On convergent series expansions at a corner in linearized elasticity and Inverse problems

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26 August 2013

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1. Introduction

- Motivation

⇒ Fracture Phenomena, Earthquakes

⇒ Difficulty in mathematical analysis

· The domain has singularities

· Solutions of the governing equation has singularity

⇒ It's very important

to analyze precise behavior of the solution at the singular points.

- Applications:

Fracture problems (crack propagation),

Inverse problems (nondestructive evaluation), etc.

2. Problem

[2] M. I. & H. I., Inverse Problems 25(2009) 105005

Domain

$\Omega \subset \mathbb{R}^2$: a bounded domain with Lipschitz boundary

homogeneous isotropic linearized elasticity

D : a polygonal open set with Lipschitz boundary

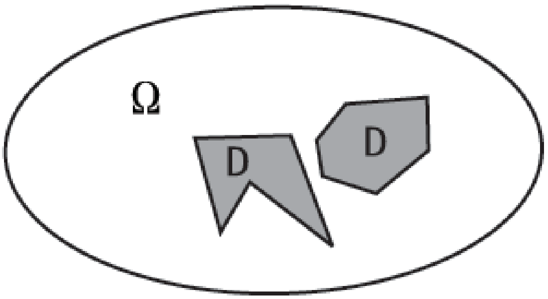
satisfying $\overline{D} \subset \Omega$ and $\Omega \setminus \overline{D}$ is connected

D is polygonal

$\overset{\text{def}}{\Leftrightarrow}$ $D = D_1 \cup \cdots \cup D_m$; each D_j is

a simply connected open set

and polygon; $\overline{D_j} \cap \overline{D_k} = \emptyset$ for $j \neq k$.



Boundary value Problem

$u = (u_i)_{i=1,2}$: the displacement vector

$\varepsilon = (\varepsilon_{ij})_{i,j=1,2}$: the strain tensor

$\sigma = (\sigma_{ij})_{i,j=1,2}$: the stress tensor

- The linearized elasticity eq. (the Navier eq.):

For $u = (u_1, u_2)^{\mathrm{T}}$

$$Au \equiv \frac{\tilde{E}}{2(1+\tilde{\nu})}\triangle u + \frac{\tilde{E}}{2(1-\tilde{\nu})}\nabla(\nabla \cdot u) = 0 \qquad \mathbf{(1)}$$

$$\tilde{E} = \begin{cases} E & \text{(plane stress)} \\ \frac{E}{1-\nu^2} & \text{(plane strain)} \end{cases} \quad \tilde{\nu} = \begin{cases} \nu & \text{(plane stress)} \\ \frac{\nu}{1-\nu} & \text{(plain strain)} \end{cases}$$

E : Young's modulus , ν : Poisson's ratio

$$E > 0 \qquad -1 < \nu < \frac{1}{2}$$

T: the boundary stress operator

$$Tu = \sigma n = \frac{\tilde{\nu}\tilde{E}}{1-\tilde{\nu}^2}(\nabla \cdot u)n + \frac{\tilde{E}}{2(1+\tilde{\nu})}\{\nabla u + (\nabla u)^{\mathrm{T}}\}n$$

$n = (n_1, n_2)^{\mathrm{T}}$: the unit outward normal to $\partial(\Omega \setminus \overline{D})$.
$F(x)k = (k_0 + k_2 x_2, k_1 - k_2 x_1)^{\mathrm{T}}$: rigid displacement with an arbitrary constant vector $k = (k_0, k_1, k_2)^{\mathrm{T}}$.

Boundary value Problem (*)

For given $g \in \{L^2(\partial\Omega)\}^2$, to find a solution u of

$$(*)\begin{cases} Au = 0 & \text{in} \quad \Omega \setminus \overline{D}, \\ Tu = 0 & \text{on} \quad \partial D, \\ Tu = g & \text{on} \quad \partial\Omega. \end{cases}$$

A weak solution $u \in \{H^1(\Omega \setminus \overline{D})\}^2$ exists under the compatibility condition; $\forall k \in \mathbb{R}^3$, $\int_{\partial\Omega} g \cdot F(x)k \, \mathrm{d}S = 0$.

3. A convergent expansion near a corner

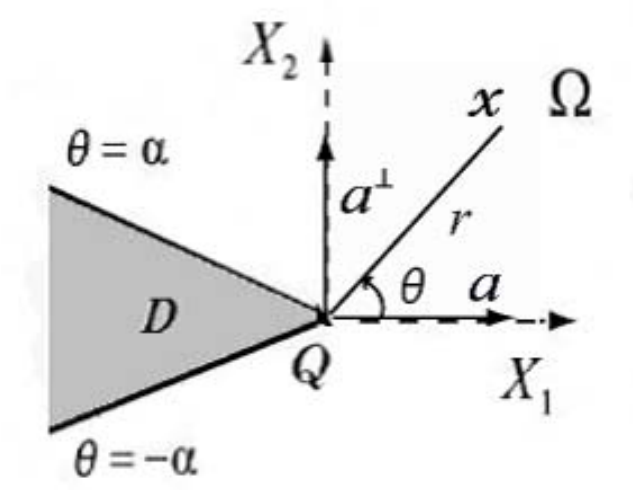
References

Costabel,M. & Dauge,M., Comm.PDE., 19(1994) 1677–1726

Grisvard, P., Elliptic problems in nonsmooth domains, 1985

Kondrat'ev, V. A. & Oleinik, O. A., Russian Math. Surveys, 38(1983), 3 − 76

A local coordinate system and some notation



Let Q be a vertex of D. $R > 0$.

