) = u^0(r, \theta) + a(\delta)\zeta(r, \theta) + \ldots$$
Construction of the asymptotics

The method of matched asymptotics consist in approximating the exact solution $u^\delta$ with a "more explicit" function defined by

$$
\tilde{u}^\delta(r, \theta) := \psi(r/\delta) \times \text{far field expansion} + \chi(r) \times \text{near field expansion}
$$

**Far field**

$$
u^\delta(r, \theta) = u^0(r, \theta) + a(\delta)\zeta(r, \theta) + \ldots
$$

**Near field**

$$
u^\delta(\delta \rho, \theta) = b(\delta)Z(\rho, \theta) + \ldots
$$
Construction of the asymptotics

The method of matched asymptotics consist in approximating the exact solution $u^\delta$ with a "more explicit" function defined by

$$\tilde{u}^\delta (r, \theta) := \psi(r/\delta) \times \text{far field expansion} + \chi(r) \times \text{near field expansion}$$

$$- \chi(r)\psi(r/\delta) \times \text{matching contribution}$$

**Far field**

$$u^\delta (r, \theta) = u^0 (r, \theta) + a(\delta)\zeta(r, \theta) + \ldots$$

**Near field**

$$u^\delta (\delta \rho, \theta) = b(\delta)Z(\rho, \theta) + \ldots$$
Far field expansion

Ansatz: \( u^\delta(r, \theta) = u^0(r, \theta) + a(\delta) \zeta(r, \theta) + \ldots \)
Far field expansion

**Ansatz:** \( u^\delta(r, \theta) = u^0(r, \theta) + a(\delta) \zeta(r, \theta) + \ldots \)

- Limit field: \( u^0 \in V_{\beta}^{\text{out}}(\Omega) \)
- Corrector: \( a(\delta) \zeta(r, \theta) \)

\[-\text{div}(\sigma^0 \nabla u^0) = f \text{ in } \Omega\]
Far field expansion

Ansatz: \( u^\delta(r, \theta) = u^0(r, \theta) + a(\delta) \zeta(r, \theta) + \ldots \)

Limit field \( u^0 \in \mathcal{V}_\beta^{out}(\Omega) \) and corrector (=?)

\[-\text{div}(\sigma^0 \nabla u^0) = f \text{ in } \Omega\]

\[f = -\text{div}(\sigma^\delta \nabla u^\delta) \approx -\text{div}(\sigma^0 \nabla u^\delta)\]
Far field expansion

Ansatz: \( u^\delta(r, \theta) = u^0(r, \theta) + a(\delta) \zeta(r, \theta) + \ldots \)

limit field
\( u^0 \in V_\beta^{\text{out}}(\Omega) \) and
\(-\text{div}(\sigma^0 \nabla u^0) = f \) in \( \Omega \)

\( f \approx -\text{div}(\sigma^0 \nabla (u^0 + a(\delta) \zeta + \ldots)) \)
Far field expansion

**Ansatz:** \( u^\delta(r, \theta) = u^0(r, \theta) + a(\delta) \zeta(r, \theta) + \ldots \)

- Limit field
- Corrector (=?)

\( u^0 \in V^\text{out}_\beta(\Omega) \) and \(-\text{div}(\sigma^0 \nabla u^0) = f \) in \( \Omega \)

\[
f \approx -\text{div}(\sigma^0 \nabla u^0) - a(\delta)\text{div}(\sigma^0 \nabla \zeta) + \ldots \quad \Rightarrow \quad -\text{div}(\sigma^0 \nabla \zeta) = 0 \text{ in } \Omega
\]

\( \zeta = 0 \text{ on } \partial \Omega, \; \zeta \neq 0 \)

**Recall:** \( V^1_{-\beta}(\Omega) \subset V^\text{out}_\beta(\Omega) \subset V^1_{+\beta}(\Omega) \)

\[
\langle A_* u, v \rangle = \int_\Omega \sigma^0 \nabla u \cdot \nabla v \, dx, \quad \text{where } * = \beta, \text{out.}
\]

**Proposition**
Ker\((A_\beta) = \text{span}\{ \zeta \} \oplus \text{Ker}(A_{-\beta}) \) with

\[
\zeta(r, \theta) = (r^{-i\mu} + c_\zeta r^{+i\mu}) \varphi_p(\theta) + \ldots \quad \text{remainder } \in V^1_{-\beta}(\Omega)
\]
Near field expansion

Next to the rounded corner, we use the fast variable $\rho = r/\delta$ so as to normalize the geometry of the perturbation.

