Instance optimality of the adaptive maximum strategy

Rob Stevenson
Korteweg-de Vries Institute for Mathematics
University of Amsterdam

joint work with

Christian Kreuzer (Uni Bochum) and Lars Diening (Uni München)

Topics

- Adaptive finite element methods
- Known convergence results
- Newest vertex bisection
- Ingredients of a proof of instance optimality of AFEM with (a modified) maximum marking strategy

(A)FEM

Elliptic bvp

$$\begin{cases} -\triangle u &= f \quad \text{on polygon } \Omega \subset \mathbb{R}^2 \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

Var. form.: Find $u \in H_0^1(\Omega)$ s.t. $\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv$ $(v \in H_0^1(\Omega))$.

(A)FEM

Elliptic bvp

$$\begin{cases} -\triangle u &= f \quad \text{on polygon } \Omega \subset \mathbb{R}^2 \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

Var. form.: Find $u \in H_0^1(\Omega)$ s.t. $\int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v \; (v \in H_0^1(\Omega))$.

FEM $\mathcal{T}$ triangulation of Ω, $\mathbb{V}(\mathcal{T}) := \{u \in H_0^1(\Omega) : u|_T \in P_1(T) \, (T \in \mathcal{T})\}$.
Find $u_\mathcal{T} \in \mathbb{V}(\mathcal{T})$ s.t. $\int_\Omega \nabla u_\mathcal{T} \cdot \nabla v_\mathcal{T} = \int_\Omega f v_\mathcal{T} \; (v_\mathcal{T} \in \mathbb{V}(\mathcal{T}))$.

Then $|u - u_\mathcal{T}|_{H^1(\Omega)} = \inf_{v_\mathcal{T} \in \mathbb{V}(\mathcal{T})} |u - v_\mathcal{T}|_{H^1(\Omega)} \lesssim h_\mathcal{T} |u|_{H^2(\Omega)}$, $h_\mathcal{T} := \max_{T \in \mathcal{T}} h_T$.

If $\max_{T \in \mathcal{T}} h_T \eqsim \min_{T \in \mathcal{T}} h_T$ (quasi-uniform mesh), then $\# \mathcal{T} \eqsim h_\mathcal{T}^{-2}$, and so $|u - u_\mathcal{T}|_{H_0^1(\Omega)} \lesssim (\# \mathcal{T})^{-\frac{1}{2}}$. This rate $\frac{1}{2}$ generally the best possible.

(A)FEM

Elliptic bvp

$$\begin{cases} -\triangle u &= f \quad \text{on polygon } \Omega \subset \mathbb{R}^2 \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

Var. form.: Find $u \in H_0^1(\Omega)$ s.t. $\int_\Omega \nabla u \cdot \nabla v = \int_\Omega fv \ (v \in H_0^1(\Omega))$.

FEM $\mathcal{T}$ triangulation of Ω, $\mathbb{V}(\mathcal{T}) := \{u \in H_0^1(\Omega) : u|_T \in P_1(T)\,(T \in \mathcal{T})\}$. Find $u_\mathcal{T} \in \mathbb{V}(\mathcal{T})$ s.t. $\int_\Omega \nabla u_\mathcal{T} \cdot \nabla v_\mathcal{T} = \int_\Omega f v_\mathcal{T} \ (v_\mathcal{T} \in \mathbb{V}(\mathcal{T}))$.

Then $|u - u_\mathcal{T}|_{H^1(\Omega)} = \inf_{v_\mathcal{T} \in \mathbb{V}(\mathcal{T})} |u - v_\mathcal{T}|_{H^1(\Omega)} \lesssim h_\mathcal{T} |u|_{H^2(\Omega)}$, $h_\mathcal{T} := \max_{T \in \mathcal{T}} h_T$.

If $\max_{T\in\mathcal{T}} h_T \eqsim \min_{T\in\mathcal{T}} h_T$ (quasi-uniform mesh), then $\#\mathcal{T} \eqsim h_\mathcal{T}^{-2}$, and so $|u - u_\mathcal{T}|_{H_0^1(\Omega)} \lesssim (\#\mathcal{T})^{-\frac{1}{2}}$. This rate $\frac{1}{2}$ generally the best possible.

However not attained with quasi-uniform meshes for Ω with re-entrant corners because $u \notin H^2(\Omega)$.

Optimal rate can be retrieved by a *proper* local refinement near the re-entrant corners. To find such a triangulation "automatically" one applies AFEM.

A posteriori error estimator With $\mathcal{S}(\mathcal{T})$ set of sides, for $S \in \mathcal{S}(\mathcal{T})$,

$$\mathcal{E}_{\mathcal{T}}^2(S) := h_S \big\| [\![\nabla u_{\mathcal{T}}]\!] \big\|_{L^2(S)}^2 + \sum_{\{T \in \mathcal{T} : T \supset S\}} h_T^2 \|f\|_{L^2(T)}^2$$

For $\tilde{\mathcal{S}} \subset \mathcal{S}(\mathcal{T})$, $\mathcal{E}_{\mathcal{T}}^2(\tilde{\mathcal{S}}) := \sum_{S \in \tilde{\mathcal{S}}} \mathcal{E}_{\mathcal{T}}^2(S)$. Then

squared *total* error $:= |u - u_{\mathcal{T}}|_{H^1(\Omega)}^2 + \underbrace{\sum_{T \in \mathcal{T}} h_T^2 \|f - \bar{f}\|_{L_2(T)}^2}_{\mathrm{osc}^2(\mathcal{T}) :=} \eqsim \mathcal{E}_{\mathcal{T}}^2(\mathcal{S}(\mathcal{T}))$.

A posteriori error estimator With $\mathcal{S}(\mathcal{T})$ set of sides, for $S \in \mathcal{S}(\mathcal{T})$,

$$\mathcal{E}_{\mathcal{T}}^2(S) := h_S \big\| [\![\nabla u_{\mathcal{T}}]\!] \big\|_{L^2(S)}^2 + \sum_{\{T \in \mathcal{T} : T \supset S\}} h_T^2 \|f\|_{L^2(T)}^2$$

For $\tilde{\mathcal{S}} \subset \mathcal{S}(\mathcal{T})$, $\mathcal{E}_{\mathcal{T}}^2(\tilde{\mathcal{S}}) := \sum_{S \in \tilde{\mathcal{S}}} \mathcal{E}_{\mathcal{T}}^2(S)$. Then

$$\text{squared } \textit{total} \text{ error} := |u - u_{\mathcal{T}}|_{H^1(\Omega)}^2 + \underbrace{\sum_{T \in \mathcal{T}} h_T^2 \|f - \bar{f}\|_{L_2(T)}^2}_{\mathrm{osc}^2(\mathcal{T}) :=} \eqsim \mathcal{E}_{\mathcal{T}}^2(\mathcal{S}(\mathcal{T})).$$

Estimator sum of local contributions. Viewing them as local error indicators suggests **AFEM** (1970's by Babuška et al.):

$$\texttt{Solve} \longrightarrow \texttt{Estimate} \longrightarrow \texttt{Mark} \longrightarrow \texttt{Refine}$$

A posteriori error estimator With $\mathcal{S}(\mathcal{T})$ set of sides, for $S \in \mathcal{S}(\mathcal{T})$,

$$\mathcal{E}^2_{\mathcal{T}}(S) := h_S \big\| [\![\nabla u_{\mathcal{T}}]\!] \big\|^2_{L^2(S)} + \sum_{\{T \in \mathcal{T} : T \supset S\}} h_T^2 \|f\|^2_{L^2(T)}$$

For $\tilde{\mathcal{S}} \subset \mathcal{S}(\mathcal{T})$, $\mathcal{E}^2_{\mathcal{T}}(\tilde{\mathcal{S}}) := \sum_{S \in \tilde{\mathcal{S}}} \mathcal{E}^2_{\mathcal{T}}(S)$. Then

$$\text{squared } \textit{total} \text{ error} := |u - u_{\mathcal{T}}|^2_{H^1(\Omega)} + \underbrace{\sum_{T \in \mathcal{T}} h_T^2 \|f - \bar{f}\|^2_{L_2(T)}}_{\operatorname{osc}^2(\mathcal{T}) :=} \eqsim \mathcal{E}^2_{\mathcal{T}}(\mathcal{S}(\mathcal{T})).$$

Estimator sum of local contributions. Viewing them as local error indicators suggests **AFEM** (1970's by Babuška et al.):

$$\texttt{Solve} \longrightarrow \texttt{Estimate} \longrightarrow \texttt{Mark} \longrightarrow \texttt{Refine}$$

Common marking strategies: *Bulk chasing*: Collect smallest $\mathcal{M} \subset \mathcal{S}(\mathcal{T})$ (*sides*) s.t. $\mathcal{E}^2_{\mathcal{T}}(\mathcal{M}) \geq \theta \mathcal{E}^2_{\mathcal{T}}(\mathcal{S}(\mathcal{T}))$; *Maximum marking*: $\mathcal{M} = \{S \in \mathcal{S}(\mathcal{T}) : \mathcal{E}^2_{\mathcal{T}}(S) \geq \theta \max_{\tilde{S} \in \mathcal{S}(\mathcal{T})} \mathcal{E}^2_{\mathcal{T}}(\tilde{S})\}$ (no sorting needed, less sensitive to choice $\theta \in (0,1]$).