$B(Q, R) = \{x \in \mathbb{R}^2 \mid |x - Q| < R\}$.

$D_R = B(Q, R) \cap (\Omega \setminus \overline{D})$.

$(X_1, X_2) = (r\cos\theta, r\sin\theta)$,

$x = Q + r(\cos\theta\, a + \sin\theta\, a^\perp)$

$0 < r < R$, $-\alpha < \theta < \alpha$, $\frac{\pi}{2} < \alpha < \pi$.

Step 1: Construction of Airy's stress function

Lemma 1 (Generalized Poincaré Lemma)

Assume $\zeta \in \{L^2(D_R)\}^2$.
If $\nabla^{\perp}\cdot\zeta \equiv (\frac{\partial}{\partial X_2}, -\frac{\partial}{\partial X_1})^{\mathrm{T}}\cdot\zeta = 0$ in D_R, then there exists $\tilde{\zeta} \in H^1(D_R)$ such that $\zeta = \nabla\tilde{\zeta}$.

From $\sigma_{ij,j} = 0$ $(Au = 0)$, $\sigma_{12} = \sigma_{21}$ and Lemma 1 Airy Stress Function $\exists U \in H^2(D_R)$ satisfying

$$\sigma_{11} = \frac{\partial^2 U}{\partial X_2^2}, \quad \sigma_{22} = \frac{\partial^2 U}{\partial X_1^2}, \quad \sigma_{12} = -\frac{\partial^2 U}{\partial X_1 \partial X_2}.$$

$$\begin{cases} Au = 0 \ \text{ in } \ \Omega \setminus \overline{D} \\ Tu = 0 \ \text{ on } \ \partial D \end{cases} \Rightarrow \begin{cases} \triangle^2 U = 0 \ \text{ in } \ D_R \\ U = \frac{\partial U}{\partial n} = 0 \ \text{ on } \ \partial D \cap \partial D_R \end{cases}$$

Step 2: The Mellin transform

$$\hat{U}(s,\theta) = \int_0^R r^{s-1} U(r,\theta)\eta(r)\,\mathrm{d}r, \qquad s \in \mathbb{C},$$

$$\hat{U}_d(s,\theta) = \int_0^R r^{s-1+d} U(r,\theta)\eta^{(d)}(r)\,\mathrm{d}r, \qquad d = 1,2,3,4.$$

$$C^\infty[0,\infty) \ni \eta(r) = \begin{cases} 1 & \text{for } 0 \le r < \rho, \\ 0 & \text{for } \quad\; r > R \end{cases} \qquad (\rho < R)$$

$\Rightarrow$ **A BVP (†) for $\hat{U}(s,\theta)$ in the transformed domain;**

$$(\dagger)\begin{cases} \left(\dfrac{\mathrm{d}^2}{\mathrm{d}\theta^2} + s^2\right)\left(\dfrac{\mathrm{d}^2}{\mathrm{d}\theta^2} + (s+2)^2\right)\hat{U} = M(s,\theta), \; -\alpha < \theta < \alpha, \\ \hat{U}(s,\theta) = \dfrac{\partial \hat{U}}{\partial\theta}(s,\theta) = 0 \qquad \theta = \pm\alpha, \end{cases}$$

$$\begin{aligned} M(s,\theta) \;=\; -\Big[&(4s+6)\frac{\partial^2 \hat{U}_1}{\partial\theta^2} + (4s^3+18s^2+24s+9)\hat{U}_1 + 2\frac{\partial^2 \hat{U}_2}{\partial\theta^2} \\ &+(6s^2+24s+23)\hat{U}_2 + (4s+10)\hat{U}_3 + \hat{U}_4\Big]. \end{aligned}$$