Near field zone $= \text{supp}\{\chi\}$
size $= o(1)$
Near field expansion

Next to the rounded corner, we use the fast variable $\rho = r/\delta$ so as to normalize the geometry of the perturbation.
Near field expansion

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Near field expansion
$\mathbb{R}^2_+ = \overline{\Xi}_1 \cup \overline{\Xi}_2$

$\sigma_N = \sigma_j$ in $\overline{\Xi}_j$
Near field expansion

Next to the rounded corner, we use the fast variable $\rho = r/\delta$ so as to normalize the geometry of the perturbation. The normalized field $U^\delta(\rho, \theta) := u^\delta(\delta \rho, \theta)$ satisfies

\[
\begin{cases}
- \text{div}(\sigma_N \nabla U^\delta) = 0 & \text{in } \mathbb{R}_+^2, \\
U^\delta = 0 & \text{on } \partial\mathbb{R}_+^2 .
\end{cases}
\]

**Ansatz:** $U^\delta(\rho, \theta) = b(\delta) Z(\rho, \delta) + \ldots$
Near field expansion

Next to the rounded corner, we use the fast variable $\rho = r/\delta$ so as to normalize the geometry of the perturbation. The normalized field $U^\delta(\rho, \theta) := u^\delta(\delta \rho, \theta)$ satisfies

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\end{cases}

Ansatz: $U^\delta(\rho, \theta) = b(\delta) Z(\rho, \delta) + \ldots$
Near field expansion

Next to the rounded corner, we use the fast variable $\rho = r/\delta$ so as to normalize the geometry of the perturbation. The normalized field $U^\delta(\rho, \theta) := u^\delta(\delta \rho, \theta)$ satisfies

$$
\begin{cases}
- \text{div}(\sigma_N \nabla Z) = 0 & \text{in } \mathbb{R}^2_+,

Z = 0 & \text{on } \partial \mathbb{R}^2_+.
\end{cases}
$$
Near field expansion

Next to the rounded corner, we use the fast variable $\rho = r/\delta$ so as to normalize the geometry of the perturbation. The normalized field $U^\delta(\rho, \theta) := u^\delta(\delta \rho, \theta)$ satisfies

$$
\begin{cases}
-\text{div}(\sigma_N \nabla Z) = 0 \quad \text{in} \quad \mathbb{R}^2_+,

Z = 0 \quad \text{on} \quad \partial \mathbb{R}^2_+.
\end{cases}
$$

Weighted Sobolev framework:

$$W^1_{\beta}(\mathbb{R}^2_+) = \{ (1 + \rho)^\beta v(\rho, \theta), \; v \in H^1_0(\mathbb{R}^2_+) \}$$

$$\langle \mathcal{A}_\beta u, v \rangle = \int_{\mathbb{R}^2_+} \sigma_N \nabla u \nabla v \; dx \quad u \in W^1_{\beta}(\mathbb{R}^2_+), \; v \in W^{-\beta}_1(\mathbb{R}^2_+)$$

Proposition

For $\beta \in (0, 2)$, we have $\text{Ker}(\mathcal{A}_-\beta) = \text{span}\{Z\} \oplus \text{Ker}(\mathcal{A}_+\beta)$ with

$$Z(\rho, \theta) = (\rho^{+i\mu} + c_Z \rho^{-i\mu}) \varphi_p(\theta) + \ldots \quad \text{remainder} \in W^\text{out}_{\beta}(\mathbb{R}^2_+).$$
Matching

We have defined $\zeta(r, \theta), Z(\rho, \theta)$, but we still have to determine the jauge functions $a(\delta), b(\delta)$. We do so by matching radial expansions.

$$u^\delta(r, \theta) = u^0(r, \theta) + a(\delta) \zeta(r, \theta) + \ldots$$
Matching

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$$u^\delta(r, \theta) = u^0(r, \theta) + a(\delta) \zeta(r, \theta) + \ldots$$

$$= (c_0 + a(\delta)c_\zeta) r^{+i\mu} \varphi_p(\theta) + a(\delta)r^{-i\mu} \varphi_p(\theta) + \ldots \quad \text{for } r \to 0.$$
Matching

We have defined $\zeta(r, \theta)$, $Z(\rho, \theta)$, but we still have to determine the jauge functions $a(\delta), b(\delta)$. We do so by matching radial expansions.