Short history of AFEM convergence theory

Convergence proof in [Dörfler '96] (before that only results in 1D).

Improved result (allowing $\mathcal{T}_0 = \mathcal{T}_\perp$) in [Morin, Nochetto, Siebert '00].

Opt. *rates* for AFEM extended with coarsening in [Binev, Dahmen, DeVore '04]: If within class of triangulations $\mathbb{T}$ (with $\overline{\cup_{\mathcal{T}\in\mathbb{T}}\mathbb{V}(\mathcal{T})} = H_0^1(\Omega)$), $\exists(\mathcal{T}_n)_n$ s.t. for some $s > 0$, $|u - u_{\mathcal{T}_n}|_{H^1(\Omega)} \lesssim (\#\mathcal{T}_n)^{-s}$, then sequence produced by this AFEM has this property.

Opt. rates without coarsening in [Ste '07]. Separate condition needed on the approximability of rhs by piecewise polynomials.

Proof that AFEM reduces the total error with opt. rate in [Cascón, Kreuzer, Nochetto, Siebert '08].

All results for bulk chasing marking strategy.

Instance optimality

Thm 1. *For* $(\mathcal{T}_k)_{k\in\mathbb{N}_0} \subset \mathbb{T}$ *produced by* `AFEM` *with (a modified) maximum marking strategy,*

$$|u - u_{\mathcal{T}_k}|^2_{H^1(\Omega)} + \operatorname{osc}^2(\mathcal{T}_k) \leq \tilde{C}\left(|u - u_{\mathcal{T}}|^2_{H^1(\Omega)} + \operatorname{osc}^2(\mathcal{T})\right)$$

for all $\mathcal{T} \in \mathbb{T}$ *with* $\#\mathcal{N}(\mathcal{T}) \setminus \mathcal{N}(\mathcal{T}_\perp) \leq \dfrac{\#\mathcal{N}(\mathcal{T}_k) \setminus \mathcal{N}(\mathcal{T}_\perp)}{C}$.

Instance optimality $\Longrightarrow$ optimal rates.

Energy minimalisation

Solving Poisson $\iff$ minimizing Dirichlet energy

$$\mathcal{J}(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 - f(v).$$

$\mathcal{J}(\mathcal{T}) := \mathcal{J}(u_\mathcal{T})$, $\mathcal{J}(\mathcal{T}^\top) := \mathcal{J}(u)$.

For $\mathcal{T} \leq \mathcal{T}_* \leq \mathcal{T}^\top$:

$$\mathcal{J}(\mathcal{T}) - \mathcal{J}(\mathcal{T}_*) = \frac{1}{2}|u_\mathcal{T} - u_{\mathcal{T}_*}|^2_{H^1(\Omega)}.$$

Energy minimalisation

Solving Poisson $\iff$ minimizing Dirichlet energy

$$\mathcal{J}(v) := \frac{1}{2}\int_\Omega |\nabla v|^2 - f(v).$$

$\mathcal{J}(\mathcal{T}) := \mathcal{J}(u_\mathcal{T})$, $\mathcal{J}(\mathcal{T}^\top) := \mathcal{J}(u)$.

For $\mathcal{T} \leq \mathcal{T}_* \leq \mathcal{T}^\top$:

$$\mathcal{J}(\mathcal{T}) - \mathcal{J}(\mathcal{T}_*) = \frac{1}{2}|u_\mathcal{T} - u_{\mathcal{T}_*}|^2_{H^1(\Omega)}.$$

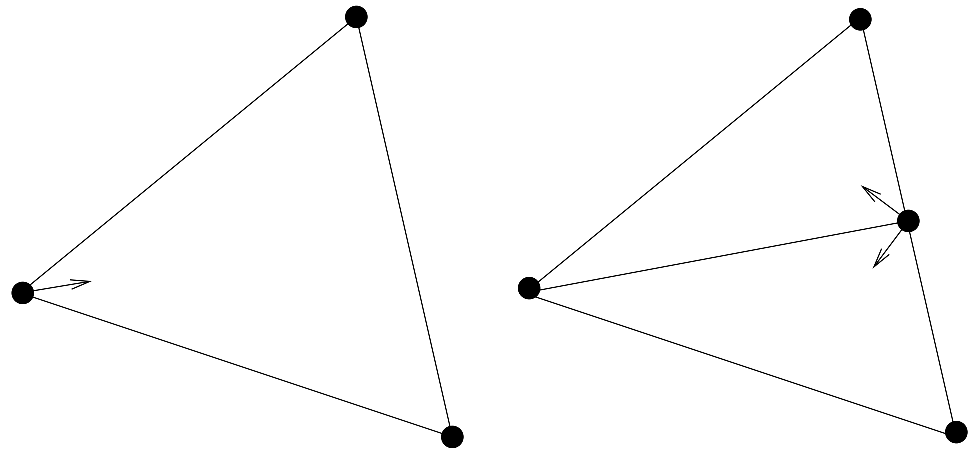
Total energy: $\mathcal{G}(\mathcal{T}) := \mathcal{J}(\mathcal{T}) + \sum_{T\in\mathcal{T}} h_T^2 \|f\|^2_{L_2(T)}$,

$$\mathcal{G}(\mathcal{T}) - \mathcal{G}(\mathcal{T}^\top) \eqsim |u - u_\mathcal{T}|^2_{H^1(\Omega)} + \mathrm{osc}^2(\mathcal{T}).$$

So suffices to prove

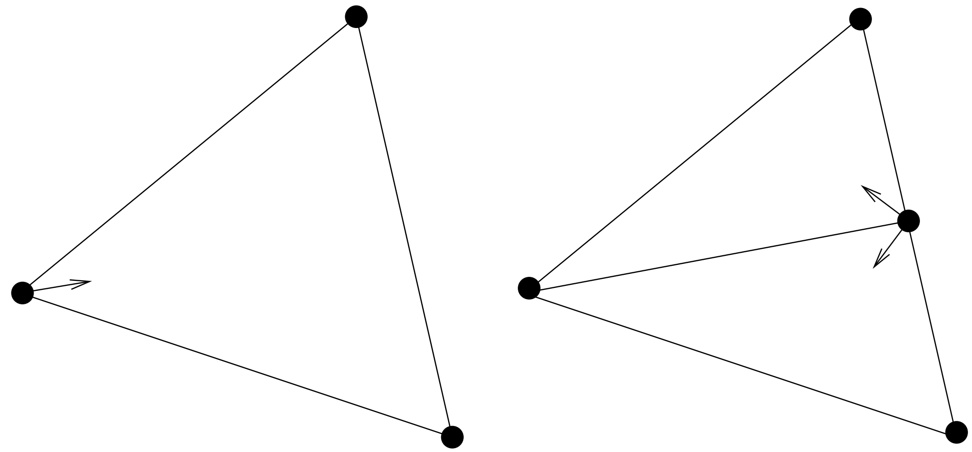
Thm 2. *For* $(\mathcal{T}_k)_{k\in\mathbb{N}_0} \subset \mathbb{T}$ *produced by* `AFEM`, $\mathcal{G}(\mathcal{T}_k) \leq \mathcal{G}(\mathcal{T})$ *for all* $\mathcal{T} \in \mathbb{T}$ *with* $\#\mathcal{N}(\mathcal{T}) \setminus \mathcal{N}(\mathcal{T}_\perp) \leq \dfrac{\#\mathcal{N}(\mathcal{T}_k) \setminus \mathcal{N}(\mathcal{T}_\perp)}{C}$.