Step 3: Green's function

Then the solution of (†) is given by

$$\hat{U}(s,\theta) = \int_{-\alpha}^{\alpha} G(\theta,\varphi,s) M(s,\varphi)\, \mathrm{d}\varphi, \qquad (2)$$

where $G(\theta,\varphi,s)$ is Green's function and explicitly described by

$$\begin{aligned} G(\theta,\varphi,s) &= -\frac{1}{8s(s+1)(s+2)\cos s\alpha \cos(s+2)\alpha} \\ &\quad \times \Big\{ (s+2)\cos(s+2)\alpha \sin s\{\alpha \pm (\theta-\varphi)\} \\ &\qquad -s\cos s\alpha \sin(s+2)\{\alpha \pm (\theta-\varphi)\} + \frac{b_1(s,\theta)b_1(s,\varphi)}{h_1(\alpha,s)} \\ &\qquad + \frac{s(s+2)a_2(s,\theta)a_2(s,\varphi)}{h_2(\alpha,s)} \Big\}. \qquad (3) \end{aligned}$$

the upper sign for $\theta < \varphi$, the lower sign for $\theta > \varphi$

In (3) the quantities are expressed by

$$
\begin{aligned}
a_2(s,\theta) &= \cos(s+2)\alpha\cos s\theta - \cos s\alpha\cos(s+2)\theta,\\
b_1(s,\theta) &= s\cos s\alpha\sin(s+2)\theta - (s+2)\cos(s+2)\alpha\sin s\theta,\\
h_1(\alpha,s) &= (s+1)\sin 2\alpha - \sin 2(s+1)\alpha,\\
h_2(\alpha,s) &= (s+1)\sin 2\alpha + \sin 2(s+1)\alpha.
\end{aligned}
$$

Poles of G

- $0,\ -1,\ -2,\ \cos s\alpha = 0,\ \cos(s+2)\alpha = 0 \Rightarrow$ Removable
- 0 and $-2 \Rightarrow$ Simple poles,

  when $\alpha = \alpha_0$ such that $2\alpha_0 = \tan 2\alpha_0$, $\left(\alpha_0 \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)\right)$.
- In $\mathrm{Re} s < -1$, roots of

$$h_1:\ \mathrm{Re} s_{1,1} \geq \mathrm{Re} s_{1,2} \geq \cdots \geq \mathrm{Re} s_{1,m} = \mathrm{Re} \bar{s}_{1,m+1} \geq \cdots$$

$$h_2:\ \mathrm{Re} s_{2,1} \geq \mathrm{Re} s_{2,2} \geq \cdots \geq \mathrm{Re} s_{2,m} = \mathrm{Re} \bar{s}_{2,m+1} \geq \cdots$$

Analysis of transcendental equations $h_1 = 0,\ h_2 = 0$

Lemma 2 For $\frac{\pi}{2} < \alpha < \pi$

1. $-2 < \mathrm{Re}\, s_{2,1} < -\frac{3}{2} \quad \left(\frac{\pi}{2} < \alpha < \alpha_0\right)$

   $-2 < \mathrm{Re}\, s_{1,1} < -\frac{\pi}{2\alpha} - 1 < \mathrm{Re}\, s_{2,1} < -\frac{3}{2} \quad (\alpha_0 < \alpha < \pi)$

2. In $\mathrm{Re}\, s < -1$ the lines $\mathrm{Re}\, s = -k\frac{\pi}{2\alpha} - 1 \ (k = 1, 2, \ldots)$ contain no roots of each $h_j(\alpha, s) = 0$.