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$$u^\delta(r, \theta) = U^\delta\left(\frac{r}{\delta}, \theta\right) = b(\delta)Z\left(\frac{r}{\delta}, \theta\right) + \ldots$$
Matching

We have defined $\zeta(r, \theta), Z(\rho, \theta)$, but we still have to determine the jauge functions $a(\delta), b(\delta)$. We do so by matching radial expansions.

$$u^\delta (r, \theta) = u^0(r, \theta) + a(\delta) \zeta(r, \theta) + \ldots$$

$$= (c_0 + a(\delta)c_\zeta) r^{+i\mu} \varphi_p(\theta) + a(\delta) r^{-i\mu} \varphi_p(\theta) + \ldots \quad \text{for } r \rightarrow 0.$$

$$u^\delta (r, \theta) = U^\delta \left(\frac{r}{\delta}, \theta\right) = b(\delta) Z \left(\frac{r}{\delta}, \theta\right) + \ldots$$

$$= b(\delta) \left(\frac{r}{\delta}\right)^{+i\mu} \varphi_p(\theta) + b(\delta) c_Z \left(\frac{r}{\delta}\right)^{-i\mu} \varphi_p(\theta) + \ldots \quad \text{for } \frac{r}{\delta} \rightarrow 0.$$
Matching

We have defined $\zeta(r, \theta), Z(\rho, \theta)$, but we still have to determine the jauge functions $a(\delta), b(\delta)$. We do so by matching radial expansions.

\[
\begin{align*}
\tilde{u}(r, \theta) &= u^0(r, \theta) + a(\delta) \zeta(r, \theta) + \ldots \\
&= (c_0 + a(\delta)c_\zeta)r^{+\mu}\varphi_p(\theta) + a(\delta)r^{-\mu}\varphi_p(\theta) + \ldots \quad \text{for } r \to 0.
\end{align*}
\]

\[
\begin{align*}
\tilde{u}(r, \theta) &= U^\delta\left(\frac{r}{\delta}, \theta\right) = b(\delta) Z(\frac{r}{\delta}, \theta) + \ldots \\
&= b(\delta) \left(\frac{r^{+\mu}}{\delta^{+\mu}}\right)\varphi_p(\theta) + b(\delta)c Z(\frac{r^{-\mu}}{\delta^{-\mu}})\varphi_p(\theta) + \ldots \quad \text{for } \frac{r}{\delta} \to 0
\end{align*}
\]
We have defined \( \zeta(r, \theta), Z(\rho, \theta) \), but we still have to determine the jauge functions \( a(\delta), b(\delta) \). We do so by matching radial expansions.

\[
\begin{align*}
    c_0 + a(\delta)c_\zeta &= b(\delta) \delta^{-i\mu} \\
    a(\delta)c_\zeta &= b(\delta)c_Z \delta^{+i\mu}
\end{align*}
\]
Matching

We have defined $\zeta(r, \theta), Z(\rho, \theta)$, but we still have to determine the jauge functions $a(\delta), b(\delta)$. We do so by matching radial expansions.

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\begin{cases}
  c_0 + a(\delta)c_\zeta = b(\delta)\delta^{-i\mu} \\
  a(\delta)c_\zeta = b(\delta)c_Z\delta^{+i\mu}
\end{cases}
\implies
\begin{align*}
  a(\delta) &= \frac{c_0 c_Z}{\delta^{-2i\mu} - c_\zeta c_Z} \\
  b(\delta) &= \frac{c_0 \delta^{-i\mu}}{\delta^{-2i\mu} - c_\zeta c_Z}
\end{align*}
\]

oscillating jauge functions!
Matching

We have defined $\zeta(r, \theta), Z(\rho, \theta)$, but we still have to determine the jauge functions $a(\delta), b(\delta)$. We do so by matching radial expansions.