Specification of $\mathbb{T}$: NVB



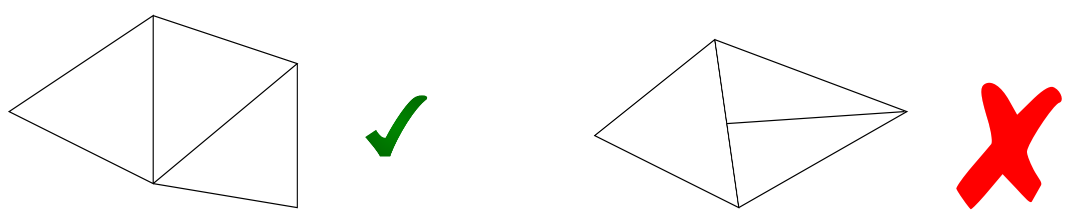
Set of all descendants organized as infinite binary tree. Uniform shape regularity.

Specification of $\mathbb{T}$: NVB



Set of all descendants organized as infinite binary tree. Uniform shape regularity.

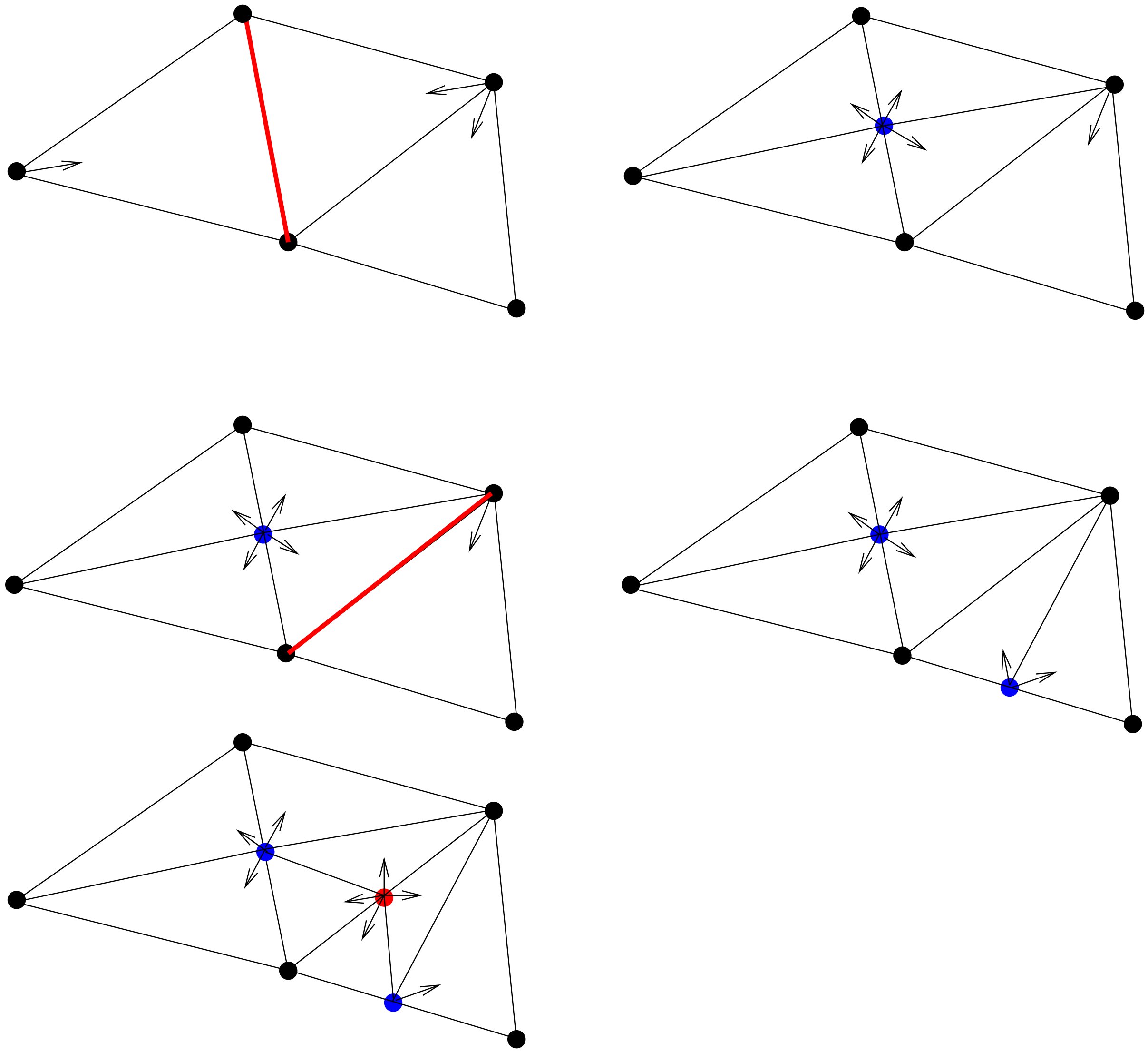
Given *conforming* initial triangulation $\mathcal{T}_\perp$,



and a selection of the newest vertices in all $T \in \mathcal{T}_\perp$, let $\mathbb{T}$ denote set of all *conforming* NVB descendants $\mathcal{T}$.

Refine

For $\mathcal{T} \in \mathbb{T}$, $\mathcal{M} \subset \mathcal{S}(\mathcal{T})$, let $\texttt{Refine}(\mathcal{T};\mathcal{M}) \in \mathbb{T}$ smallest refinement of $\mathcal{T}$ in which all $S \in \mathcal{M}$ have been bisected.



Thm 3 (Binev, Dahmen, DeVore '04). *With suitable choice of the newest vertices in* $\mathcal{T}_\perp$, $(\mathcal{T}_k)_k \subset \mathbb{T}$ *defined by* $\mathcal{T}_0 = \mathcal{T}_\perp$ *and* $\mathcal{T}_{k+1} = \texttt{Refine}(\mathcal{T}_k; \mathcal{M}_k)$ *for some* $\mathcal{M}_k \subset \mathcal{S}(\mathcal{T}_k)$, $k = 0, 1, \ldots$, *satisfies*

$$\#\mathcal{N}(\mathcal{T}_k) \setminus \mathcal{N}(\mathcal{T}_\perp) \eqsim \sum_{i=0}^{k-1} \#\mathcal{M}_i.$$

Sufficient results for instance optimality

Let $\mathcal{T}_m^{\mathrm{opt}}$ minimizes $\mathcal{G}$ among all $\mathcal{T} \in \mathbb{T}$ with $\#\mathcal{N}(\mathcal{T}) \setminus \mathcal{N}(\mathcal{T}_\perp) \leq m$.

Lem 1. *Let* $(\mathcal{T}_k)_{k\in\mathbb{N}_0} \subset \mathbb{T}$ *prod. by* `AFEM`*, with* $(\mathcal{M}_k)_{k\in\mathbb{N}_0}$ *seq. of marked sides. With* m *s.t.* $\mathcal{G}(\mathcal{T}_m^{\mathrm{opt}}) \geq \mathcal{G}(\mathcal{T}_k) > \mathcal{G}(\mathcal{T}_{m+1}^{\mathrm{opt}})$*, one has*

$$\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_{k+1}) \geq \gamma\, \#\mathcal{M}_k \left(\mathcal{G}(\mathcal{T}_m^{\mathrm{opt}}) - \mathcal{G}(\mathcal{T}_{m+1}^{\mathrm{opt}})\right).$$

E.g. $\#\mathcal{M}_k \equiv 1$ and, say $\gamma = \frac{1}{2}$, then $\mathcal{G}(\mathcal{T}_k) \leq \mathcal{G}(\mathcal{T}_{\lfloor k/2 \rfloor}^{\mathrm{opt}})$, and, using the BDD result, $\#\mathcal{N}(\mathcal{T}_k) \setminus \mathcal{N}(\mathcal{T}_\perp) \lesssim k$, which means instance optimality.

Sufficient results for instance optimality

Let $\mathcal{T}_m^{\mathrm{opt}}$ minimizes $\mathcal{G}$ among all $\mathcal{T} \in \mathbb{T}$ with $\#\mathcal{N}(\mathcal{T}) \setminus \mathcal{N}(\mathcal{T}_\perp) \leq m$.