3. The strip $-(k+1)\dfrac{\pi}{2\alpha} - 1 < \mathrm{Re}\, s < -k\dfrac{\pi}{2\alpha} - 1$

   contains two roots of $h_1(\alpha, s) = 0$ $(k :$ odd$)$,

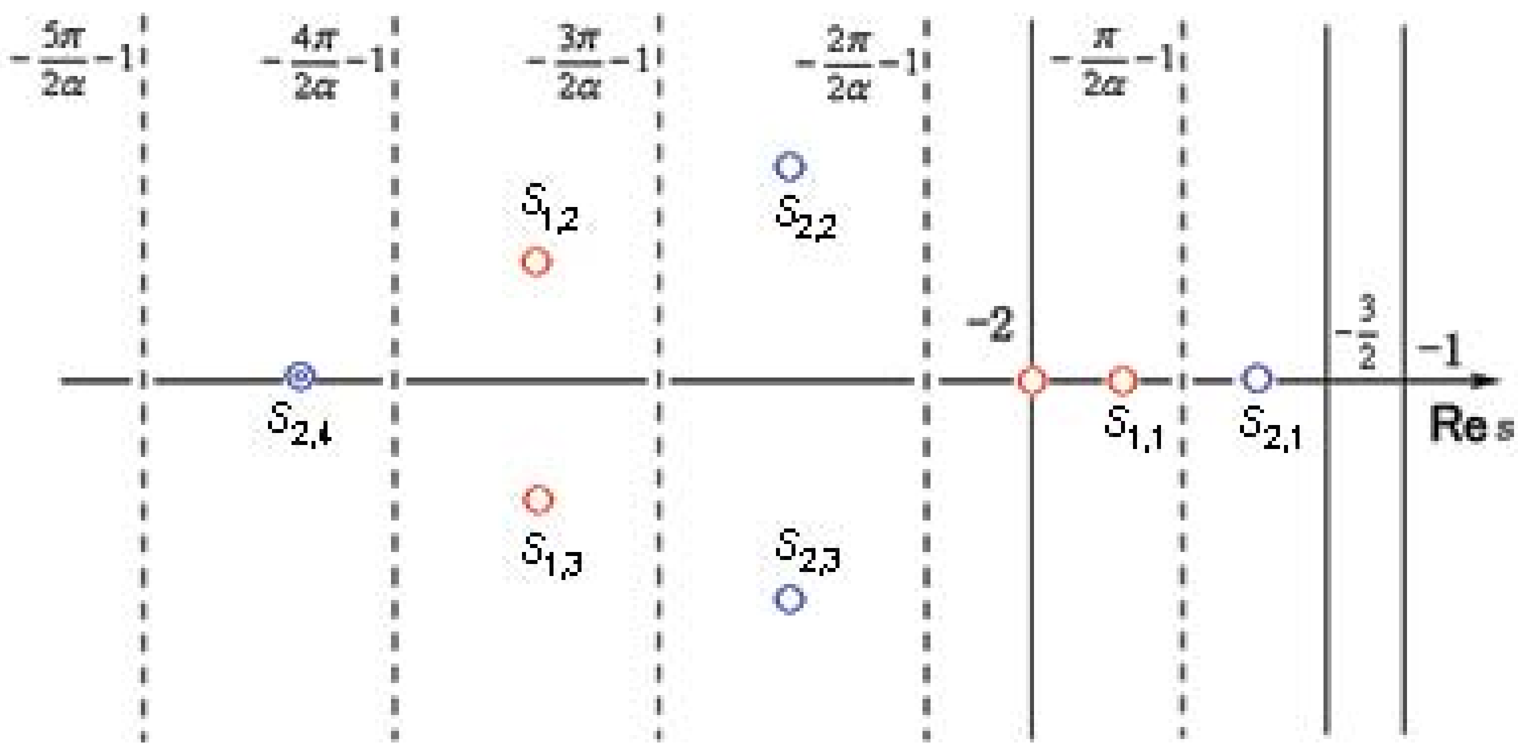
   two roots of $h_2(\alpha, s) = 0$ $(k :$ even$)$

   (including complex conjugate and multiplicity).

4. The multiplicity of the roots of each $h_j(\alpha, s) = 0$ is not greater than two. Moreover, the double roots are all real and at most two.

5. The roots in $\mathrm{Re}s < -2$ are not integer.

6. The roots in $s(s+2) \neq 0$ satisfy $\sin s\alpha \cos s\alpha \neq 0$.



Estimate of G

Lemma 3

1. For given $L > 1$ $\exists c_0(L), c_1(L), c_2(L) > 0$ (depend on L) s. t. if $1 < |\mathrm{Re}s| \le L$ and $|\mathrm{Im}s| \ge c_0(L)$, then

$$|h_j| \ge c_2(L) e^{2|\mathrm{Im}s|\alpha} \quad (j = 1, 2), \quad |G| \le c_1(L)|s|^{-1}.$$

2. $\exists c_0, c_1, c_2 > 0$ (independent of k) s. t.
if $\mathrm{Re}s = -\dfrac{k\pi}{2\alpha} - 1$ $(k \in \mathbb{N})$ and $|\mathrm{Im}s| \ge c_0$, then

$$|h_j| \ge c_2 e^{2|\mathrm{Im}s|\alpha} \quad (j = 1, 2), \quad |G| \le c_1|s|^{-1};$$

if $\mathrm{Re}s = -\dfrac{k\pi}{2\alpha} - 1$ $(k \in \mathbb{N})$ and $|\mathrm{Im}s| < c_0$, then

$$|h_j| \ge c_2 k \quad (j = 1, 2), \quad |G| \le c_1 k^{-2}.$$

Step 4: <u>The inverse Mellin transform</u>

(2) for the solution $\hat{U}(s,\theta)$ is valid for $\mathrm{Re}s > 0$.

$U\eta \in H_0^2(D_R)$, $U\eta$ also belongs to a weighted Sobolev space $P^2(D_R) = \{f | r^{-2+|\tilde{d}|} D^{\tilde{d}} f \in L^2(D_R)\}$ for $|\tilde{d}| \le 2$

$$\begin{aligned}
\left|\int_0^R r^{s-1} U(r,\theta)\eta(r)\,\mathrm{d}r\right| &\le \left|\int_0^R r^{\mathrm{Re}s+1}(r^{-2}U(r,\theta)\eta(r))\,\mathrm{d}r\right| \\
&\le \int_0^R r^{\mathrm{Re}s+1} r^{-\frac{1}{2}} \left| r^{-2+\frac{1}{2}} U(r,\theta)\eta(r))\right| \,\mathrm{d}r \\
&\le \|r^{\mathrm{Re}s}\|_{L^2(D_R)} \cdot \|r^{-2}U\eta\|_{L^2(D_R)} < \infty
\end{aligned}$$

for $\mathrm{Re}s > -1$.