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\end{cases}
\]

**Proposition**
The coefficients $c_\zeta, c_Z \in \mathbb{C}$ systematically verify $|c_\zeta| = |c_Z| = 1$. oscillating jauge functions!
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We have defined $\zeta(r, \theta), Z(\rho, \theta)$, but we still have to determine the jauge functions $a(\delta), b(\delta)$. We do so by matching radial expansions.

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Proposition
The coefficients $c_\zeta, c_Z \in \mathbb{C}$ systematically verify $|c_\zeta| = |c_Z| = 1$.

Consequence: The matched asymptotic expansion is well defined only under the condition that

\[
\delta \notin J = \{ \delta \in (0, 1) \mid \delta^{-2i\mu} = c_\zeta c_Z \}
\]

Unfortunately $J$ admits $\delta = 0$ as accumulation point.
(Simplified) convergence estimate

**Theorem** [Chesnel,Claeys,Nazarov]

Assume that $\kappa_\sigma \in \mathcal{I}$, and consider a datum $f \in H^{-1}(\Omega)$ such that $f = 0$ near $r = 0$. Assume in addition:

- $\lim_{n \to \infty} \delta_n = 0$ with $\inf_{n \geq 0} |\delta_n^{-2} - c_\zeta c_\zeta| > 0$,
- $\text{Ker}(A_{-\beta}) = \{0\}$,
- $\text{Ker}(\mathcal{A}_{\beta}) = \{0\}$.

Then $\forall \epsilon \in (0, 2), \exists C_\epsilon > 0$ independent of $\delta$ such that

$$
\|u^\delta - \tilde{u}^\delta\|_{H^1_0(\Omega)} \leq C_\epsilon \delta^{2-\epsilon} \|f\|_{H^{-1}(\Omega)} \quad \forall \delta \in (0, 1)
$$

where $\tilde{u}^\delta(r, \theta)$, the matched expansion of $u^\delta(r, \theta)$, is defined by:

$$
\tilde{u}^\delta(r, \theta) = \psi(r/\delta) \left( u^0(r, \theta) + a(\delta) \zeta(r, \theta) \right) + \chi(r) b(\delta) Z(r/\delta, \theta) - \chi(r) \psi(r/\delta) \left( b(\delta)(r/\delta)^{i\mu} + a(\delta)r^{-i\mu} \right) \varphi_p(\theta)
$$
(Simplified) convergence estimate

**Theorem** [Chesnel, Claeys, Nazarov]

Assume that \( \kappa \in \mathcal{I} \), and consider a datum \( f \in H^{-1}(\Omega) \) such that \( f = 0 \) near \( r = 0 \). Assume in addition:

- \( \lim_{n \to \infty} \delta_n = 0 \) with \( \inf_{n \geq 0} |\delta_n^{-2i\mu} - c_\zeta c_z| > 0 \),
- \( \ker(A_{-\beta}) = \{0\} \),
- \( \ker(A_{\beta}) = \{0\} \).

Then \( \forall \epsilon \in (0, 2), \exists C_\epsilon > 0 \) independent of \( \delta \) such that

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+ \chi(r) b(\delta) Z(r/\delta, \theta) \\
- \chi(r) \psi(r/\delta) \left( b(\delta)(r/\delta)^{i\mu} + a(\delta)r^{-i\mu} \right) \varphi_p(\theta)
\]
Numerical illustration

\[ u^\delta \in H_0^1(\Omega^\delta) \text{ satisfies} \]
\[ -\text{div}(\sigma \nabla u^\delta) = f \text{ in } \Omega^\delta \]

with \[ f = \begin{cases} 
1 & \text{for } x < 0 \\
0 & \text{for } x \geq 0 
\end{cases} \]

We represent \( \Re \{ u^\delta \} \) as \( \delta \to 0 \) for two values of \( \kappa_\sigma = \sigma_2/\sigma_1 \):

a) \( \kappa_\sigma = -1.0001 \notin [-1, -1/3] \)

b) \( \kappa_\sigma = -0.9999 \in [-1, -1/3] \)
Thank you for your attention