Lem 1. *Let* $(\mathcal{T}_k)_{k\in\mathbb{N}_0} \subset \mathbb{T}$ *prod. by* `AFEM`*, with* $(\mathcal{M}_k)_{k\in\mathbb{N}_0}$ *seq. of marked sides. With* m *s.t.* $\mathcal{G}(\mathcal{T}_m^{\mathrm{opt}}) \geq \mathcal{G}(\mathcal{T}_k) > \mathcal{G}(\mathcal{T}_{m+1}^{\mathrm{opt}})$*, one has*

$$\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_{k+1}) \geq \gamma\, \#\mathcal{M}_k \big(\mathcal{G}(\mathcal{T}_m^{\mathrm{opt}}) - \mathcal{G}(\mathcal{T}_{m+1}^{\mathrm{opt}})\big).$$

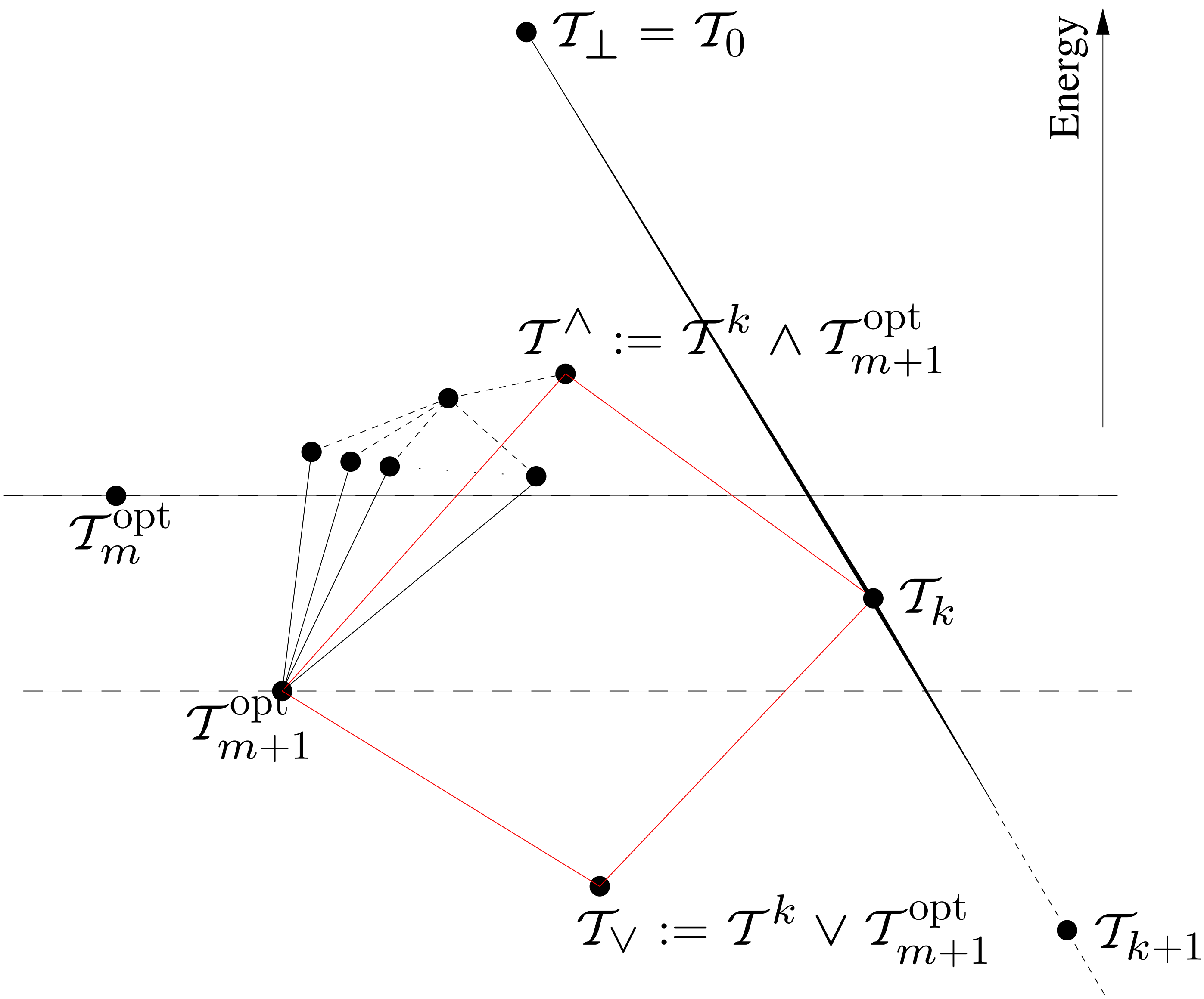
E.g. $\#\mathcal{M}_k \equiv 1$ and, say $\gamma = \frac{1}{2}$, then $\mathcal{G}(\mathcal{T}_k) \leq \mathcal{G}(\mathcal{T}_{\lfloor k/2 \rfloor}^{\mathrm{opt}})$, and, using the BDD result, $\#\mathcal{N}(\mathcal{T}_k) \setminus \mathcal{N}(\mathcal{T}_\perp) \lesssim k$, which means instance optimality.

For $\#\mathcal{M}_k$ “large”, we have (with a similar proof):

Lem 2. $\mathcal{G}(\mathcal{T}_{k+1}) \leq \mathcal{G}(\mathcal{T}_{m+\lfloor \frac{\#\mathcal{M}_k}{K} \rfloor}^{\mathrm{opt}})$.

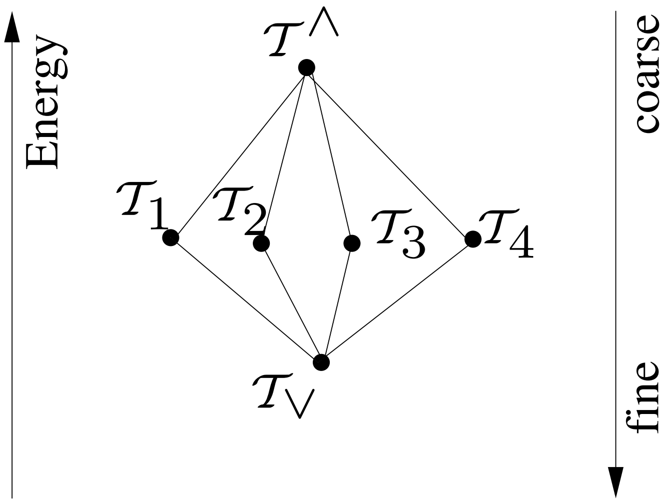
Together Lemmas 1 and 2 give the instance optimality result Thm 2.
Focus on Lemma 1.

Illus. with Lemma 1



The lower diamond estimate

Def 1. For $\{\mathcal{T}_1, \ldots, \mathcal{T}_m\} \subset \mathbb{T}$, we call $(\mathcal{T}^\wedge, \mathcal{T}_\vee; \mathcal{T}_1, \ldots, \mathcal{T}_m)$ a *lower diamond* of size m when $\mathcal{T}^\wedge = \bigwedge_{j=1}^m \mathcal{T}_j$, $\mathcal{T}_\vee = \bigvee_{j=1}^m \mathcal{T}_j$, and the areas of *coarsening* $\Omega(\mathcal{T}_j \setminus \mathcal{T}_\vee)$ are pairwise disjoint.

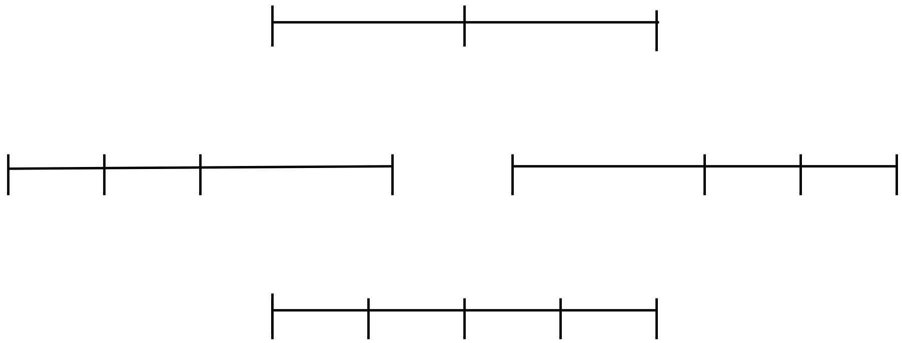


Rem 1. Last condition is fulfilled automatically when $m = 2$.