Thus, for $r < \rho$

$$U(r,\theta) = \frac{1}{2\pi i}\int_{\ell_1 - i\infty}^{\ell_1 + i\infty} r^{-s}\hat{U}(s,\theta)\,\mathrm{d}s \tag{4}$$

with $\ell_1 = -1+\epsilon$, where $\epsilon > 0$ (arbitrarily small).

Step 5: Changing the integration path

First, one can see that $|\hat{U}_d(s,\theta)|$ is rapidly decreasing as $|s| \to \infty$ and therefore $|M(s,\theta)|$ is also.
Lemma 3-1 $\Rightarrow$ as $|\mathrm{Im}s| \to \infty$,

$$|\hat{U}(s,\theta)| \leq \int_{-\alpha}^{\alpha} |G(\theta,\varphi,s)||M(s,\varphi)|\,\mathrm{d}\varphi \leq c\int_{-\alpha}^{\alpha} |s|^{-1}|s|^{-\infty}\,\mathrm{d}\varphi \to 0. \quad (5)$$

Meromorphy of $\hat{U}(s,\theta)$, (5) and Lemma 2-2 $\Rightarrow$

$$\begin{aligned}
&U(r,\theta)\\
&= \frac{1}{2\pi i}\int_{\ell_1-i\infty}^{\ell_1+i\infty} r^{-s}\hat{U}(s,\theta)\,\mathrm{d}s\\
&= \frac{1}{2\pi i}\int_{\ell_2-i\infty}^{\ell_2+i\infty} r^{-s}\hat{U}(s,\theta)\,\mathrm{d}s + \lim_{M\to\infty}\left\{\frac{1}{2\pi i}\int_{\ell_2+iM}^{\ell_1+iM} r^{-s}\hat{U}(s,\theta)\,\mathrm{d}s\right\}\\
&\quad + \lim_{M'\to+\infty}\left\{\frac{1}{2\pi i}\int_{\ell_1-iM'}^{\ell_2-iM'} r^{-s}\hat{U}(s,\theta)\,\mathrm{d}s\right\} + \sum_{\ell_2<\mathrm{Re}s<\ell_1}\mathrm{Res}(r^{-s}\hat{U}(s,\theta))\\
&= \frac{1}{2\pi i}\int_{\ell_2-i\infty}^{\ell_2+i\infty} r^{-s}\hat{U}(s,\theta)\,\mathrm{d}s + \sum_{\ell_2<\mathrm{Re}s<\ell_1}\mathrm{Res}(r^{-s}\hat{U}(s,\theta)),
\end{aligned}$$

where $\ell_2 \equiv -\frac{\pi}{2\alpha} - 1$ and $\mathrm{Res}(r^{-s}\hat{U}(s,\theta))$ is the residue.

Repeating this process yields

$$U(r,\theta) = \sum_{\mathrm{Re}\, s > \ell_{k+1}} \mathrm{Res}(r^{-s}\hat{U}(s,\theta)) + \frac{1}{2\pi i}\int_{\ell_{k+1}-i\infty}^{\ell_{k+1}+i\infty} r^{-s}\hat{U}(s,\theta)\,\mathrm{d}s,$$

where $\ell_{k+1} = -k\frac{\pi}{2\alpha} - 1 \ (k = 1,2,3,\cdots)$.
From Lemma 3-2 it holds that for $r < \rho < 1$ that

$$\begin{aligned}
&\left|\int_{\ell_{k+1}-i\infty}^{\ell_{k+1}+i\infty} r^{-s}\hat{U}(s,\theta)\,\mathrm{d}s\right| \\
&\le \ r^{-\ell_{k+1}}\left(\int_{\ell_{k+1}-i\infty}^{\ell_{k+1}-ic_0} |\hat{U}|\,\mathrm{d}s + \int_{\ell_{k+1}-ic_0}^{\ell_{k+1}+ic_0} |\hat{U}|\,\mathrm{d}s + \int_{\ell_{k+1}+ic_0}^{\ell_{k+1}+i\infty} |\hat{U}|\,\mathrm{d}s\right) \\
&\le \ cr^{-\ell_{k+1}}\left(\int_{\ell_{k+1}-i\infty}^{\ell_{k+1}-ic_0} |s|^{-1}|s|^{-\infty}\,\mathrm{d}s + \int_{\ell_{k+1}-ic_0}^{\ell_{k+1}+ic_0} |k|^{-2}|s|^{-\infty}\,\mathrm{d}s + \int_{\ell_{k+1}+ic_0}^{\ell_{k+1}+i\infty} |s|^{-1}|s|^{-\infty}\,\mathrm{d}s\right) \\
&\quad \to 0 \quad \text{as} \quad k \to \infty.
\end{aligned}$$