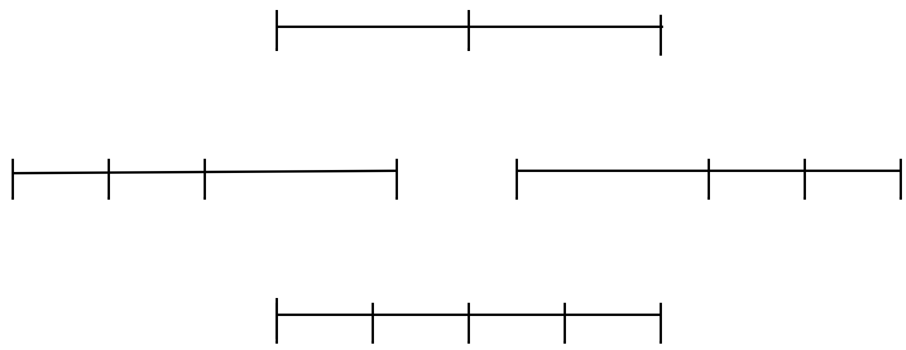
Thm 4. *Let* $(\mathcal{T}^\wedge, \mathcal{T}_\vee; \mathcal{T}_1, \ldots, \mathcal{T}_m)$ *be a lower diamond. Then*

$$|u_{\mathcal{T}_\vee} - u_{\mathcal{T}^\wedge}|^2_{H^1(\Omega)} \eqsim \sum_{j=1}^m |u_{\mathcal{T}_\vee} - u_{\mathcal{T}_j}|^2_{H^1(\Omega)}$$

Error in best approx equiv to error in stable projection. Interpolation in 1D



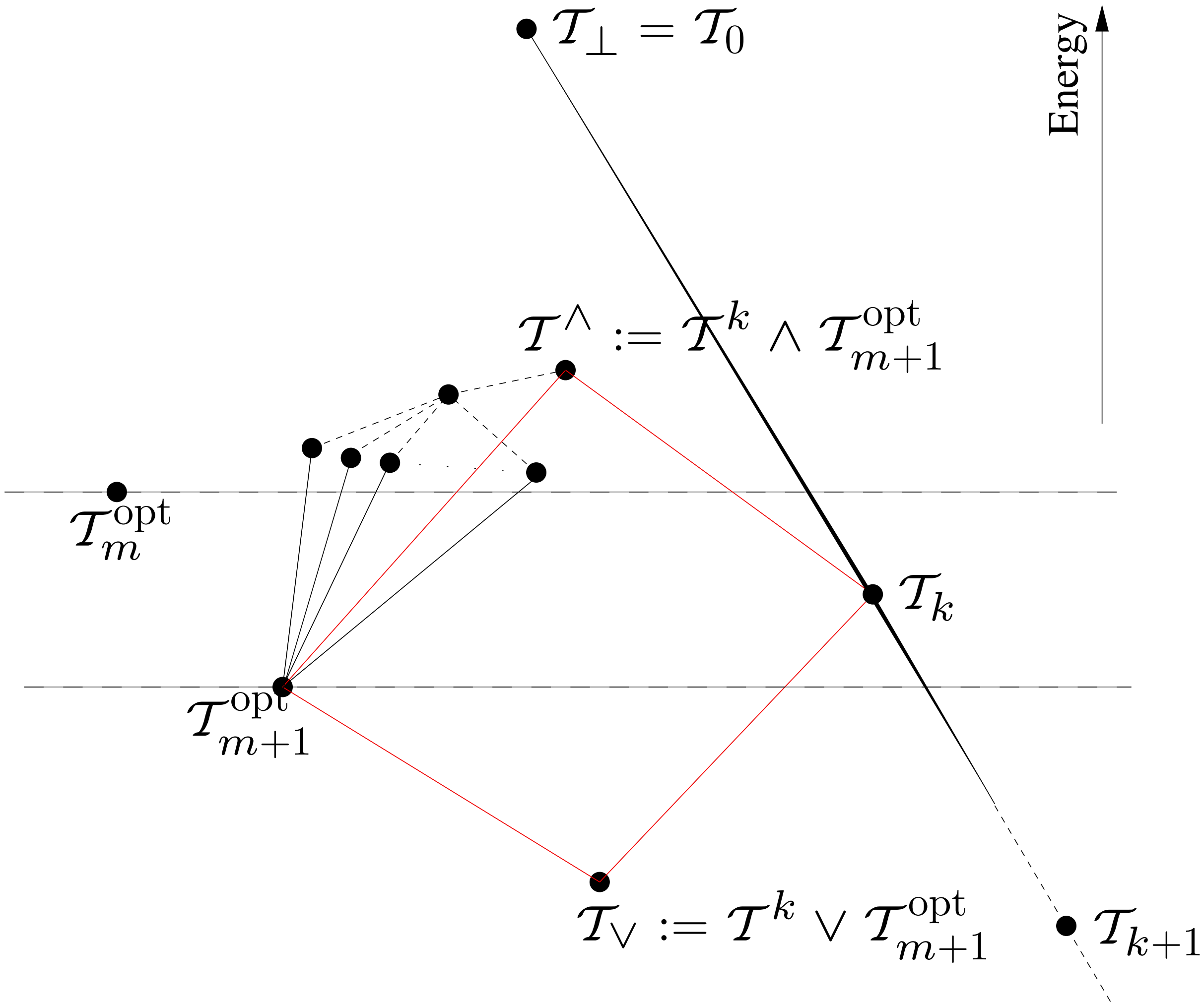
Error in best approx equiv to error in stable projection. Interpolation in 1D



In nD, use Scott-Zhang.

Corol 1. *Let* $(\mathcal{T}^{\wedge}, \mathcal{T}_{\vee}; \mathcal{T}_1, \ldots, \mathcal{T}_m)$ *be a lower diamond, then*

$$\mathcal{G}(\mathcal{T}^{\wedge}) - \mathcal{G}(\mathcal{T}_{\vee}) \eqsim \sum_{j=1}^{m} \mathcal{G}(\mathcal{T}_j) - \mathcal{G}(\mathcal{T}_{\vee})$$



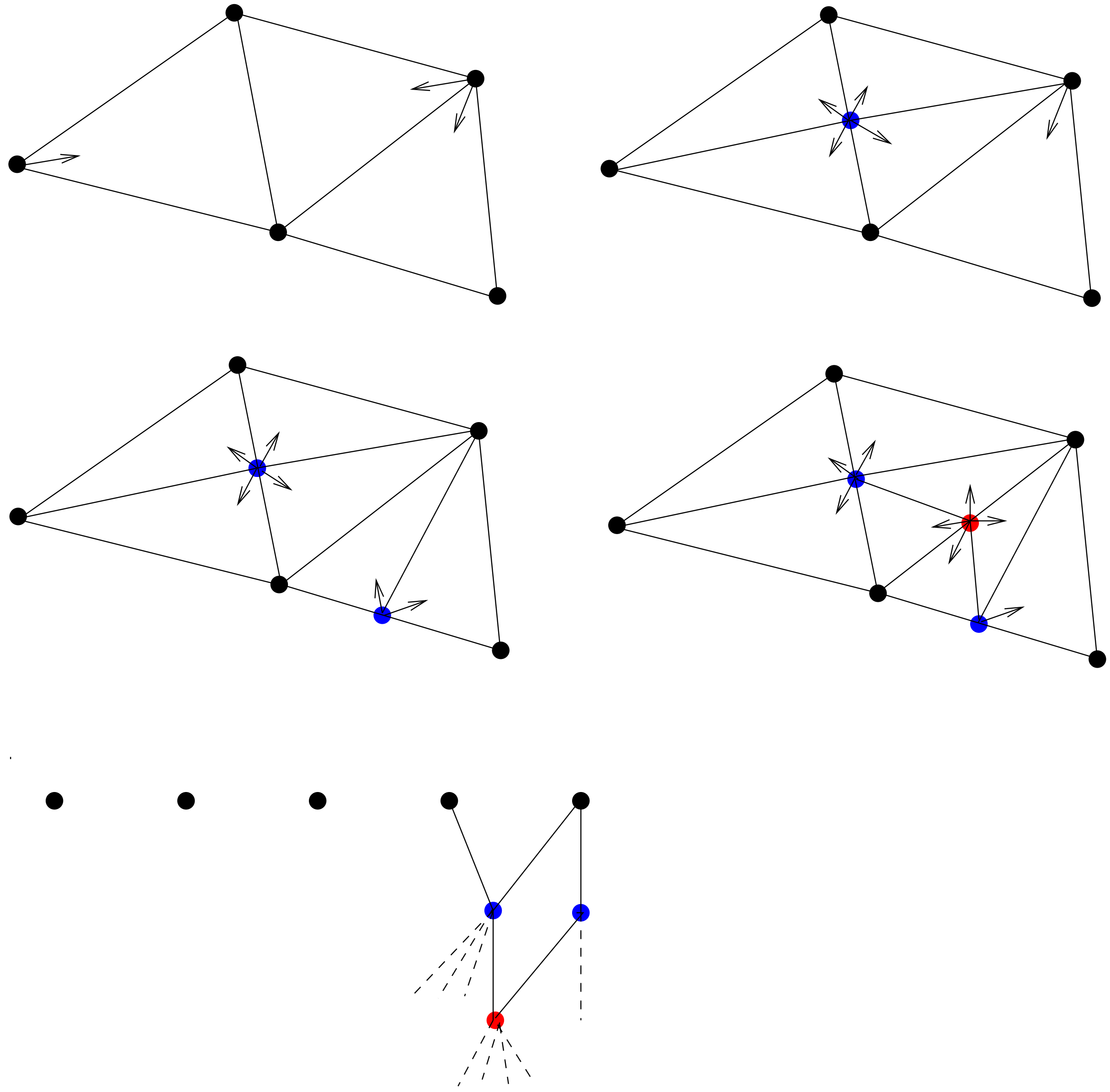
$$\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_\vee) \geq \frac{1}{2}\Big(\big(\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_\vee)\big) + \big(\mathcal{G}(\mathcal{T}^{\text{opt}}_{m+1}) - \mathcal{G}(\mathcal{T}_\vee)\big)\Big)$$
$$\eqsim \mathcal{G}(\mathcal{T}^\wedge) - \mathcal{G}(\mathcal{T}_\vee) \geq \mathcal{G}(\mathcal{T}^\wedge) - \mathcal{G}(\mathcal{T}^{\text{opt}}_{m+1}).$$