$\Rightarrow$ an uniformly in D_ρ and in $H^2(D_\rho)$ convergent infinite series: $(j = 1,2,\ m = 1,2,3,\cdots)$

$$U(r,\theta) = \sum_{j,m} r^{-s_{j,m}} c_{j,m} a_j(\theta) + \sum_{j,m} \frac{\mathrm{d}}{\mathrm{d}s}\left(r^{-s}\tilde{c}_{j,m}(s) a_j(s,\theta)\right)\Big|_{s=\tilde{s}_{j,m}}.$$

Residue

$$\mathrm{Res}(r^{-s}\hat{U}(s,\theta))|_{s=s_{j,m}} = r^{-s_{j,m}} c_{j,m} a_j(\theta),$$

where

$$c_{j,m} = \frac{-1}{8(s_{j,m}+1)I_{j,m}} \int_{-\alpha}^{\alpha} M(s_{j,m},\theta)a_j(\theta)\ \mathrm{d}\theta$$

with for $j = 1$

$$\begin{aligned}
I_{1,m} &= D_{1,m} \sin s_{1,m}\alpha \sin(s_{1,m}+2)\alpha, \\
D_{1,m} &= \sin 2\alpha - 2\alpha \cos 2(s_{1,m}+1)\alpha, \\
a_1(\theta) &= a_1(s_{1,m},\theta) \\
&\equiv \sin(s_{1,m}+2)\alpha \sin s_{1,m}\theta \\
&\quad - \sin s_{1,m}\alpha \sin(s_{1,m}+2)\theta
\end{aligned}$$

and for $j = 2$

$$\begin{aligned}
I_{2,m} &= D_{2,m} \cos s_{2,m}\alpha \cos(s_{2,m}+2)\alpha, \\
D_{2,m} &= \sin 2\alpha + 2\alpha \cos 2(s_{2,m}+1)\alpha, \\
a_2(\theta) &= a_2(s_{2,m},\theta).
\end{aligned}$$

$$\mathrm{Res}(r^{-s}\hat{U}(s,\theta))|_{s=\tilde{s}_{j,m}} = \frac{\mathrm{d}}{\mathrm{d}s}\left(r^{-s}\tilde{c}_{j,m}(s)a_j(s,\theta)\right)\Big|_{s=\tilde{s}_{j,m}},$$

where

$$\tilde{c}_{j,m}(s) = \frac{-1}{8(s+1)\tilde{I}_{j,m}} \int_{-\alpha}^{\alpha} M(s,\theta)a_j(s,\theta)\ \mathrm{d}\theta$$

with for $j = 1$

$$\begin{aligned}
\tilde{I}_{1,m} &= \tilde{D}_{1,m} \sin s\alpha \sin(s+2)\alpha, \\
\tilde{D}_{1,m} &= \frac{h_1(\alpha,s)}{(s-\tilde{s}_{1,m})^2}
\end{aligned}$$

and for $j = 2$

$$\begin{aligned}
\tilde{I}_{2,m} &= \tilde{D}_{2,m} \cos s\alpha \cos(s+2)\alpha, \\
\tilde{D}_{2,m} &= \frac{h_2(\alpha,s)}{(s-\tilde{s}_{2,m})^2}.
\end{aligned}$$

The relation between the displacement u and U is

$$\begin{cases} \dfrac{\partial u_r}{\partial r} = \dfrac{1}{\tilde{E}}\left\{\dfrac{1}{r}\dfrac{\partial U}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 U}{\partial \theta^2} - \tilde{\nu}\dfrac{\partial^2 U}{\partial r^2}\right\}, \ (u_r + iu_\theta) = e^{-i\theta}(u_1 + iu_2), \\ \qquad \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} = \dfrac{1}{\tilde{E}}\left\{\dfrac{\partial^2 U}{\partial r^2} - \tilde{\nu}\dfrac{1}{r}\dfrac{\partial U}{\partial r} - \tilde{\nu}\dfrac{1}{r^2}\dfrac{\partial^2 U}{\partial \theta^2}\right\} - \dfrac{u_r}{r}, \\ \qquad \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} + \dfrac{\partial u_\theta}{\partial r} - \dfrac{u_\theta}{r} = \dfrac{2(1+\tilde{\nu})}{\tilde{E}}\left\{\dfrac{1}{r^2}\dfrac{\partial U}{\partial \theta} - \dfrac{1}{r}\dfrac{\partial^2 U}{\partial r \partial \theta}\right\}. \end{cases}$$

Proposition 4 When $\alpha \neq \alpha_0$, $\exists A_{j,m}$, $B_{j,m}(s) \in \mathbb{C}$ $(j = 1, 2, m = 1, 2, 3, \cdots)$ and a constant vector k s. t.