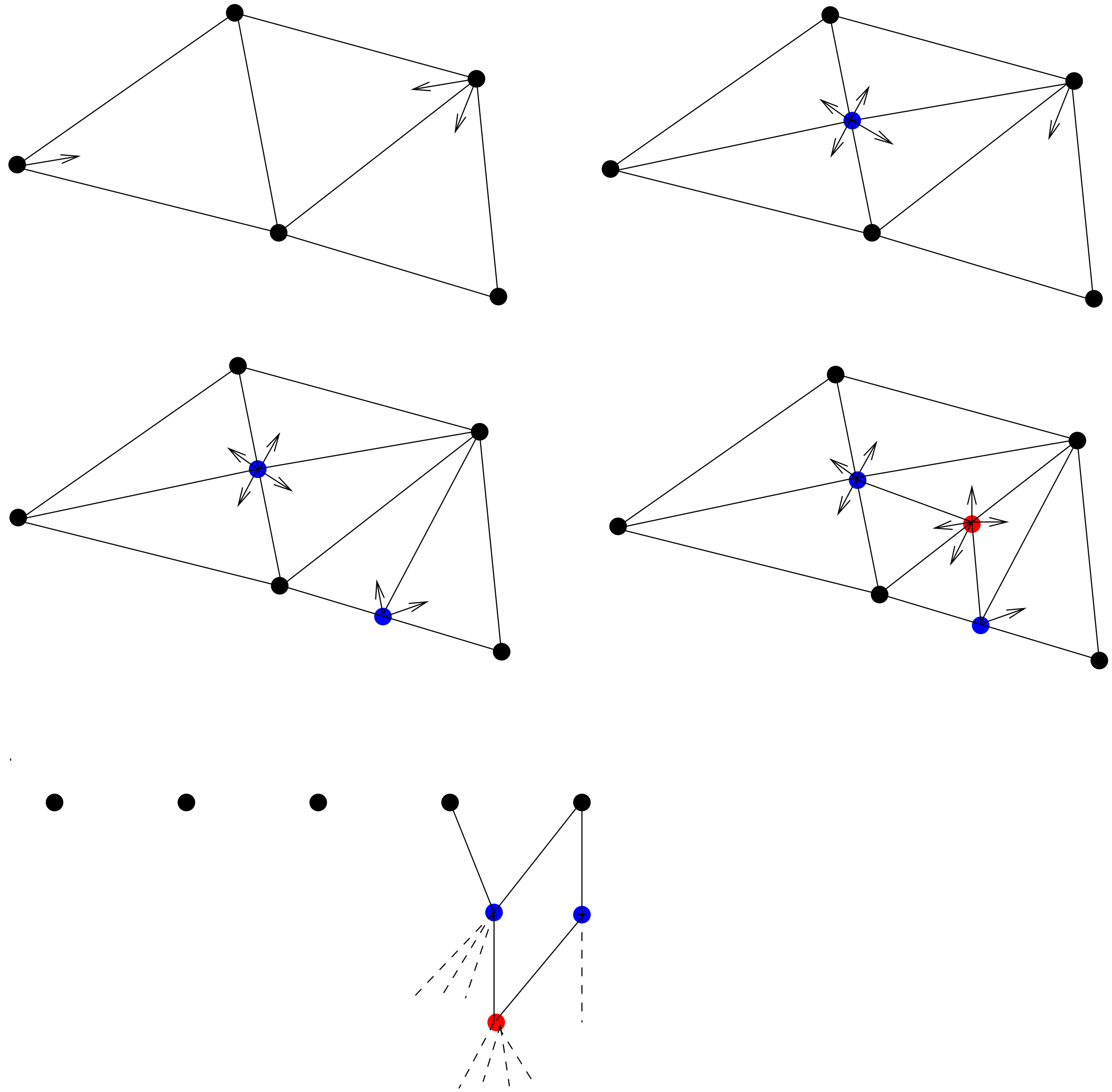
Family tree, and free nodes

Let $\mathfrak{N} := \bigcup_{\mathcal{T} \in \mathbb{T}} \mathcal{N}(\mathcal{T})$. Every $v \in \mathfrak{N} \setminus \mathcal{N}(\mathcal{T}_\perp)$ is midpoint edge $e = T \cap \tilde{T}$ opposite to the 2 (1 if $v \in \partial\Omega$) newest vertices of T and $\tilde{T}$.

Calling these vertices parent(s) v, a (family) tree structure is defined on $\mathfrak{N}$.

For $\mathcal{T} \in \mathbb{T}$, $\mathcal{N}(\mathcal{T})$ is a subtree of this infinite master tree.





Elements of $\mathcal{N}(\mathcal{T}) \setminus \mathcal{N}(\mathcal{T}^{\perp})$ without children are called the free nodes of $\mathcal{T}$. A free node can be removed whilst retaining a triangulation in $\mathbb{T}$.

Proof of Lemma 1

(Modified) maximum strategy

and

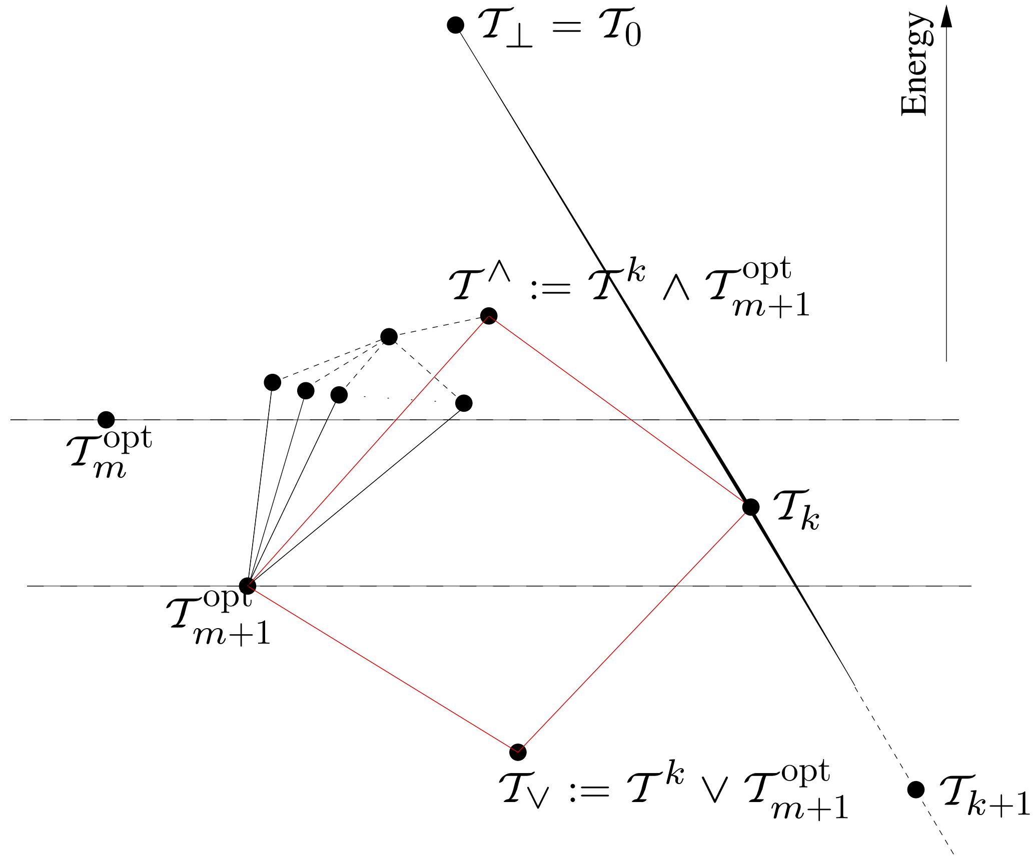
Thm 5. *Let* $\mathcal{T}, \mathcal{T}_* \in \mathbb{T}$ *with* $\mathcal{T} \le \mathcal{T}_*$. *Then*

$$\mathcal{G}(\mathcal{T}) - \mathcal{G}(\mathcal{T}_*) \eqsim \mathcal{E}^2_{\mathcal{T}}(\mathcal{S}(\mathcal{T}) \setminus \mathcal{S}(\mathcal{T}_*)).$$

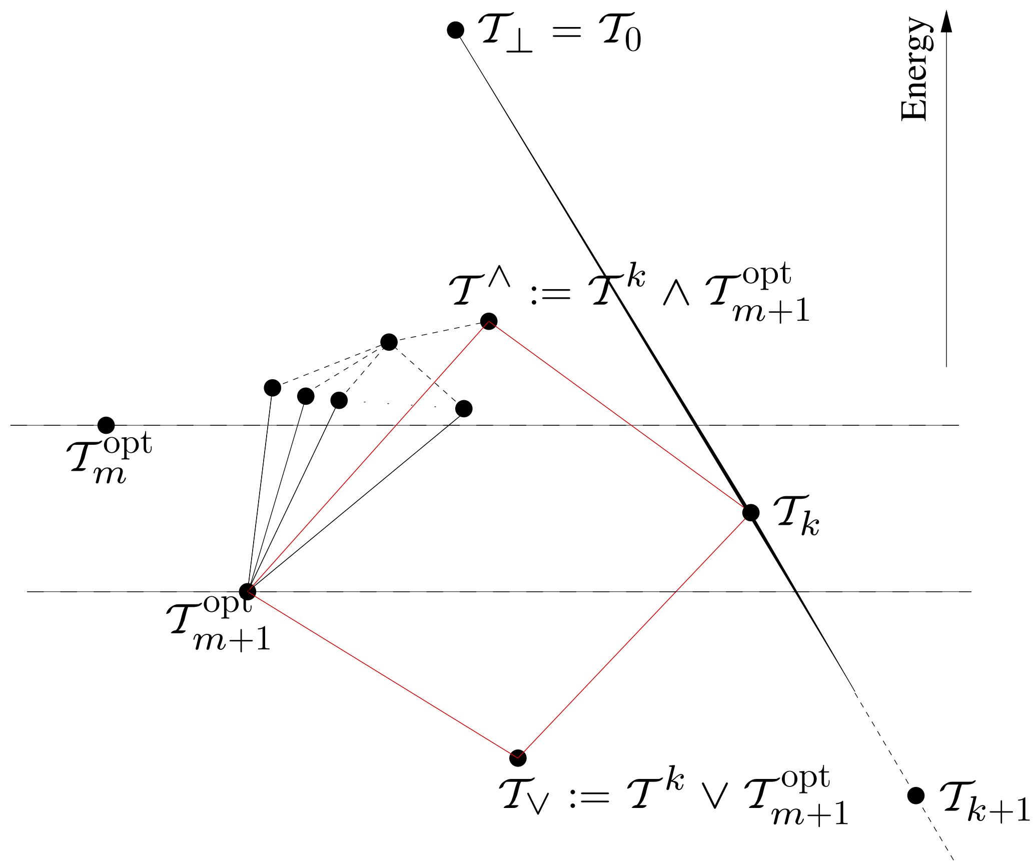
and

Thm 6. *For* $\mathcal{T} \le \mathcal{T}_*$, $\#\mathtt{free}(\mathcal{N}(\mathcal{T})) \lesssim \#\mathtt{free}(\mathcal{N}(\mathcal{T}_*))$.

$$\rightsquigarrow \quad \mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_{k+1}) \gtrsim \frac{\#\mathcal{M}_k}{\#\mathtt{free}(\mathcal{N}(\mathcal{T}_\vee) \setminus \mathcal{N}(\mathcal{T}_k))} \big(\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_\vee)\big).$$

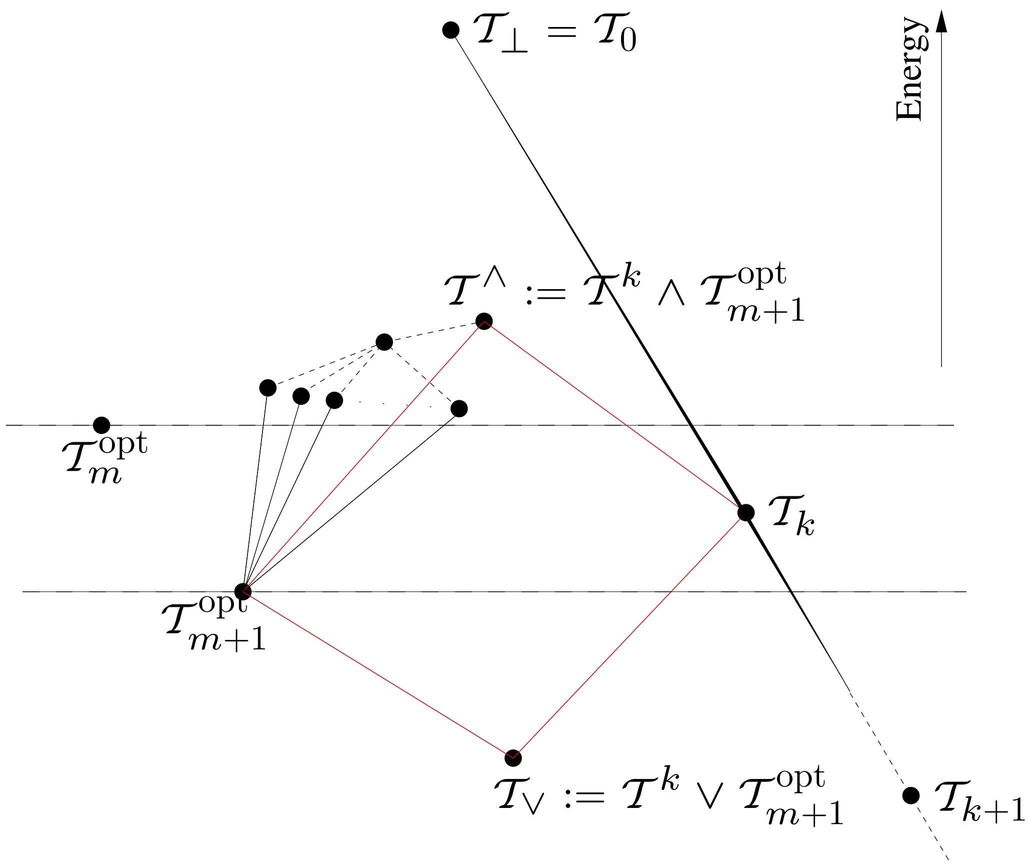


$$\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_{k+1}) \gtrsim \frac{\#\mathcal{M}_k}{\#\mathtt{free}(\mathcal{N}(\mathcal{T}_\vee) \setminus \mathcal{N}(\mathcal{T}_k))} \big(\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_\vee)\big)$$