$$\begin{aligned} u(r,\theta) &= \sum_{j,m} A_{j,m} r^{-s_{j,m}-1}\Phi_j(s_{j,m},\theta) \\ &\quad + \sum_{j,m} \frac{\partial}{\partial s}\left(r^{-s-1}B_{j,m}(s)\Phi_j(s,\theta)\right)\Big|_{s=\tilde{s}_{j,m}} + F(X)k, \end{aligned}$$

where the coefficients can be given by

$$A_{j,m} = \frac{c_{j,m}}{\tilde{E}}, \qquad B_{j,m}(s) = \frac{\tilde{c}_{j,m}(s)}{\tilde{E}}.$$

The series is convergent, absolutely in $H^1(D_\rho)$ and uniformly in D_ρ.
Here Φ_j is the following vector field :

$$\Phi_1(s,\theta) = \begin{pmatrix} (1+\tilde{\nu})s\sin(s+2)\alpha\sin(s+1)\theta + (\tilde{\nu}-3)\sin s\alpha\sin(s+1)\theta \\ \quad -(1+\tilde{\nu})(s+1)\sin s\alpha\sin(s+3)\theta \\ -(1+\tilde{\nu})s\sin(s+2)\alpha\cos(s+1)\theta + (\tilde{\nu}-3)\sin s\alpha\cos(s+1)\theta \\ \quad +(1+\tilde{\nu})(s+1)\sin s\alpha\cos(s+3)\theta \end{pmatrix},$$

$$\Phi_2(s,\theta) = \begin{pmatrix} (1+\tilde{\nu})s\cos(s+2)\alpha\cos(s+1)\theta + (\tilde{\nu}-3)\cos s\alpha\cos(s+1)\theta \\ \quad -(1+\tilde{\nu})(s+1)\cos s\alpha\cos(s+3)\theta \\ (1+\tilde{\nu})s\cos(s+2)\alpha\sin(s+1)\theta - (\tilde{\nu}-3)\cos s\alpha\sin(s+1)\theta \\ \quad -(1+\tilde{\nu})(s+1)\cos s\alpha\sin(s+3)\theta \end{pmatrix}.$$

Remark 1 Since $U(r,\theta)$ is real valued, $A_{j,m}$ and $B_{j,m}(s)$ have the following properties:

1. If $s_{j,m} \in \mathbb{R}$, then the corresponding $A_{j,m} \in \mathbb{R}$
2. If $s_{j,m} \in \mathbb{C} \setminus \mathbb{R}$ s.t. $s_{j,m+1} = \overline{s}_{j,m}$, then $A_{j,m+1} = \overline{A}_{j,m}$
3. $B_{j,m}(\tilde{s}_{j,m}),\ B'_{j,m}(\tilde{s}_{j,m}) \in \mathbb{R}$

Remark 2 If $\alpha = \alpha_0$, then $s = -2$ becomes a simple pole of $\hat{U}(s,\theta)$ and the following additional term $u^*(r,\theta)$ should be taken into account to the series in Proposition 4 ;

$$u^*(r,\theta) = rA^* \begin{pmatrix} (1+\tilde{\nu})(1+\cos 2\alpha_0)\sin\theta + 2(1-\tilde{\nu})\cos 2\alpha_0\theta\cos\theta \\ +4\log r\cos 2\alpha_0\sin\theta \\ (1+\tilde{\nu})(1-\cos 2\alpha_0)\cos\theta + 2(1-\tilde{\nu})\cos 2\alpha_0\theta\sin\theta \\ -4\log r\cos 2\alpha_0\cos\theta \end{pmatrix}$$

with a real constant A^*.

4. Application

Inverse Problem(Reconstruction) :

"Extract information(position, size, shape) of unknown polygonal D from measured data on $\partial\Omega$"

$\Downarrow$ The enclosure method by Ikehata

The Main Result : One can extract the convex hull of D uniquely from a single set of the surface displacement field and traction on $\partial\Omega$!!

We do not require any other assumptions for the unknown D and boundary data.

Mathematical tools of the Enclosure method

Let S^1 denote the unit circle.

• The support function h_D:

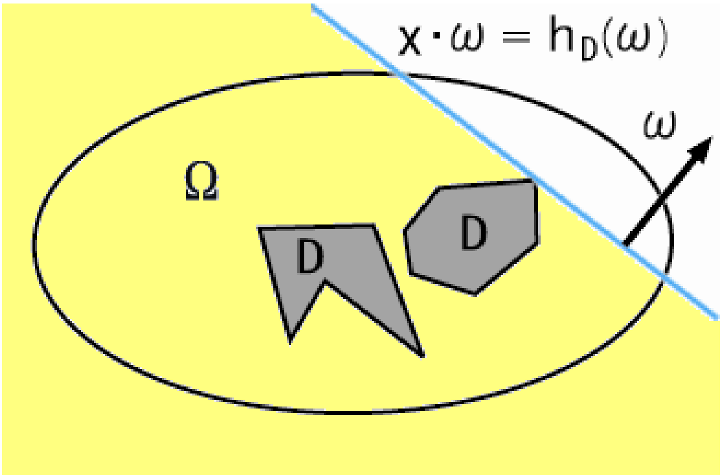
$$h_D(\omega) = \sup_{x \in D} x \cdot \omega, \qquad \omega \in S^1$$