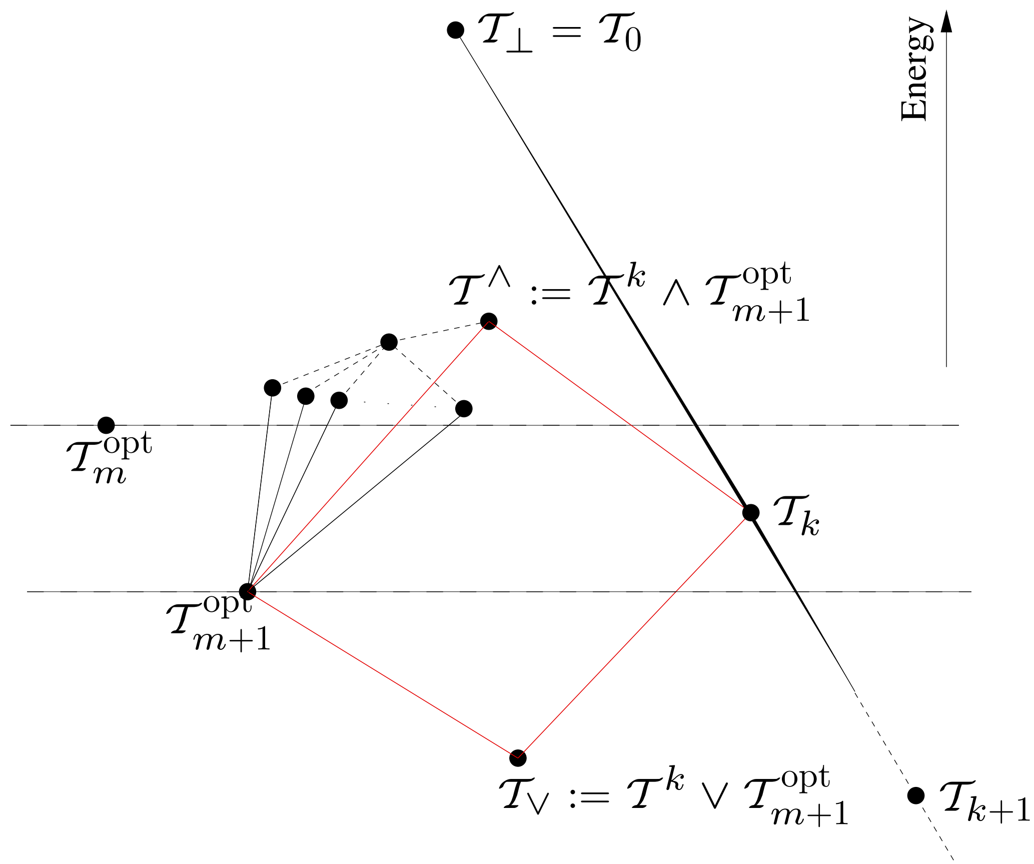


$$\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_{k+1}) \gtrsim \frac{\#\mathcal{M}_k}{\#\mathtt{free}(\mathcal{N}(\mathcal{T}_\vee) \setminus \mathcal{N}(\mathcal{T}_k))} \big(\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_\vee)\big)$$

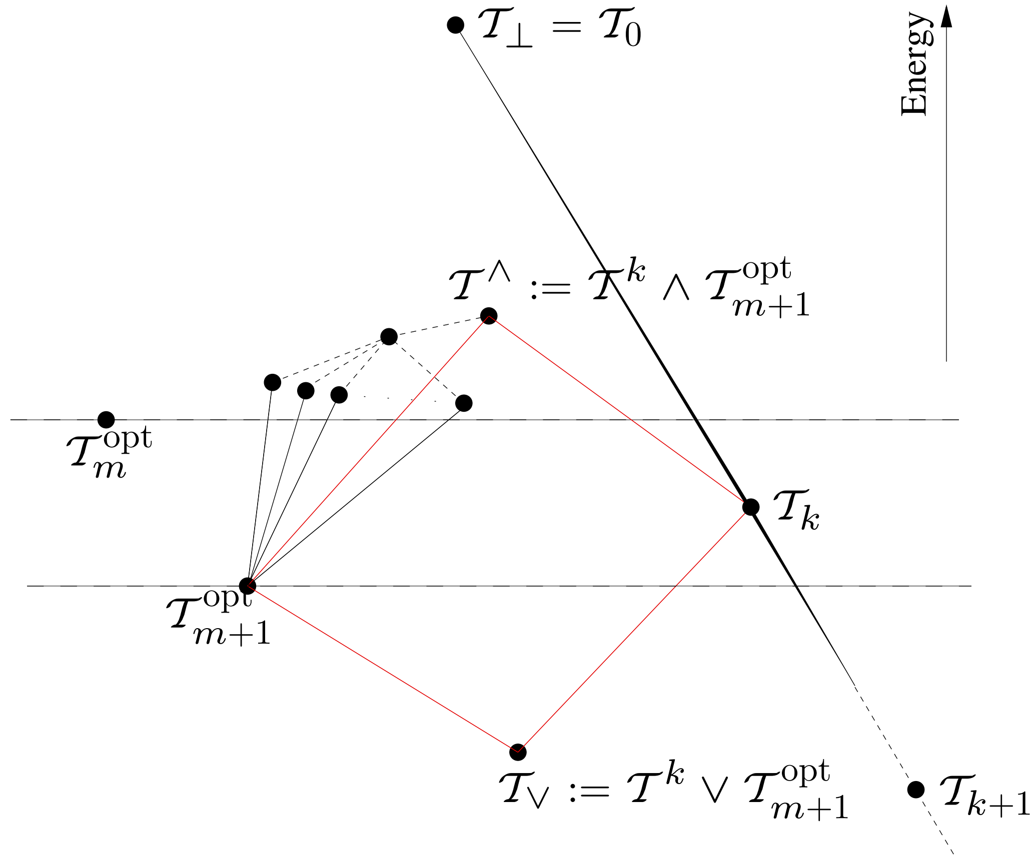
$$= \frac{\#\mathcal{M}_k}{\#\mathtt{free}(\mathcal{N}(\mathcal{T}_{m+1}^{\text{opt}} \setminus \mathcal{N}(\mathcal{T}^\wedge))} \big(\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_\vee)\big)$$



$$\begin{aligned}
\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_{k+1}) &\gtrsim \frac{\#\mathcal{M}_k}{\#\mathtt{free}(\mathcal{N}(\mathcal{T}_\vee) \setminus \mathcal{N}(\mathcal{T}_k))}\big(\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_\vee)\big) \\
&= \frac{\#\mathcal{M}_k}{\#\mathtt{free}(\mathcal{N}(\mathcal{T}_{m+1}^{\text{opt}} \setminus \mathcal{N}(\mathcal{T}^\wedge))}\big(\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_\vee)\big) \\
&\gtrsim \frac{\#\mathcal{M}_k}{\#\mathtt{free}(\mathcal{N}(\mathcal{T}_{m+1}^{\text{opt}}) \setminus \mathcal{N}(\mathcal{T}^\wedge))}\big(\mathcal{G}(\mathcal{T}^\wedge) - \mathcal{G}(\mathcal{T}_{m+1}^{\text{opt}})\big)
\end{aligned}$$



$$\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_{k+1}) \gtrsim \frac{\#\mathcal{M}_k}{\#\mathtt{free}(\mathcal{N}(\mathcal{T}_{m+1}^{\text{opt}}) \setminus \mathcal{N}(\mathcal{T}^\wedge))}\big(\mathcal{G}(\mathcal{T}^\wedge) - \mathcal{G}(\mathcal{T}_{m+1}^{\text{opt}})\big)$$



$$\mathcal{G}(\mathcal{T}_k) - \mathcal{G}(\mathcal{T}_{k+1}) \gtrsim \frac{\#\mathcal{M}_k}{\#\mathtt{free}(\mathcal{N}(\mathcal{T}_{m+1}^{\text{opt}}) \setminus \mathcal{N}(\mathcal{T}^\wedge))}\big(\mathcal{G}(\mathcal{T}^\wedge) - \mathcal{G}(\mathcal{T}_{m+1}^{\text{opt}})\big)$$

$$\mathcal{G}(\mathcal{T}^\wedge) - \mathcal{G}(\mathcal{T}_{m+1}^{\text{opt}}) \gtrsim \#\mathtt{free}(\mathcal{N}(\mathcal{T}_{m+1}^{\text{opt}} \setminus \mathcal{N}(\mathcal{T}^\wedge))\big(\mathcal{G}(\mathcal{T}_m^{\text{opt}}) - \mathcal{G}(\mathcal{T}_{m+1}^{\text{opt}})\big).$$

Summary

`AFEM` with a (modified) maximum marking strategy is *instance optimal* for reducing the total error.

Summary

`AFEM` with a (modified) maximum marking strategy is *instance optimal* for reducing the total error.

Thank you for your attention!