$\Downarrow$

One can extract

the convex hull of D as

$$\bigcap_{\omega \in S^1} \{x \in \mathbb{R}^2 \mid x \cdot \omega < h_D(\omega)\}.$$



•**Assumptions for ω**

(A): $\{x \in \mathbb{R}^2 \mid x \cdot \omega = h_D(\omega)\} \cap \partial D$: **only one point Q.**

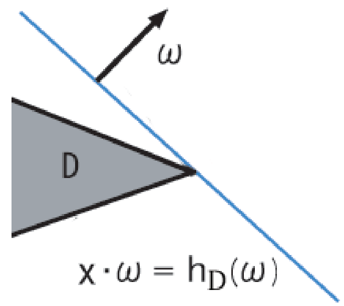


Fig. (A) and (A')

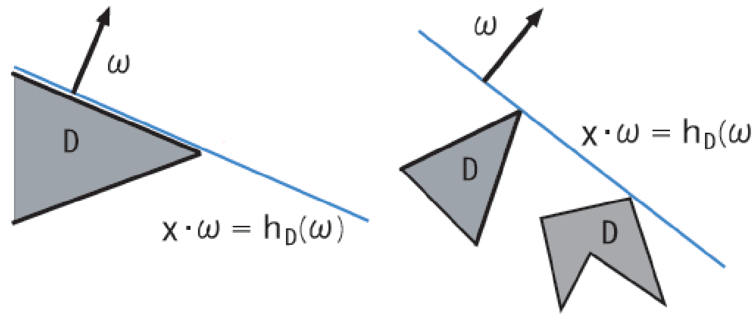


Fig. not (A)

(A'): ω **satisfies (A) and that the interior angle bisector of D at the point Q is not perpendicular to the line $x \cdot \omega = h_D(\omega)$**

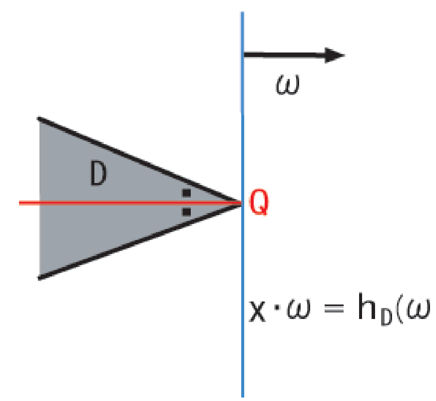


Fig. (A), not (A')

Remark 3

For given polygonal D the set of directions violating (A') is a finite set.

Theorem Assume that D is polygonal. Let u be not a rigid displacement. Under the assumption (A')

$$h_D(\omega) = \lim_{\tau \longrightarrow \infty} \frac{1}{\tau} \log |I_\omega(\tau, 0)| \, .$$

• The special solution v of $Av = 0$ in $\mathbb{R}^2$: for $\tau > 0$

$$v(x) = (\omega + i\omega^\perp) e^{\tau x \cdot (\omega + i\omega^\perp)}$$

• the indicator function

$$I_\omega(\tau, t) = e^{-\tau t} \left\{ \int_{\partial\Omega} (g \cdot v - u \cdot Tv) \, \mathrm{d}S \right\}$$

for $\tau > 0$ and $t \in \mathbb{R}$, where u is a weak solution of $(*)$.

Importance of the convergent series expansion

$$\frac{\log|I_\omega(\tau,0)|}{\tau} = h_D(\omega) + \frac{\log|I_\omega(\tau,h_D(\omega))|}{\tau}$$

Indicator ft. : $I_\omega(\tau,t) = e^{-\tau t}\left\{\int_{\partial\Omega} (Tu\cdot v - u\cdot Tv)\ \mathrm{d}S\right\}$

$\Downarrow$ integrating by parts

$$I_\omega(\tau,t) = e^{-\tau t}\int_{\partial D} Tv\cdot(u - F(x)k)\ \mathrm{d}S \quad \forall k\in\mathbb{R}^3$$

$\Downarrow$ $\tau\to\infty$

$$I_\omega(\tau,h_D(\omega)) = e^{-\tau h_D(\omega)}\int_{\partial D\cap B_R(Q)} Tv\cdot u\ \mathrm{d}S + O(\tau e^{-\tau\delta})$$

5. Conclusion

Assumptions:

1. $\Omega \subset \mathbb{R}^2$ is a bounded homogeneous isotropic linearized elastic plate.
2. $\partial\Omega$ is Lipschitz.
3. D is an unknown open subset of Ω and polygonal.
4. u is a weak solution of $(*)$ and not a rigid displacement.

Convergent series expansion $\Downarrow$ Enclosure method

Result: We can extract the convex hull of D uniquely from a single set of boundary data (u, Tu)!!